Generating Functions for Enumerating Chains of Partitions with Distinct Parts

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Abstract

This paper continues investigations by Stanley and Butler in which they enumerate chains of partitions. More specifically, we consider pairs of partitions composed of distinct parts and their corresponding Young diagrams and look for interesting properties of their generating functions. A proof of the generating function is provided when the difference between a pair of diagrams is fixed in one of three ways: a single box, k boxes in a row, and one box added to two consecutive rows whose lengths differ by one. In particular, we provide an explicit, rational formula when these generating functions are divided by the generating function for partitions made of distinct parts. We conclude with a conjecture for the generating function of adding k disjoint boxes to a Young diagram with distinct parts.

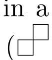
1 Introduction

Partitions of natural numbers, though a seemingly simple concept, have nonetheless captured the attention of mathematicians for centuries due to their complex patterns and properties. Some of the brightest mathematical minds, like Euler, G.H. Hardy, and Ramanujan, have dedicated parts of their careers to understanding their properties. These mathematicians, and many others, have discovered interesting applications of partitions to many branches of math. For example, the denominator of the generating function for partitions is Euler's function, which also has interesting properties in complex analysis. In addition, the theory of partitions is included in the study of symmetric functions and representation theory, among other areas.

More recently, Richard Stanley has also spent time studying properties of partitions and one of their visual representations, Young diagrams [3]. One of his investigations is the study of chains of partitions ordered by containment of their Young diagrams. In particular, let $S = \{a_1, ..., a_j\}$ and $\lambda^0 < ... < \lambda^j$ be a chain of partitions such that $|\lambda^0| = n$ and $|\lambda^i| = n + a_i$ for $1 \leq i \leq j$. Stanley showed that the generating function for the number of these chains, $\sum_{n\geq 0} f_S(n)q^n$, divided by the generating function for partitions is a rational function. More specifically, denote the generating function for the number of partitions as $P(q) = \prod_{i\geq 1}(1 - q^i)^{-1}$. Then, Stanley's result states that

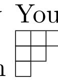
$\dfrac{\sum_{n\geq 0} f_S(n)q^n}{P(q)} = A_S(q)$ where $A_S(q)$ is a rational function whose denominator can be written as $(1-q)(1-q^2)...(1-q^{a_j})$. Furthermore, when the Young diagrams of the specified partitions differ by a single connected component, σ, Stanley provides an explicit formula for $A_S(q)$. After presenting his proof of the first claim, he also poses the question of whether or not a simpler proof is possible. The exact details of his proof will be discussed later.

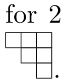
Lynne Butler continued Stanley's investigation of the rational function $A_S(q)$ [2]. She provides a bijective proof of Stanley's claim when σ is a single connected component and generalizes his claim for all pairs of partitions. In addition, when σ is composed of k connected components, she proves that if the numerator of $A_S(q)$ is divided by $(1-q)^{|\sigma|-k}$ then a polynomial with nonnegative coefficients is produced. Finally, she applies her results to investigate the enumeration of chains $\lambda^0 < ... < \lambda^j$ described in the previous paragraph.

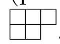
The aim of this paper is to describe an investigation similar to Stanley and Butler's. Instead of considering general partitions, we investigate partitions composed of distinct parts. In particular, we look for interesting properties of the generating functions for pairs of partitions with distinct parts. We present proofs for explicit formulas of these generating functions when σ is one box, k connected boxes in a row, and two boxes added to consecutive rows whose lengths differ by 1 (). Finally, we provide a conjecture of the formula for adding k boxes to k distinct rows.

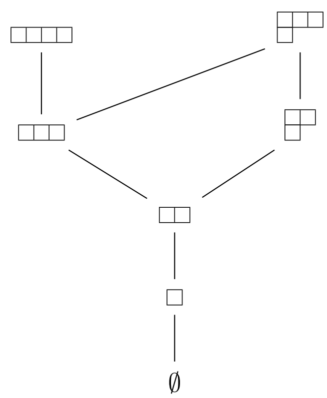
2 Background

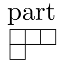
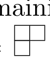
Before beginning the investigation described in the previous section, we describe the notation used throughout the paper. Given a positive integer n, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is a partition of n if the λ_i are weakly decreasing positive integers (if $i < j$ then $\lambda_i \geq \lambda_j$) and $\sum_{i=1}^{k} \lambda_i = n$. In this case, we write $\lambda \vdash n$. For example, $\lambda = (3, 2, 1)$ is a partition of 6.

