Outcome Separation Logic: Local Reasoning for Correctness and Incorrectness with Computational Effects

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The compositionality and local reasoning properties of separation logic have led to significant advances in scalable static analysis. But new requirements for program analysis have emerged—many programs display computational effects (such as randomization) and, orthogonally, static analysis for incorrectness has proved to be very effective. We present Outcome Separation Logic (OSL), the first variant of separation logic that is sound for both correctness and incorrectness reasoning with varying computational effects. OSL has a frame rule that resembles that of standard Separation Logic, however we make different underlying assumptions in order to lift restrictions imposed by SL that preclude reasoning about incorrectness and effects.

Building on this fundamental theory, we also define symbolic execution algorithms that use bi-abduction to derive specifications for programs with effects. This involves a new tri-abduction procedure to analyze programs whose execution branches due to effects such as nondeterministic or probabilistic choice. This work furthers the compositionality promised by separation logic by opening up the possibility for greater reuse of analysis tools across two dimensions: bug-finding and verification across programs with varying effects.

1 INTRODUCTION

Compositional reasoning using separation logic [Reynolds 2002] and bi-abduction [Calcagno et al. 2009] has helped scale static analysis to industrial software with hundreds of millions of lines of code, making it possible to analyze code changes without disrupting the fast-paced engineering culture that developers are accustomed to [Calcagno et al. 2015; Distefano et al. 2019].

While the ideal of fully automated program verification remains elusive, analysis tools can boost confidence in code correctness by ensuring that a program will not go wrong in a variety of ways. In languages like C or C++, this includes ensuring that a program will not crash due to a segmentation fault or leak memory. However, a static analyzer failing to prove the absence of bugs does not imply that the program is incorrect; it could be a false positive.

Many programs are, in fact, incorrect. Analysis tools capable of finding bugs are thus in some cases more useful than verification tools, since the reported errors lead directly to tangible code improvements [Le et al. 2022]. Motivated by the need to identify bugs, Incorrectness Logic [O’Hearn 2019] and Incorrectness Separation Logic (ISL) [Raad et al. 2020, 2022] were recently introduced.

While ISL enjoys compositionality just like separation logic, the semantics of SL and ISL are incompatible—specifications and analysis tools cannot readily be shared between them. Further, the soundness of local reasoning in separation logic relies on the following three assumptions, limiting its applicability [Yang and O’Hearn 2002].

**Nondeterminism.** Memory allocation must be nondeterministic to ensure that running a program in a larger heap results in fewer possible addresses for allocated pointers, preserving the truth of the postcondition. As such, the promise of local reasoning and, by extension, scalable program analysis cannot immediately extend to alternative execution models such as probabilistic computation.

**Must properties.** Separation logic can only express properties that must occur, not ones that may occur, making it inept for incorrectness, since many bugs only occur some of the time.
**Safe preconditions.** The precondition must contain enough information to ensure that all program paths do not encounter memory faults. This means that bug-finding analyses must examine the entire program, even if a bug occurs only in certain traces.

Our key insight is that local reasoning is sound under different assumptions, which do not force any particular evaluation model, and are compatible with incorrectness reasoning as well. To that end, we introduce *Outcome Separation Logic* (OSL), a single program logic for locally reasoning about both correctness and incorrectness with varied computational effects. OSL builds on *Outcome Logic* (OL), which was already shown to support correctness and incorrectness reasoning with effects [Zilberstein et al. 2023], but not local reasoning. Using OSL as a logical foundation, we present symbolic execution algorithms that are able to infer both correctness and incorrectness specifications using a shared set of procedure summaries. Our contributions are as follows.

**Outcome Separation Logic.** While Zilberstein et al. [2023] embedded separation logic in OL, we go further by making heap assertions a first class part of OL. This culminates in a frame rule that resembles that of standard separation logic, but relies on different underlying assumptions for its soundness, allowing us to lift the three aforementioned restrictions of SL that make it unsuitable for reasoning about effects and incorrectness. The OSL semantics is based on an algebraic representation of *choice*; we show how this can encode deterministic, nondeterministic, and probabilistic programs.

**Tri-abduction for parallel composition.** Bi-abduction enables scalable reasoning for sequential programs by reconciling the postcondition of one precomputed spec with the precondition of the next. However, programs with effects are not purely sequential, but rather have control flow branching that arises from, e.g., nondeterministic or probabilistic choice. Calcagno et al. [2009, 2011] handle nondeterminism by generating candidate preconditions for each program trace, and then re-evaluating the program to check whether each one is valid for all paths. This approach has two downsides: it misses some valid preconditions and it must do extra work to re-check the ones it does find. To solve this problem, we introduce the parallel composition analogue of bi-abduction, which we call *tri-abduction* because it infers three assertions rather than two (a precondition, and a leftover frame for each of the two effectful branches).

**Symbolic execution algorithms.** Building on the previous two contributions, we present symbolic execution algorithms to analyze C-like pointer programs. The core algorithm finds all the reachable outcomes, and is therefore capable of reasoning about both correctness and incorrectness.

We also define a *single-path* variant, which generates specifications in which the postcondition is just one of the (possibly many) outcomes. It is similar to bug-finding algorithms based on Incorrectness Logic (Pulse [Raad et al. 2020] and Pulse-X [Le et al. 2022]) in its ability to drop paths for increased scalability, but with the added benefit that it can soundly re-use procedure summaries generated by the correctness algorithm. Computing and storing procedure summaries in large codebases is resource intensive, so the fact that such summaries can be shared between bug-finding and verification tools opens the possibility for increased performance and scalability.

In Section 2, we lay out the key ideas for the paper by showing how the assumptions of separation logic prevent reasoning about arbitrary effects and incorrectness, and how we lift those assumptions in OSL. Next, in Sections 3 and 4 we define Outcome Separation Logic (OSL), show three instantiations, and prove the soundness of the frame rule. In Section 5 we define tri-abduction, which is inspired by bi-abduction but is used for parallel composition rather than sequential composition. Tri-abduction does not replace bi-abduction, we use both together in Section 6 to define symbolic execution algorithms. In Section 7 we examine some case studies to show the applicability of these algorithms and finally we conclude in Sections 8 and 9 by discussing related work and next steps.
2 \hspace{1em} \textbf{KEY IDEAS: LOCAL REASONING FOR MORE TYPES OF PROGRAMS}

We begin our investigation by examining how the local reasoning principles of separation logic, along with bi-abductive inference, underly powerful and scalable analysis techniques. The goal of such analyses is to symbolically execute a program and report the result as a Hoare Triple \( \{ P \} C \{ Q \} \): running the program \( C \) in a state satisfying the precondition \( P \) will result in a state satisfying the postcondition \( Q \) [Hoare 1969]. Hoare triples are compositional; a specification for the sequence of two program commands can be constructed given specifications for each command.

\[
\begin{array}{c}
\{ P \} C_1 \{ Q \} \quad \{ Q \} C_2 \{ R \} \\
\{ P \} C_1 ; C_2 \{ R \}
\end{array}
\]

The \textsc{Sequence} rule is a good starting point for building scalable program analyses, but it is not quite compositional enough. Since the postcondition of \( C_1 \) must exactly match the precondition of \( C_2 \), it may be difficult to apply this rule, particularly if \( C_1 \) and \( C_2 \) are procedure calls for which we already have pre-computed summaries (in the form of Hoare Triples), none of which exactly match.