We may represent a partition visually by a Young diagram. Given $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$, the Young diagram of shape λ with size n is a diagram of n boxes having k left-justified rows with row i containing λ_i boxes for $1 \leq i \leq k$. Because the parts of a partition are weakly decreasing, row i cannot extend further to the right than row j for any $i > j$ in any Young diagram. For example, $\lambda = (3, 2, 1)$ corresponds to the Young diagram  .

We call a Young diagram shifted if row i is shifted $i-1$ spaces to the right for $2 \leq i \leq k$. The shifted Young diagram corresponding to $\lambda = (3, 2, 1)$ is . To maintain the property that row i cannot extend further to the right than row j for $i > j$ for shifted Young diagrams, the components of λ must be strictly decreasing ($i < j$ then $\lambda_i > \lambda_j$) instead of weakly decreasing. In other words, shifted Young diagrams can only be produced from partitions with distinct parts. (If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, then $\lambda_i \neq \lambda_j$ for $i \neq j$.) The notion of shiftedness led us to our investigation of Young diagrams composed of distinct parts.

These Young diagrams form a partially ordered set (poset) where the diagrams are ordered by containment. For example,  $\subset$ . We can visualize this relation by constructing the lattice formed via Young diagrams with distinct parts under this containment. An edge is drawn between 2 diagrams in the lattice if the size of the diagrams differs by 1 and the smaller diagram is contained in the larger. The bottom 5 levels of this lattice are shown here:



We designate the row containing the empty diagram to be row 0, so that row n corresponds to the row of Young diagrams with distinct parts of size n. Furthermore, given a chain of diagrams $\lambda^0 < ... < \lambda^j$, the skew Young diagram of (λ^j, λ^0) denoted by λ^j/λ^0 is the part of λ^j remaining after λ^0 has been removed from it. For example, if $\lambda =$  and $\mu =$  then $\lambda/\mu = \square$. Now, fix a skew diagram $\sigma = (\sigma_1, ..., \sigma_k)$. When we enumerate pairs of diagrams (λ, μ) such that $\lambda/\mu = \sigma$, we are counting the number of pairs of diagrams with skew Young diagram σ.

This paper investigates generating functions for certain pairs of these Young diagrams with distinct parts. In general, given a sequence of coefficients $(a_n)_{n\geq 0} = a_0, a_1, a_2, ...$ that enumerate sets of combinatorial objects, we say that $f(x) = \sum_{n\geq 0} a_n x^n$ is the generating function for those objects. For this paper, we will use the fact that the generating function for the number of partitions with distinct parts is $D(q) = \prod_{i\geq 1}(1 + x^i)$, in which the coefficient of x^n is the number of $\lambda \vdash n$ with distinct parts.

Lastly, Young diagrams with distinct parts may be referred to as YDDPs throughout the paper, and unless specified we will assume partitions and diagrams being discussed are composed of distinct parts.

3 Generating Function for Adding One Box

3.1 The Claim

The first result we present gives the generating function for the number of pairs of YDDPs (λ, μ) such that $|\lambda| = n+1, |\mu| = n$, and λ/μ is one box. The coefficient of x^n is the number of such pairs.

Proposition 3.1. *If we denote $f(n)$ as the coefficient of x^n as described above, then* $\dfrac{\sum_{n\geq 0} f(n)q^n}{D(q)} = \dfrac{1}{1-q^2}$.