In response to this problem, separation logic offers a second form of compositionality via the \textsc{Frame} rule, an inference rule that allows information about additional (unused) program resources \( F \) to be added to the pre- and postcondition of a completed proof.

\[
\begin{array}{c}
\{ P \} C \{ Q \} \\
\{ P * F \} C \{ Q * F \}
\end{array}
\]

But the ability to \textit{frame} in information about additional resources does not answer the question of how to compose specifications. Given \( \{ P_1 \} C_1 \{ Q_1 \} \) and \( \{ P_2 \} C_2 \{ Q_2 \} \), it is not immediately clear what—if anything—we can add to make \( Q_1 \) match \( P_2 \). This is where bi-abduction comes in—a technique that finds a missing resource \( M \) and a leftover frame \( F \) to make the entailment \( Q_1 * M \vdash P_2 * F \) hold. With the help of bi-abduction, we get the following more usable sequence rule that stitches together two precomputed summaries without reexamining either piece of code.

\[
\begin{array}{c}
\{ P_1 \} C_1 \{ Q_1 \} \quad Q_1 * M \vdash P_2 * F \\
\{ P_2 \} C_2 \{ Q_2 \} \\
\{ P_1 * M \} C_1 ; C_2 \{ Q_2 * F \}
\end{array}
\]

But the ability to \textit{frame} in information about additional resources does not answer the question of how to compose specifications. Given \( \{ P_1 \} C_1 \{ Q_1 \} \) and \( \{ P_2 \} C_2 \{ Q_2 \} \), it is not immediately clear what—if anything—we can add to make \( Q_1 \) match \( P_2 \). This is where bi-abduction comes in—a technique that finds a missing resource \( M \) and a leftover frame \( F \) to make the entailment \( Q_1 * M \vdash P_2 * F \) hold. With the help of bi-abduction, we get the following more usable sequence rule that stitches together two precomputed summaries without reexamining either piece of code.

\[
\begin{array}{c}
\{ P_1 \} C_1 \{ Q_1 \} \quad Q_1 * M \vdash P_2 * F \\
\{ P_2 \} C_2 \{ Q_2 \} \\
\{ P_1 * M \} C_1 ; C_2 \{ Q_2 * F \}
\end{array}
\]

Unfortunately, as it stands now, bi-abduction does not apply to programs with effects such as probabilistic choice. In the remainder of this section, we will examine why this is the case, and then explain how the logic we describe in this paper lifts several restrictions in order to extend bi-abductive analysis to more programs (e.g., ones with computational effects) and program properties (e.g., reachability and incorrectness) that current bi-abductive analyzers cannot support.

2.1 \hspace{1em} \textbf{Interlude: Reasoning about Effects and Incorrectness}

Before describing why the frame rule does not support arbitrary computational effects and incorrectness, we take a brief detour to explain how those properties differ from conventional program analysis. Incidentally, computational effects and incorrectness have an intricate interaction. Identifying bugs in a pure (effectless) program is not so hard, as demonstrated by the following example in which the program crashes because it attempts to write into a null pointer \( x \).

\[
\{ \text{ok} : x = \text{null} \} \ [x] \leftarrow 1 \quad \{ \text{er} : x = \text{null} \}
\]

The situation becomes more complicated once effects are involved.\footnote{Even nontermination is an effect, so looping programs must be handled delicately when it comes to incorrectness reasoning.} Rather than dereferencing a pointer that is \textit{known} to be invalid, suppose we dereference a pointer that \textit{might} be invalid, and—crucially—whether or not it is allocated comes down to nondeterminism. The following is one
such scenario; now the program begins in the empty heap (emp) and x is obtained using malloc, which nondeterministically either returns a valid pointer or null. In Hoare Logic, the best we can do is specify this program using a disjunction.

\{ \text{ok : emp} \} \ x := \text{malloc}() ; [x] \leftarrow 1 \ {((\text{ok : x} \leftarrow 1) \lor (\text{er : x} = \text{null}))}

While the above specification hints that the program has a bug, it is in fact inconclusive since the disjunctive postcondition does not guarantee that both outcomes are reachable by actual program executions. Hoare Logic is fundamentally unable to characterize this bug, since the postcondition must describe all possible end states of the program; we cannot express something that may happen.

Two solutions for characterizing true bugs have been proposed. The first one is Incorrectness Logic (IL), which uses an alternative semantics to express that all states described by the postcondition are reachable from a state described by the precondition [O’Hearn 2019]. Specifying the aforementioned bug is possible using Incorrectness Logic; the semantic of the following triple is that all the states described by the post are reachable, including ones where the error occurs.

\{ \text{ok : emp} \} \ x := \text{malloc}() ; [x] \leftarrow 1 \ {((\text{ok : x} \leftarrow 1) \lor (\text{er : x} = \text{null}))}

IL has a sound frame rule [Raad et al. 2020] and can underly bi-abductive symbolic execution algorithms [Le et al. 2022]. However—just like separation logic—IL is specialized to nondeterministic programs and is not suitable for other effects. In addition, being inherently under-approximate, the semantics of IL cannot capture correctness properties, which must cover all the reachable outcomes. As such, different analyses and procedure summaries must be used for verification vs bug-finding.

In this paper, we take a different approach based on Outcome Logic (OL), which is compatible with both correctness and incorrectness while also supporting a variety of monadic effects [Zilberstein et al. 2023]. OL is similar to Hoare Logic, but the pre- and postconditions of triples describe collections of states rather than individual ones. A new logical connective —the outcome conjunction—can guarantee the reachability of multiple outcomes. For instance, the aforementioned bug can be characterized using the following OL specification by replacing the disjunction in the postcondition.