Proof. First note that this claim is the same as $\sum_{n\geq 0} f(n)q^n = D(q)+\sum_{n\geq 0} f(n)q^{n+2}$. This states that the number of pairs (λ, μ) such that $|\lambda| = n+1, |\mu| = n, \mu \subset \lambda$ and λ/μ is one box is equal to the number of pairs (γ, τ) such that $|\gamma| = n-1, |\tau| = n-2, \tau \subset \gamma$, and γ/τ is one box plus the number of YDDPs of size n. As a result, finding a bijection between $A = \{(\lambda, \mu) : |\lambda| = n+1, |\mu| = n, \mu \subset \lambda, \lambda/\mu = \square \}$ and $B = \{(\gamma, \tau) : |\gamma| = n-1, |\tau| = n-2, \tau \subset \gamma, \gamma/\tau = \square \} \cup \{\phi : |\phi| = n\}$ is sufficient to prove the claim.

Considering the poset of YDDPs as previously described, it is clear to see each pair $(\lambda, \mu) \in A$ corresponds bijectively to a path between the n^{th} and $n+1^{st}$ rows in the lattice, and similarly for $(\gamma, \tau) \in B$. As such, we will not differentiate between a pair (λ, μ) and its associated path. Now the proof reduces to bijectively mapping each such path $(\lambda, \mu) \in A$ to either a path $(\gamma, \tau) \in B$ or to a YDDP of size n.

First observe that if we fix a diagram $\mu = (\mu_1, ..., \mu_k)$ such that $|\mu| = n$ and define $\lambda = (\mu_1 + 1, \mu_2, ..., \mu_k)$, then $\mu \subset \lambda$ and $\lambda/\mu = \square$, so $(\lambda, \mu) \in A$ is a path in the lattice. Since this λ is uniquely determined by μ, we map (λ, μ) to μ.

Therefore, the claim will follow when we exhibit a one-to-one correspondence from $A' = A \setminus \{(\lambda = (\mu_1 + 1, \mu_2, ..., \mu_k), \mu = (\mu_1, ..., \mu_k))\}$ to $B' = B \setminus \{\phi : |\phi| = n\}$.

Next, we make another observation. Let $(\lambda, \mu) \in A'$. Then $\mu = (\mu_1, \mu_2, ..., \mu_k)$ with $|\mu| = n$ and $\lambda = (\mu_1, ..., \mu_i + 1, ...\mu_k)$ for some $i \neq 1$. Now $(\mu, \gamma = (\mu_1, ..., \mu_{i-1} - 1, ..., \mu_k))$ is also an edge in the lattice. To see this, note that λ being a YDDP implies that $\mu_{i-1} > \mu_i + 1$. If this were not the case, then $\mu_i + 1 = \mu_{i-1}$ and λ would not be a YDDP. It follows that γ is a YDDP and $\gamma \subset \mu$ such that $\mu/\gamma = \square$. Conversely, let $(\mu' = (\mu'_1, ..., \mu'_j, ...\mu'_l)$, $\gamma' = (\mu'_1, ..., \mu'_j - 1, ...\mu'_l))$ be an edge between the $n-1^{st}$ and n^{th} rows where $j \neq l$ if $\mu'_l = 1$. Then, $(\lambda' = (\mu'_1, ..., \mu'_{j+1} + 1, ...\mu'_l), \mu')$ is an edge in A'. Because γ' is a YDDP, $\mu'_j - 1 > \mu'_{j+1}$, so $\mu'_j > \mu'_{j+1} + 1$. Therefore, λ' is also a YDDP.

Equipped with this fact, we proceed by temporarily associating the path $(\lambda, \mu) \in A'$ to the path (μ, γ) where λ, μ, γ are as defined above. Note, by the uniqueness of γ, that this establishes a bijection between A' and $C' = \{$ edges between the $n-1^{st}$ and n^{th} rows of the lattice that do not arise from starting a new row in the diagram $\}$ (in other words, C' contains the edges between YDDPs with the same number of rows).

Thus, it suffices to give a bijective mapping from C' to B'. To do so, we slightly amend the above procedure. Previously, we were not concerned with pairs of the form $(\lambda = (\mu_1 + 1, \mu_2, ..., \mu_k), \mu = (\mu_1, ..., \mu_k))$ because these were mapped to Young diagrams and weren't in A'. To adjust for this, if $(\mu, \gamma = (\gamma_1, ..., \gamma_k)) \in C'$ corresponds to an edge of this form, we map it to the edge $(\gamma, \tau = (\gamma_1, ..., \gamma_k - 1)) \in B'$. Now we have a bijective map from C' to B'. Given $(\mu = (\gamma_1, ..., \gamma_i + 1, ...\gamma_k), \gamma = (\gamma_1, ..., \gamma_k) \in C'$, map (μ, γ) to $(\gamma, \tau = (\gamma_1, ..., \gamma_k - 1))$ if $i = 1$ or to $(\gamma, \tau = (\gamma_1, ..., \gamma_{i-1} - 1, ...\gamma_k))$ if $i \neq 1$. In doing so, we complete the correspondence. This proves the claim.