\{ \text{ok : emp} \} \ x := \text{malloc}() ; [x] \leftarrow 1 \ {((\text{ok : x} \leftarrow 1) \oplus (\text{er : x} = \text{null}))}

The above triple stipulates that running the program in the empty heap will result in two reachable outcomes. In this case, the program is nondeterministic and its semantics is accordingly characterized by a set of program states \( S \). The outcome conjunction tells use that there exist nonempty sets \( S_1 \) and \( S_2 \) with \( S = S_1 \cup S_2 \) such that \( S_1 \not\subseteq (\text{ok : x} \leftarrow 1) \) and \( S_2 \not\subseteq (\text{er : x} = \text{null}) \). Since both outcomes are satisfied by nonempty sets, we know that they are both reachable by a real program trace.

However, for efficiency, specifying the bug above should not require recording information about the ok outcome. In incorrectness reasoning, it is desirable to drop outcomes so as to only explore some of the program paths [O’Hearn 2019; Le et al. 2022]. We achieve this in OL by replacing the extraneous outcome with \( \top \), ensuring that the ok program path will not continue to be analyzed.

\{ \text{ok : emp} \} \ x := \text{malloc}() ; [x] \leftarrow 1 \ {((\text{er : x} = \text{null}) \oplus \top)}

The concept of outcomes is more general than nondeterministic choice. For example, Outcome Logic can also be used to reason about probabilistic programs, where the (weighted) outcome conjunction additionally quantifies the likelihoods of outcomes. For example, the following program attempts to ping an IP address, which succeeds 99% of the time, and fails with probability 1% due to an unreliable network connection.

\{ \text{ok : true} \} \ x := \text{ping}(192.0.2.1) \ {((\text{ok : x} = 0) \oplus_{99\%} (\text{er : x} = 1))}

Our goal in this paper is to augment Outcome Logic with a sound frame rule, and to use the resulting theory to build bi-abductive symbolic execution algorithms both for correctness (finding
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all reachable outcomes) and incorrectness (only exploring one program path at a time). We will next see why the frame rule of separation logic precludes reasoning about arbitrary effects and incorrectness before exploring our new solution based on Outcome Separation Logic.

2.2 Overcoming the Restrictions of the Frame Rule

Unfortunately, local reasoning does not come for free; separation logic imposes several restrictions which makes it unsuitable for our goal of reasoning about computational effects, reachability, and dropping program paths. We now walk through the three core restrictions imposed by separation logic, and how we soundly relax them in Outcome Separation Logic.

Soundness relies on nondeterminism. Separation logic requires that the underlying program semantics, and—in particular—memory allocation be nondeterministic. To see what goes wrong without nondeterminism, let us consider a semantics in which the memory allocator always deterministically picks the next available heap address. The following specification is then valid; if \( x \) is allocated in the empty heap, then it must be given the first address, i.e., \( x = 1 \).

\[
\{\text{emp} \} \ x := \text{alloc}() \ {x = 1 \land x \rightarrow -}
\]

Now, using the frame rule, we could infer the following (incorrect) specification.

\[
\{\text{emp} \} \ x := \text{alloc}() \ {x = 1 \land x \rightarrow -} \quad \text{FRAME}
\]

\[
\{y \rightarrow 1\} \ x := \text{alloc}() \ {x = 1 \land x \rightarrow - \land y \rightarrow 1}
\]

It is possible that \( y \) has address 1, in which case \( x \) (a freshly allocated pointer) cannot also be equal to 1. So, this application of the frame rule is clearly unsound.

Separation logic’s response to this issue is to force the semantics of memory allocation to be nondeterministic [Yang and O’Hearn 2002]. That is, in the above program, \( x \) could be assigned any address that is not already allocated. Effectively, this means that the postcondition cannot say anything specific about the address of \( x \); we cannot conclude that \( x = 1 \), but rather we could only conclude that \( x = 1 \lor x = 2 \lor \cdots \), which remains true after framing in \( y \rightarrow 1 \). Nondeterminism has a delicate interaction with the next restriction; we will show how both can be relaxed together.

Must properties prevent incorrectness reasoning. If we concede for a moment that memory allocation will be nondeterministic and focus just on the unification of verification and bug-finding, separation logic still imposes undesirable constraints. The soundness of framing also relies on the fact that Hoare Logic postconditions are must properties (they describe outcomes that must occur), and not may properties (that sometimes occur). Outcome Logic can express may properties, as is required for incorrectness. In fact, the bug that we saw in Section 2.1 required a may property.

The problematic interaction between framing and reachability is displayed in the following example, where the premise explicitly states that \( x = 1 \) is a reachable outcome of allocating \( x \) in the empty heap (note that allocation is still assumed to be nondeterministic).

\[
\langle \text{ok : emp} \rangle \ x := \text{alloc}() \ \langle (\text{ok : x = 1}) \oplus (\text{ok : x \neq 1}) \rangle \quad \text{FRAME}
\]

\[
\langle y \rightarrow 1 \rangle \ x := \text{alloc}() \ \langle (\text{ok : x = 1} \land y \rightarrow 1) \oplus (\text{ok : x \neq 1} \land y \rightarrow 1) \rangle
\]

This inference is invalid; there are states satisfying the precondition in which \( y \) has the address 1, in which case the outcome where \( x = 1 \) is no longer reachable.

In summary, it is the combined interaction between nondeterministic memory allocation and the semantics of Hoare Logic that makes framing sound. However, this delicate interplay is ensuring a more direct property about local reasoning: assertions cannot be too specific about the addresses of pointers, since those addresses may change after framing. In fact, Yang and O’Hearn [2002] already postulated that these restrictions could be dropped if assertions were invariant under address renaming and Baktiev [2006] subsequently proved exactly that for a deterministic separation logic.
We take a similar approach in OSL by proving that the symbolic heaps that are typically used in symbolic execution algorithms [Berdine et al. 2005a,b; Calcagno et al. 2009, 2011] are invariant under renaming (Section 4.2). More specifically, if \((s, h) \models P\), then \((\pi(s), \pi(h)) \models P\) where \(\pi\) permutes the heap addresses via some bijection. Our underlying program semantics is parametric on an allocator, showing that our frame rule is sound for any allocation semantics. Since real world allocators are not truly nondeterministic, this model captures the semantics more accurately, while also paving the way for reasoning about programs with alternative evaluation models.

**Safe preconditions.** The final restriction dictated by the frame rule relates to safety: framing cannot affect whether or not a program will encounter a memory fault. The reason for this restriction is because in typical partial correctness logics, any postcondition is valid if the program encounters a fault. So, the triple \(\{\text{emp}\} [x] \leftarrow 1 \{\text{emp}\}\) is valid since the program is guaranteed to fault.

Now, using the frame rule, we can add information about \(y\) to obtain \(\{y \mapsto 2\} [x] \leftarrow 1 \{y \mapsto 2\}\), which is untrue in the case that \(y\) aliases \(x\). In response, separation logic requires the precondition to be safe for all program paths; any state satisfying the precondition will not encounter a fault.