□

3.2 Example

We now provide an example to illustrate the previous claim and proof. Consider the case when $n = 6$, so we consider the paths between the 6^{th} and 7^{th} rows. In Figure 1, we use a dashed line to represent paths mapped to the corresponding YDDP in the 6^{th} row. The remaining paths are color-coded to show the resulting mappings via the method explained in the proof.

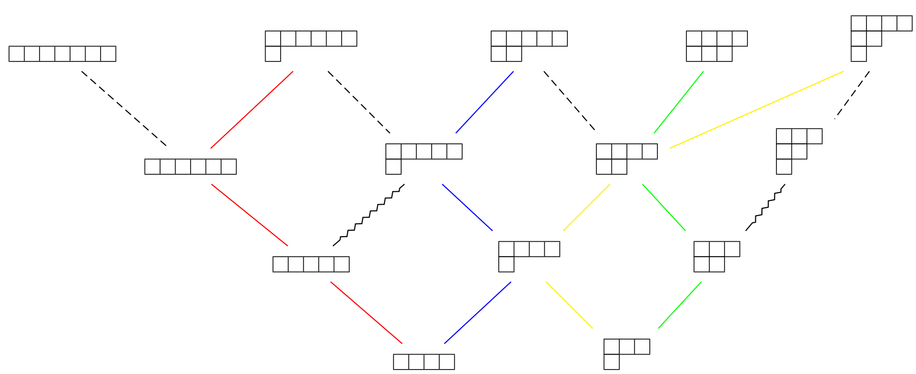


Figure 1: The complete 4^{th} through 7^{th} rows of Young's lattice for YDDPs. The dashed and colored lines correspond to the mapping describe in the proof of the previous proposition. The squiggly lines are edges not counted.

4 Additional Methods

In order to present the next proofs, we will need to utilize a few results proved by Richard Stanley and George Andrews. This section will provide a general overview of these results.

4.1 General Form of Rational Function

The first result addressed here is from Stanley in his book *Enumerative Combinatorics Volume I* presented as Exercise 21 at the end of Chapter 4 [3]. As mentioned in the introduction, he begins an investigation of the function $A_S(q)$ such that $\sum_{n\geq 0} f(n)q^n = P(q)A_S(q)$, where $P(q) = \prod_{i\geq 1}(1-q^i)^{-1}$ is the generating function for all Young diagrams. Stanley provides an argument for the rationality of $A_S(q)$ and shows the denominator can be written as $\phi_{a_j}(q) = (1-q)(1-q^2)...(1-q^{a_j})$. The method Stanley utilizes to show this claim will be implemented in our proofs. An overview of his argument will be provided here.

We will describe the idea behind his argument for a simple example. Consider the case where we want to count the ways to add k boxes to one row in any Young diagram of size n (not necessarily made of distinct parts). Let c denote the length of the row after the k boxes are adjoined. If this row was removed from the diagram, the remaining diagram, D, would have no parts with length $c-1, c-2, ..., c-k+1$. (If it did, when we add the row back into the diagram with the k new boxes, the rows would not be weakly decreasing.) For a fixed c, each such pair (c, D) occurs exactly once. Therefore, we can write the following equation:

$$\sum_{n\geq 0} f(n)q^{n+k} = P(q)\sum_{c\geq k} q^c(1-q^{c-1})(1-q^{c-2})...(1-q^{c-k+1})$$

The left hand side of the equation represents the generating function for adding k boxes to a a diagram of size n where the exponent is $n+k$ because this is the total number of boxes after the addition. On the right hand side of the equation, we sum over the the possible values of c (which must be at least k since it includes at least the k added boxes). Then, for each value of c, we consider the number of diagrams that could have a row of length c removed. This accounts for the $(1-q^{c-i})$ terms which indicate that the initial diagrams cannot have rows of length $c-i$. Then c boxes are inserted back into the diagram, so we multiply each term in the summation by q^c. After doing so, we may divide both sides by q^k to find the desired generating function. Stanley completes his claim by summing over the right hand side and showing the desired denominator occurs.

This approach will be implemented in later proofs as a method of writing a desired generating function in terms of an infinite sum.

4.2 The Half ${}_1\psi_1$

A second tool we will be using in our proofs is the "half", ${}_1\psi_1$, formula introduced to us by Dennis Stanton through a personal communication [4]. Stanton provides a proof from George Andrews who utilized the result to prove a formula stated without proof in Srinivasa Ramanujan's "Lost" Notebook [1]. We will simply utilize the result. To understand the statement of the formula, we provide the following necessary notation:

$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1})$.

The statement of the formula for given a, b, N is:

$$\sum_{n\geq 0} \frac{(-aq;q)_n}{(bq;q)_n} q^{(N+1)n} =$$

$$\frac{b^{-N-1}}{1+aq^{N+1}/b} \frac{(q;q)_N}{(-aq/b;q)_N} \left(\frac{(-aq;q)_\infty}{(bq;q)_\infty} - (1-b) \sum_{k=0}^{N} \frac{(-aq/b;q)_k}{(q;q)_k} b^k \right) \quad (1)$$

In the case where $N = 0$, the formula can be simplified to:

$$(b+aq) \sum_{n\geq 0} \frac{(-aq;q)_n}{(bq;q)_n} q^n = \frac{(-aq;q)_\infty}{(bq;q)_\infty} - 1 + b. \quad (2)$$

5 Generating Function for Adding k Boxes in a Row

In our first proposition, we presented the generating function for adding one box to a YDDP. Now, we consider the generating function for adding k boxes to a row in a YDDP.

Proposition 5.1. *Let $f_k(n)$ be the coefficient of the n^{th}-degree term of the generating function for legally adding k boxes to a row of a YDDP. Then,*

$$\frac{\sum_{n\geq 0} f_k(n) q^n}{D(q)} = \frac{(1+q)(1+q^2)...(1+q^k) - q^k}{(1+q)(1+q^2)...(1+q^k)(1-q^k)}.$$

Proof. First note that, by using Stanley's method [3], we can write

$$\frac{\sum_{n\geq 0} f_k(n) q^{n+k}}{D(q)} = \frac{q^k}{(1+q)...(1+q^k)} + \sum_{c>k} \frac{q^c}{(1+q^{c-k})...(1+q^c)}.$$

Because we are considering YDDPs instead of general Young diagrams, to indicate a partition may not have a component of size i, we divide by $(1+q^i)$ instead of multiplying by $(1-q^i)$. (This is a direct result of the difference in generating functions. Whereas for general diagrams we have the generating function $P(q) = \prod_{i\geq 1} \frac{1}{(1-q^i)}$, for YDDPs we have $D(q) = \prod_{i\geq 1}(1+q^i)$). The first term on the right side represents when $c = k$ indicating the k boxes are added to a new row on the bottom. In this case, we drop the $(1+q^{c-k})$ term from the denominator since we do allow empty rows.

Now we proceed by reindexing the right side of the equation. By substituting in $c+k$ for c and then dividing both sides by q^k, we find

$$\frac{\sum_{n\geq 0} f_k(n)q^n}{D(q)} = \frac{1}{(1+q)...(1+q^k)} + \sum_{c\geq 1} \frac{q^c}{(1+q^c)...(1+q^{c+k})}. \tag{3}$$

Now, to find the desired generating function, we manipulate the summation on the right hand side. To do so, we will use equation (2).