Safety of the precondition is undesirable for our purposes; it requires that we examine the entire program, which is at odds with dropping paths for reasoning about incorrectness. Fortunately, as we show in Section 4.3, OSL preconditions need not be safe since OSL is not a partial correctness logic. Following the previous example, \(\langle \text{ok : emp} \rangle [x] \leftarrow 1 \langle \text{ok : emp} \rangle\) is not a valid OSL specification since \(\text{ok : emp}\) is not a reachable outcome of running the program. Rather, if the precondition of an OSL triple is unsafe, then the postcondition can only be \(\top\), i.e., \(\langle \text{ok : emp} \rangle [x] \leftarrow 1 \langle \top \rangle\). Framing information into the latter triple is perfectly safe since the post will absorb any outcome including undefined behavior, nontermination, and successful termination \(\langle \text{ok : y \mapsto 2} \rangle [x] \leftarrow 1 \langle \top \rangle\).

OSL allows us to decide how much of the memory footprint to specify. In a correctness analysis that covers all paths, the precondition must be safe for the entire program. If we instead want to reason about incorrectness and drop paths, then it must only be safe for the paths we explore.

### 2.3 Symbolic Execution and Tri-Abduction

Providing a logical foundation for symbolic execution algorithms was one of the primary motivations for developing OSL, and—given that OSL can express both correctness and incorrectness properties—those algorithms will be capable of both verification and bug-finding.

Our approach takes inspiration from industrial strength bi-abductive analyzers (Abductor [Calcagno et al. 2009] and Infer [Calcagno et al. 2015]), but with greater care taken to handle computational effects. The aforementioned tools can analyze nondeterministic programs, but they are not guaranteed to find specifications for programs with control flow branching.

To see what goes wrong, let us examine a program that uses disjoint resources in the two nondeterministic branches: \(\{(x \leftarrow 1) \} + \{[y] \leftarrow 2\}\). Using bi-abduction, we could conclude that \(x \mapsto \) is a valid precondition for the first branch whereas \(y \mapsto \) is valid for the second, but there is no straightforward way to find a precondition valid for both branches. As a result, the program must be re-evaluated with each candidate precondition to ensure that they are safe for all branches.

Calcagno et al. [2011, §4.3] acknowledged this issue, and mentioned a possible fix that involves re-running the abduction procedure until nothing more can be added to each precondition. Rather than using two passes (as Abductor already does), this approach would require a pass for each combination of nondeterministic program choices, which is exponential in the worst case. We take a different approach, acknowledging that choice operations are fundamentally different from sequential composition and therefore require a new type of inference, which we call tri-abduction.

As its name suggests, tri-abduction infers three components (to bi-abduction’s two). Given \(P_1\) and \(P_2\)—preconditions for two program branches—the goal is to find a single anti-frame \(M\) and...
two leftover frames \( F_1 \) and \( F_2 \) such that \( M \models P_1 \ast F_1 \) and \( M \models P_2 \ast F_2 \), allowing us to compose the summaries for two program branches in a parallel fashion, as demonstrated below.\(^2\)

\[
\frac{(P_1) \ C_1 \ (Q_1) \quad P_1 \ast F_1 \models M \models P_2 \ast F_2 \quad (P_2) \ C_2 \ (Q_2)}{(M) \ C_1 +_a C_2 \ ((Q_1 \ast F_1) \oplus_a (Q_2 \ast F_2)) \quad \text{TRI-ABDUCTIVE COMPOSITION}}
\]

Note that the choice operator \(+_a\) and the outcome conjunction \(\oplus_a\) are parameterized by a weight. In Section 3, we describe interpretations of these weights that allow us to algebraically represent both nondeterministic and probabilistic choice in a uniform way. Tri-abduction does not replace bi-abduction, but rather they work together in complementary ways—bi-abduction is used to compose commands in a sequence whereas tri-abduction composes branches arising from effects.

In addition, we are interested in bug-finding algorithms, which—similar to Pulse and Pulse-X [Raad et al. 2020; Le et al. 2022]—do not traverse all the program paths. We achieve this using a single-path version of the algorithm, producing summaries of the form \( \langle \text{ok} : P \rangle \ C \langle (\text{ter} : Q) \oplus \top \rangle \), with only a single outcome specified and the remaining ones covered by \( \top \). The soundness of the single-path approach is motivated by the fact that \( P \oplus Q \Rightarrow P \oplus \top \); extraneous outcomes can be converted to \( \top \), ensuring that those paths will not be explored. Just like in Pulse and Pulse-X, this ability to drop outcomes allows the analysis to retain less information for increased scalability.

We have now seen an overview of how OSL is built from the ground up to support local reasoning with only a single outcome specified and the remaining ones covered by an interpretation, select either \( C \) described in Section 3.1. For deterministic programs, \( \text{GCL} \), the choice operator \( \text{skip} \) containing Commands each instance’s computational effects. The syntax of the language is given below.

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We have now seen an overview of how OSL is built from the ground up to support local reasoning for correctness and incorrectness in the presence of effects, and how tri-abduction aids in building symbolic execution algorithms for programs with control flow branching. In the remainder of the paper, we formalize these concepts. In Sections 3 and 4, we define a program semantics and Outcome Separation Logic and prove the soundness of the frame rule. Tri-abduction is defined in Section 5 and symbolic execution in Section 6, before we examine case studies in Section 7.

### 3 PROGRAM SEMANTICS

We begin the technical development by defining the semantics for the underlying programming language of Outcome Separation Logic. All instances of OSL share the same program syntax, but this syntax is interpreted in different ways corresponding to the choice mechanisms dictated by each instance’s computational effects. The syntax of the language is given below.

\[
\begin{align*}
C \in \text{Cmd} &::= \text{skip} \mid C_1 ; C_2 \mid C_1 +_a C_2 \mid \text{if } e \text{ then } C_1 \text{ else } C_2 \mid \text{while } e \text{ do } C \mid c \\
\end{align*}
\]

\[
\begin{align*}
c \in \text{Instr} &::= x := e \mid x := \text{alloc()} \mid \text{free}(e) \mid [e_1] \leftarrow e_2 \mid x \leftarrow [e] \mid \text{error()} \mid f(\vec{e}) \\
e \in \text{Exp} &::= e_1 = e_2 \mid \neg e \mid x \mid X \mid \kappa \quad (x \in \text{Var}, X \in \text{LVar}, \kappa \in \text{Const}, f \in \text{Proc}, a \in \mathcal{A})
\end{align*}
\]

Commands \( C \in \text{Cmd} \) are similar to those of Dijkstra’s [1975] Guarded Command Language (GCL), containing skip, sequencing \((C_1 ; C_2)\), and the usual control flow structures if and while. Unlike GCL, the choice operator \( C_1 +_a C_2 \) is parameterized by \( a \in \mathcal{A} \), which has certain algebraic properties described in Section 3.1. For deterministic programs, \( \mathcal{A} = \{0, 1\} \), so \( a \in \mathcal{A} \) is a Boolean that can select either \( C_1 \) or \( C_2 \). The nondeterministic interpretation also uses \( \mathcal{A} = \{0, 1\} \), but \( a = 1 \) selects both branches (the program is denotationally interpreted as a set of outcomes). In the probabilistic interpretation, \( \mathcal{A} = [0, 1] \) and \( C_1 \) is chosen with probability \( a \) and \( C_2 \) with probability \( 1 - a \).