First, let $a = \frac{1}{q}$ and $b = -q^k$ so that $-aq = -1$ and $bq = -q^{k+1}$. Now, through an application of (2), we derive the following.

$$\begin{aligned}
&(1-q^k)\sum_{n\geq 0} \frac{2(1+q)...(1+q^{n-1})}{(1+q^{k+1})...(1+q^{k+n})}q^n \\
&= (1-q^k)\left[1 + \sum_{n\geq 1} \frac{2(1+q)...(1+q^{n-1})}{(1+q^{k+1})...(1+q^{k+n})}q^n\right] \\
&= (1-q^k) + (1-q^k)\sum_{n\geq 1} \frac{2(1+q)...(1+q^{n-1})}{(1+q^{k+1})...(1+q^{k+n})}q^n \\
&= \frac{\prod_{i\geq 0}(1+q^i)}{\prod_{i\geq 1}(1+q^{k+i})} - 1 - q^k
\end{aligned}$$

Furthermore, through some algebraic reduction we find:

$$(1-q^k)\sum_{n\geq 1} \frac{(1+q)...(1+q^{n-1})}{(1+q^{k+1})...(1+q^{k+n})}q^n = (1+q)...(1+q^k) - 1$$

Now, notice that if we factor $(1+q)...(1+q^k)$ out of the numerator of the summation, then the remaining $(1+q^{k+1})...(1+q^{n-1})$ term will cancel with the first terms of the denominator. As a result, we have

$$(1+q)...(1+q^k)(1-q^k)\sum_{n\geq 1} \frac{q^n}{(1+q^n)...(1+q^{k+n})} = (1+q)...(1+q^k) - 1.$$

And lastly, a final division results in

$$\sum_{n\geq 1} \frac{q^n}{(1+q^n)...(1+q^{k+n})} = \frac{(1+q)...(1+q^k) - 1}{(1+q)...(1+q^k)(1-q^k)} \tag{4}$$

We may now combine equations (3) and (4) to see that

$$\frac{\sum_{n\geq 0} f_k(n)q^n}{D(q)} = \frac{1}{(1+q)...(1+q^k)} + \sum_{c\geq 1} \frac{q^c}{(1+q^c)...(1+q^{c+k})}$$
$$= \frac{1}{(1+q)...(1+q^k)} + \frac{(1+q)...(1+q^k)-1}{(1+q)...(1+q^k)(1-q^k)}.$$

It then follows that

$$\frac{\sum_{n\geq 0} f_k(n)q^n}{D(q)} = \frac{(1+q)(1+q^2)...(1+q^k)-q^k}{(1+q)(1+q^2)...(1+q^k)(1-q^k)} \tag{5}$$

and the proof is complete.

□

Note: If $k=1$, equation (5) tells us

$$\frac{\sum_{n\geq 0} f_1(n)q^n}{D(q)} = \frac{(1+q)(1+q^2)...(1+q^k)-q^k}{(1+q)(1+q^2)...(1+q^k)(1-q^k)}$$
$$= \frac{(1+q)-q}{(1+q)(1-q)}$$
$$= \frac{1}{1-q^2}$$

which is the same as Proposition 3.1.

6 Generating Function for Adding 2 Boxes Adjoined at Corner

The next result we prove is the generating function, $\sum_{n\geq 0} f_{1,1}(n)$, for legally adding two boxes where one box is added to two consecutive rows whose lengths differ by 1. Similar to the other propositions, we find a rational function.

Proposition 6.1. *Let $\sum_{n\geq 0} f_{1,1}(n)$ be as described above. Then,*

$$\frac{\sum_{n\geq 0} f_{1,1}(n)q^n}{D(q)} = \frac{q}{(1+q)(1-q^4)}.$$

Proof. For this proof, we follow a similar strategy as previously presented. First note, in the prior application of Stanley's result, we only had to account for adding boxes to one row for a total length of c [3]. In this case, we are adding a box to two rows, so the total addition will be $2c-1$ since the lengths of the rows differ by 1. This will be accounted for when we write the $\sum_{n\geq 0} f_{1,1}(n)q^n$ as a sum.

Using this adjustment to Stanley's method we get

$$\frac{\sum_{n\geq 0} f_{1,1}(n)q^{n+2}}{D(q)} = \frac{q^3}{(1+q)(1+q^2)} + \sum_{c>2} \frac{q^{2c-1}}{(1+q^{c-2})(1+q^{c-1})(1+q^c)}.$$

The first term on the right represents when $c = 2$ indicating the boxes are added to the bottom two rows. In this case, we remove the $(1 + q^{c-2})$ from the denominator to indicate we do allow empty rows.

Next, we divide by q^2 to have the generating function in the desired form, and reindex the summation to start at $c = 0$.

$$\frac{\sum_{n\geq 0} f_{1,1}(n)q^{n}}{D(q)} = \frac{q}{(1+q)(1+q^2)} + q^3 \sum_{c\geq 0} \frac{q^{2c}}{(1+q^{c+1})(1+q^{c+2})(1+q^{c+3})}. \quad (6)$$

Now, let $N = 1, a = 1$, and $b = -q^3$ so that $-aq = -q$ and $bq = -q^4$. Then, by (1), we have

$$\begin{aligned}
&\sum_{c\geq 0} \frac{(1+q)(1+q^2)...(1+q^c)}{(1+q^4)(1+q^5)...(1+q^{c+3})} q^{2c} \\
&= \frac{1/q^6}{(1-1/q)} \frac{(1-q)}{(1-1/q^2)} \left[\frac{\prod_{i\geq 1}(1+q^i)}{\prod_{i\geq 4}(1+q^i)} - (1+q^3) \sum_{k=0}^{1} \frac{(1/q^2;q)_k}{(q;q)_k} (-q^3)^k \right] \\
&= \frac{(1-q)}{q^3(q-1)(q^2-1)} \left[(1+q)(1+q^2)(1+q^3) - (1+q^3) \left[1 - \frac{(1-1/q^2)}{(1-q)} q^3 \right] \right] \\
&= \frac{(1-q)}{q^3(q-1)(q^2-1)} \left[(1+q^3) \left[(1+q)(1+q^2) - 1 + \frac{q(q^2-1)}{(1-q)} \right] \right] \\
&= \frac{-1}{q^3(q^2-1)} \left[(1+q^3) \left[1 + q^2 + q + q^3 - 1 - q^2 - q \right] \right] \\
&= \frac{(1+q^3)}{(1-q^2)} \qquad (7)
\end{aligned}$$

Now, factor $(1 + q)(1 + q^2)(1 + q^3)$ out of the numerator of the summation so that we have

$$(1+q)(1+q^2)(1+q^3) \sum_{c\geq 0} \frac{q^{2c}}{(1+q^{c+1})(1+q^{c+2})(1+q^{c+3})} = \frac{(1+q^3)}{(1-q^2)}.$$

From this we conclude that

$$\sum_{c\geq 0} \frac{q^{2c}}{(1+q^{c+1})(1+q^{c+2})(1+q^{c+3})} = \frac{1}{(1+q)(1+q^2)(1-q^2)}. \quad (8)$$

By combining (6) and (8), we see that

$$\begin{aligned}
\frac{\sum_{n\geq 0} f_{1,1}(n)q^n}{D(q)} &= \frac{q}{(1+q)(1+q^2)} + q^3 \sum_{c\geq 0} \frac{q^{2c}}{(1+q^{c+1})(1+q^{c+2})(1+q^{c+3})} \\
&= \frac{q}{(1+q)(1+q^2)} + q^3 \frac{1}{(1+q)(1+q^2)(1-q^2)} \\
&= \frac{q}{(1+q)(1-q^4)}.
\end{aligned}$$

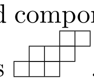
□

7 Conjecture

We conclude this paper with a conjecture for the general form of the generating function for adding k disjoint boxes to a YDDP. If this generating function is defined as $\sum_{n\geq 0} g_k(n)q^n$, then we believe this can be expressed as

$$\frac{\sum_{n\geq 0} g_k(n)q^n}{D(q)} = \frac{q^{\binom{k}{2}}}{\prod_{i=1}^{k}(1-q^{2i})}.$$

We are currently working to confirm this formula for the case when $k = 2$ by an approach similar to our proof of Proposition 3.1.

Another direction for future work is generalizing the generating function for legally adding any connected component to a YDDP. For example, counting the ways to add a shape such as  .

In addition, we could study a generalization of legally adding k boxes that are adjacent at the corner to a YDDP.

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