Instructions \( c \in \text{Instr} \) can assign to variables \( x := e \), allocate \( (x := \text{alloc()}\) and deallocate \( (\text{free}(e)\) memory, write \( ([e_1] \leftarrow e_2)\) and read \( (x \leftarrow [e])\) pointers, crash \( (\text{error}())\), and call procedures \( f(\vec{e})\). Expressions have a limited syntax, containing equalities, negation, program variables \( x \in \text{Var} \), logical variables \( X \in \text{LVar} \), and constants \( \kappa \in \text{Const} \) (e.g., integers and Booleans). This ensures that

\(^2\)We use the word parallel to contrast the sequential nature of bi-abduction. In this case, we are not talking about concurrent branches, but rather branches that result from computational effects—namely nondeterministic and probabilistic choice. It is likely that tri-abduction could be used to compose concurrent branches too, although that is out of scope for this paper.
entailments containing expressions are decidable, which is nontrivial even after ruling out pointer arithmetic [Berdine et al. 2005a], but is necessary for bi-abduction [Calcagno et al. 2009].

In the remainder of this section, we will formally define denotational semantics for the language above. This will first involve discussing the algebraic properties of the program weights $a \in \mathcal{A}$, after which we can define a (monadic) execution model to interpret sequential composition.

### 3.1 Algebraic Preliminaries

The program semantics is parametric on an algebraic interpretation of choice. We now recall the definitions of some algebraic structures that will be used to instantiate the program semantics into different execution models. First, we use monoids to model combining and scaling outcomes.

**Definition 3.1 (Monoid).** A monoid $\langle A,+,\emptyset \rangle$ consists of a carrier set $A$, an associative binary operator $+: A \times A \rightarrow A$, and an identity element $\emptyset \in A$ such that $a + \emptyset = \emptyset + a = a$ for all $a \in A$. Additionally, a monoid is partial if $+$ is partial ($+: A \times A \rightarrow A$) and it is commutative if $a + b = b + a$.

For example, $\langle \{0,1\},+,\emptyset,1 \rangle$ is a partial commutative monoid that is commonly used in probabilistic computation since probabilities come from the interval $[0,1]$ and addition is undefined if the sum is greater than 1. Scalar multiplication $\langle \{0,1\},\cdot,1 \rangle$ is another monoid with with same carrier set, but it is total rather than partial. These two monoids can be combined to form a semiring, as follows.

**Definition 3.2 (Semiring).** A semiring $\langle A,+,\cdot,\emptyset,1 \rangle$ consists of a carrier set $A$, along with an addition operator $+$, a multiplication operator $\cdot$ and two elements $\emptyset,1 \in A$ such that:

1. $\langle A,+,\emptyset \rangle$ is a commutative monoid.
2. $\langle A,\cdot,1 \rangle$ is a monoid (we sometimes omit $\cdot$ and write $a \cdot b$ as $ab$).
3. Multiplication distributes over addition: $a \cdot (b + c) = ab + ac$ and $(a + b) \cdot c = ac + bc$
4. $\emptyset$ is the annihilator of multiplication: $a \cdot \emptyset = \emptyset \cdot a = \emptyset$

A semiring is partial if $\langle A,+,\emptyset \rangle$ is instead a partial commutative monoid (PCM), but multiplication remains total. In the case of a partial semiring, distributive rules only apply if the sum is defined.

**Definition 3.3 (Natural Ordering).** The natural order of a semiring $\langle A,+,\cdot,\emptyset,1 \rangle$ is defined to be $a \leq b$ if $\forall a' \in A.a + a' = b$. A semiring is naturally ordered if the natural order is a partial order. Note that $\leq$ is trivially reflexive and transitive, so it remains only to show that it is anti-symmetric. That is, if $a \leq b$ and $b \leq a$, then $a = b$.

For the probabilistic semiring, the natural order corresponds to real number comparison. We now define Outcome Algebras that give the interpretation of choice. The carrier set $A$ is used to represent the weight of an outcome. In deterministic and nondeterministic evaluation models, this weight can be 0 or 1 (a Boolean), but in the probabilistic model, it can be any probability in $[0,1]$. The rules for combining these weights vary by execution model. As we will later see, the semantics of loops uses fixed points, requiring the semirings to be Scott continuous (defined in Appendix B).

**Definition 3.4 (Outcome Algebra).** An Outcome Algebra is a structure $\langle A,+,\cdot,\neg,\emptyset,1 \rangle$ in which $\langle A,+,\cdot,\emptyset,1 \rangle$ is a Scott continuous, naturally ordered, partial semiring with the following properties:

1. Complementation: $\neg: A \rightarrow A$ is a partial unary operation such that if $\neg a$ is defined, then $a + \neg a = 1$ and $\neg \neg a = a$.
2. Supremum: $\sup(A) = 1$, making $\langle A,\leq \rangle$ a complete lattice.
3. Scaling sums: if $a + b$ is defined, then there exist $u,v \in A$ such that $a = (a + b) \cdot u$ and $b = (a + b) \cdot v$ and $u + v = 1$.

---

3Our expression syntax differs slightly from that of Berdine et al. [2005a], which included $\not=$ rather than logical negation. The two are nonetheless equally expressive since $e_1 \not= e_2$ is equivalent to $\neg(e_1 = e_2)$ and $\neg e$ is equivalent to $e = \text{false}$. 


Outcome Algebras can encode the following three interpretations of choice:

**Definition 3.5 (Deterministic Outcome Algebra).** A deterministic program has at most one outcome (zero if it diverges). To encode this, we use an Outcome Algebra $\langle \{0, 1\}, +, \cdot, 0, 1 \rangle$ where the elements $\{0, 1\}$ are Booleans indicating whether or not an outcome has occurred. The sum operation is usual integer addition, but is undefined for $1 + 1$, since two outcomes cannot simultaneously occur in a deterministic setting. In addition, $\cdot$ is typical integer multiplication, and $\overline{a} = 1 - a$.

**Definition 3.6 (Nondeterministic Outcome Algebra).** The nondeterministic Outcome Algebra is $\langle \{0, 1\}, V, \land, \neg, 0, 1 \rangle$. Similar to the previous case, the elements are Booleans indicating whether an outcome has occurred, but now the semiring addition is a logical disjunction, indicating that outcomes can be combined. We also define $\overline{1} = 1$, and $\overline{0}$ is not defined.

**Definition 3.7 (Probabilistic Outcome Algebra).** Let $\langle \{0, 1\}, +, \cdot, 0, 1 \rangle$ be an outcome algebra where $+$ is real-valued addition (and undefined if $a + b > 1$), $\cdot$ is real-valued multiplication, and $\overline{a} = 1 - a$. The carrier set $\{0, 1\}$ indicates that each outcome has a probability of occurring.

In the style of Moggi [1991], the language semantics is monadic in order to sequence effects. We now show how to construct a monad given any Outcome Algebra.

**Definition 3.8 (Outcome Monad).** Given an Outcome Algebra $\langle A, +, \cdot, 0, 1 \rangle$, we define an outcome monad $\langle \mathcal{W}(A, -), \text{unit}, \text{bind} \rangle$, where $\mathcal{W}(A, S) = \{ m : S \to_{\text{fin}} A \mid \sum_{s \in S} m(s) \leq 1 \}$ is the set of finitely supported weighting functions and the monad operations are defined as follows:

\[
\text{unit}(s)(t) = \begin{cases} 1 & \text{if } s = t \\ \emptyset & \text{if } s \neq t \end{cases} \quad \text{bind}(m, f)(t) = \sum_{s \in \text{supp}(m)} m(s) \cdot f(s)(t)
\]

We also let $\text{supp}(m) = \{ s \mid m(s) \neq \emptyset \}$ and $|m| = \sum_{s \in \text{supp}(m)} m(s)$. This monad is very similar to the Giry [1982] monad, but it is generalized to work over any partial semiring, rather than probabilities $[0, 1] \subseteq \mathbb{R}$. It fairly easy to see that $\mathcal{W}$ obeys the monad laws, given the semiring laws.

So, a weighting function $m \in \mathcal{W}(A, S)$ assigns a weight $a \in A$ to each program state $s \in S$. Definitions 3.5 to 3.7 gave interpretations for $A$ in which $\mathcal{W}(A, S)$ encodes deterministic, nondeterministic, and probabilistic computation, respectively. In the (non)deterministic cases, $m(s) \in \{0, 1\}$, indicating whether or not $s$ is present in the collection of outcomes $m$. Due to the interpretation of $+$ in Definition 3.5, the constraint that $\sum_{s \in S} m(s) \leq 1$ guarantees that $m$ can contain at most one outcome, whereas in the nondeterministic case, $m$ can contain arbitrarily many. In the probabilistic case, $m(s) \in [0, 1]$ and gives the probability of the outcome $s$ in the distribution $m$.

The semiring operations can be lifted to weighting functions. We will overload some notation to also refer to pointwise liftings as follows: $m_1 + m_2 = \lambda s.(m_1(s) + m_2(s)), \emptyset = \lambda s.\emptyset$, and $a \cdot m = \lambda s.(a \cdot m(s))$. When the nondeterministic algebra (Definition 3.6) is lifted in this way, the result is isomorphic to the powerset monad with $m_1 + m_2 = m_1 \cup m_2$ and $\emptyset = \emptyset$.

Now, in order to represent errors and undefined states in the language semantics, we will define a monad transformer [Liang et al. 1995] based on the coproduct $S + E + 1$ where $S$ is the set of program states, $E$ is the set of errors, and we additionally include an undefined symbol. We define the following three injection functions, plus shorthand for the undefined element:

\[
1_{\text{ok}} : S \to S + E + 1 \quad 1_{\text{er}} : E \to S + E + 1 \quad 1_{\text{undef}} : 1 \to S + E + 1 \quad \text{undef} = 1_{\text{undef}}(\star)
\]

Borrowing the notation of Incorrectness Logic [O’Hearn 2019], we use ok and er to denote states in which the program terminated successfully or crashed, respectively. We will also write $1_e$ to refer to one of the above injections, where $e \in \{\text{ok}, \text{er}\}$.
Noam Zilberstein, Angelina Saliling, and Alexandra Silva

\begin{document}
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{denotational_semantics.png}
\caption{Denotational semantics of program commands, parametric on an outcome algebra \(\langle A, +, \cdot, 0, 1 \rangle\), an allocator function \(\text{alloc} : S \times H \rightarrow W(A, \text{Addr})\), and a procedure table \(P : \text{Proc} \rightarrow \text{Cmd} \times \text{Var}\).}
\end{figure}

\textbf{Definition 3.9 (Error Monad Transformer).} Let \(\langle W(A, -), \text{bind}_W, \text{unit}_W \rangle\) be the outcome monad described in Definition 3.8 and let \(E\) be some set of error states. We now define a new monad \(\langle W(A, - + E + 1), \text{bind}, \text{unit} \rangle\) where the monad operations are defined as follows:

\[
\text{unit}(s) = \text{unit}_W(1_{\text{ok}}(s)) \quad \text{bind}(m, f) = \text{bind}_W\left(m, \lambda \sigma. \begin{cases} f(s) & \text{if } \sigma = 1_{\text{ok}}(s) \\ \text{unit}_W(\sigma) & \text{otherwise} \end{cases}\right)
\]

\section{Denotational Semantics}

We are now ready to give semantics to the language we introduced at the beginning of this section. As is usual in separation logic, program states consist of variable stores \(S = \{s : \text{Var} \cup \text{LVar} \rightarrow \text{Val}\}\), and heaps \(H = \{h : \text{Addr} \rightarrow \text{Val} \cup \{\perp\}\}\). In the style of Raad et al. [2020], a heap is both a \textit{partial} mapping and also includes \(\perp\) in the codomain, allowing us to distinguish between cases where we have no information about an address \((\ell \notin \text{dom}(h))\) and cases where we explicitly know that \(\ell\) is deallocated \((h(\ell) = \perp)\). The set of addresses Addr is opaque, but we assume null \(\in \text{Const} \subseteq \text{Val}\) and \(\text{Addr} \subseteq \text{Val}\) and \(\text{Addr} \cap \text{Const} = \emptyset\). Both stacks and heaps are finite (but unbounded).

In addition to being parametric on an outcome algebra \(\langle A, +, \cdot, 0, 1 \rangle\), the semantics is also parametric on an allocator function, showing that the remainder of the technical development is agnostic to the details of how memory is allocated. An allocator \(\text{alloc} : S \times H \rightarrow W(A, \text{Addr})\) is a function such that \(|\text{alloc}(s, h)| = 1\) and \(\text{supp}(\text{alloc}(s, h))\) is disjoint from all the addresses found in \(s\) and \(h\). A deterministic allocator \(\text{alloc}(s, h) = \min(\text{Addr} \setminus (\text{dom}(h) \cup \text{im}(s) \cup \text{im}(h)))\) that always
picks the first unused address is valid in all OSL instances.\textsuperscript{4} To interpret procedure calls, we use $P : \text{Proc} \to \text{Cmd} \times \text{Var}$, a global procedure table that returns a command and vector of variable names (the arguments) given a procedure name $f \in \text{Proc}$. We assume that all procedures used in programs are defined and pass the correct number of arguments.

The semantics is shown in Figure 1. The set of program states is $\text{St} = S \times \mathcal{H} + S \times \mathcal{H} + 1$ and program configurations come from $\mathcal{W}(A, \text{St})$. The monad operations (Definition 3.9) are used to give semantics to skip and $\;\downarrow$ in the standard way, if statements select the appropriate branch according to the value of the Boolean guard, and while loops are defined using a least fixed point.

Choice $C_1 +_a C_2$ is defined using a generic weighted sum whose specific meaning depends on the algebraic interpretation. Choice is not so useful in the deterministic model (Definition 3.5) since $C_1 + C_2$ is equivalent to just evaluating evaluating $C_1$ and $C_1 +_0 C_2$ is equivalent to $C_2$. In the nondeterministic model (Definition 3.6), since $\emptyset$ is undefined, we write the choice operator as $C_1 + C_2$ instead of $C_1 +_1 C_2$, which will evaluate to the set of all the outcomes of both $C_1$ and $C_2$.

Finally, in the probabilistic model (Definition 3.7), $C_1 +_a C_2$ evaluates to a probability distribution where $C_1$ is chosen with probability $a$ and $C_2$ is chosen with probability $1 - a$.

We define two operations before giving the semantics of instructions: error $(s, h)$ constructs an error state and update $(s, h, t, s', h')$ returns $(s', h')$ if the address $t$ is allocated in $h$, it returns an error if $t$ is deallocated, and is undefined if $t \notin \text{dom}(h)$. Assignment is defined in the usual way by updating the program store; memory allocation uses the alloc operation to obtain a fresh address (or collection thereof) and initializes the value to 0; deallocation, reads, and writes are implemented using update and errors use error. Procedure names are looked up in $P$ to obtain $C$ and $\vec{x}$ before running $C$ on a store updated by setting $\vec{x}$ to have the values of the inputs $\vec{e}$.

Despite using the partial sum operation and a least fixed point, the semantics of programs is total. We prove this in Appendix B. We will additionally occasionally use the monadic extension of the semantics, which is defined as $[C]^\dagger(m) = \text{bind}(m, [C])$.

### 4 OUTCOME SEPARATION LOGIC

We now proceed to define Outcome Separation Logic (OSL), present a frame rule, and prove its soundness. First, we define an assertion logic that will be used as the pre- and postconditions of outcome triples. These assertions are based on the outcome assertions of Zilberstein et al. [2023], using the symbolic heaps of Berdine et al. [2005a,b] as basic predicates.

**Symbolic Heaps.** The syntax for symbolic heaps is shown below and the semantics is standard as defined by Berdine et al. [2005b, §2].

\[
P := \exists \vec{X}. \Delta \quad \text{(Symbolic Heaps)} \quad \Pi := \text{true} \mid \Pi_1 \land \Pi_2 \mid e \quad \text{(Pure Assertions)}
\]

\[
\Delta := \Pi \land \Sigma \quad \text{(Quantifier-Free)} \quad \Sigma := \text{true} \mid \text{emp} \mid \Sigma_1 * \Sigma_2 \mid e_1 \mapsto e_2 \mid \text{ls}(e_1, e_2) \quad \text{(Spacial Assertions)}
\]

A symbolic heap $P$ consists of existentially quantified logical variables, along with a pure part $\Pi$ and a spacial part $\Sigma$. A pure assertion is a conjunction of Boolean valued expressions $e \in \text{Exp}$ (equalities and inequalities), whereas a spacial assertion is a sequence of heap assertions joined by separating conjunctions. The separating conjunction requires that the heap can be split into two disjoint components to satisfy the two assertions separately.

\[(s, h) \vdash \Sigma_1 * \Sigma_2 \iff \exists h_1, h_2. \ h = h_1 \uplus h_2 \text{ and } (s, h_1) \vdash \Sigma_1 \text{ and } (s, h_2) \vdash \Sigma_2\]

The points-to predicate $e_1 \mapsto e_2$ specifies a singleton heap in which the address $e_1$ points to the value $e_2$. We also define negative heap assertions $e \not\mapsto$ as syntactic sugar for $e \mapsto \bot$. These assertions were introduced in Incorrectness Separation Logic to express that a pointer is invalidated in order

\[\text{The image of a function is defined as follows } \text{im}(f) = \{ f(x) \mid x \in \text{dom}(f) \}.\]
to prove that a program crashes due to a memory error [Raad et al. 2020]. Finally, we have an
inductive list segment predicate \( \text{ls}(e_1, e_2) \), which states that there is a sequence of pointers starting
with \( e_1 \) and ending with \( e_2 \). Formally, it is the least solution of:

\[
\text{ls}(e_1, e_2) \iff (e_1 = e_2 \land \text{emp}) \lor (\exists X. e_1 \mapsto X \mid \text{ls}(X, e_2))
\]

We also provide overloaded definitions of the \(*\) and \(\land\) operators.

\[
(\exists X. \Pi \land \Sigma) \ast (\exists Y. \Pi' \land \Sigma') \triangleq \exists X \exists Y. (\Pi \land \Pi') \land (\Sigma \land \Sigma') \land (\Pi \ast \Sigma) \land (\Pi \ast \text{emp})
\]

Though symbolic heaps have limited expressivity—particularly for pure assertions—they have a
comeplete decision procedure [Berdine et al. 2005a], which is necessary for bi-abductive analysis
algorithms. These same symbolic heaps are used by Calcagno et al. [2009, 2011].

**Outcome Assertions.** OSL assertions are based on the outcome assertions of Zilberstein et al.
[2023]. They use symbolic heaps as basic predicates and rely on an outcome algebra for their
interpretation. The syntax for these assertions is below and their semantics is in Figure 2.

\[
\varphi ::= \top | \top^\circ | \varphi \lor \psi | \varphi \odot_a \psi | \epsilon : P \quad \epsilon ::= \text{ok} | \text{er}
\]

Outcome assertions include some familiar constructs such as \( \top \), which is always true, and disjunctions \( \varphi \lor \psi \). Additionally, we have \( \top^\circ \), which asserts that there are no outcomes.

The weighted outcome conjunction \( \varphi \odot_a \psi \) splits the program configuration \( m \) into two pieces \( m_1 \)
and \( m_2 \) whose weighted sum is equal to \( m \) (see Figure 2, note that this connective requires \( a \) to be defined). Using the nondeterministic interpretation (Definition 3.6), the only valid choice of \( a \) is 1, thus we will omit the \( a \) in this case and simply write \( \varphi \odot \psi \), which means that \( \varphi \) and \( \psi \) are outcomes that can arise due to nondeterminism. In the probabilistic interpretation, for any \( p \in [0, 1] \), the assertion \( \varphi \odot_p \psi \) states that \( \varphi \) occurs with probability \( p \) and \( \psi \) occurs with probability \( 1 - p \).

Finally, basic assertions \( \text{ok} : P \) and \( \text{er} : Q \) require that all states in \( \text{supp}(m) \) terminated
successfully and satisfy \( P \) or crashed and satisfy \( Q \), respectively, where \( P \) and \( Q \) are symbolic heaps. The requirement that \( |m| = 1 \) ensures that the set of outcomes is nonempty in the (non)deterministic cases (Definitions 3.5 and 3.6) and that the total mass is 1 in the probabilistic case (Definition 3.7).

We were motivated to pick this particular set of outcome assertions in light of our goal to define
symbolic execution algorithms in the style of Calcagno et al. [2009], which compute procedure
summaries of the form \( \{P\} \text{f}(\overline{a}) \{Q_1 \lor \cdots \lor Q_n\} \) and the disjunctive post indicates a series of possible outcomes. In our case, we will exchange those disjunctions for outcome conjunctions in the cases where the outcomes arise due to computational effects. We include \( \top \) in order to drop outcomes using the assertion \( \varphi \odot_a \top \) (as discussed in Section 2). Finally, we include the standard disjunction to express joins of outcomes that occur due to logical choice, and also to express partial correctness; while \( \epsilon : P \) guarantees reachability, \( \epsilon : P \lor \top^\circ \) also permits nontermination.

**Remark 1 (Disproving Outcome Assertions).** Given that a goal of this paper is to create symbolic
execution algorithms for both correctness and incorrectness, it makes sense to ask whether it is possible to disprove triples using these outcome assertions as pre- and postconditions. This may seem dubious given that the syntax does not include logical negation, however Zilberstein et al. [2023, §5] showed that outcome assertions that are expressed as a sequence atomic assertions
separated by \( \oplus \) can be disproven using a different sequence of atoms separated by \( \ominus \)—logical negation is, in fact, unnecessary. We can then use various other facts to disprove atoms, for example 
\[
(\text{er} : Q) \Rightarrow \neg(\text{ok} : P) \quad \text{and} \quad (x \triangleright \triangleright) \Rightarrow \neg(x \mapsto v).
\]

**OSL Triples.** The semantics of OSL triples is the same as those of standard OL, and—crucially—makes no mention of safety that is typically needed to guarantee the soundness of the frame rule [Yang and O’Hearn 2002]. As we discussed in Section 2 and discuss further in Section 4.3, the safety requirement is incompatible with dropping program paths, so for OSL it is important to omit it.

**Definition 4.1 (Outcome Separation Logic Triples).** Given an outcome algebra \( \langle A, +, \cdot, \neg, 0, \top \rangle \) and an allocator function, the validity of OSL triples is defined as follows:
\[
\vDash \langle \psi \rangle C \langle \phi \rangle \iff \forall m \in \mathcal{W}(A, \mathcal{St}). \ m \vDash \phi \implies [C]^{\top}(m) \vDash \psi
\]

**The Frame Rule.** In the remainder of this section, we build the necessary foundations to introduce and prove the soundness of the frame rule. Before doing so, we need a separating conjunction for outcome assertions. Rather than adding the separating conjunction as a logical connective, we define it inductively as a transformation on outcome assertions below. We use the symbol \( \oplus \) to distinguish it from the usual separating conjunction \( \cdot \) on symbolic heaps. Unlike \( \cdot \), \( \oplus \) is asymmetric; the left hand side is an outcome assertion, whereas the right hand side is a symbolic heap.

\[
\begin{align*}
T \ominus F & \triangleq T \\
T^\ominus \ominus F & \triangleq T^\ominus \\
(\phi \lor \psi) \ominus F & \triangleq (\phi \ominus F) \lor (\psi \ominus F) \\
(e : P) \ominus F & \triangleq e : P \oplus F
\end{align*}
\]

So, \( \ominus \) has no effect on \( T \) and \( T^\ominus \), it distributes over \( \lor \) and \( \ominus P \), and for basic assertions \( e : P \), we simply join \( P \oplus F \) with the usual separating conjunction. We can now express the OSL frame rule:
\[
\frac{\langle \phi \rangle C \langle \psi \rangle \mod(C) \cap \mathsf{fv}(F) = \emptyset}{\langle \phi \ominus F \rangle C \langle \psi \ominus F \rangle}^{\text{FRAME}}
\]

This rule resembles the frame rule of separation logic [O’Hearn et al. 2001], with the same side condition stating \( F \) must not mention any modified program variables. However, unlike the standard frame rule, it can be used to reason about multiple outcomes simultaneously. For example, suppose we wanted to add information about a pointer \( y \) to the specification we saw in Section 2.1.

\[
\begin{align*}
\langle \text{ok} : \text{emp} \rangle x := \text{malloc}() \uplus [x] & \leftarrow 1 \langle (\text{ok} : x \mapsto 1) \ominus (\text{er} : x = \text{null} \land \text{emp}) \rangle \\
\langle \text{ok} : y \mapsto 2 \rangle x := \text{malloc}() \uplus [x] & \leftarrow 1 \langle (\text{ok} : x \mapsto 1 \cdot y \mapsto 2) \ominus (\text{er} : x = \text{null} \land y \mapsto 2) \rangle
\end{align*}
\]

As we discussed in Section 2.2, proving the soundness of the OSL frame rule requires different underlying assumptions. The key to the soundness proof is to show that \([C] (s, h \uplus h’)\) yields a result that could also be obtained from \([C] (s, h)\) by joining the additional memory footprint and performing other operations that do not affect the truth of the post. These operations will be expressed as **rewriting functions**, which redistribute the weight of a program configuration \( m \) according to a relation \( R \). This concept is defined formally below.

**Definition 4.2 (Rewriting Functions).** Given a relation \( R \subseteq X \times Y \) and \( m \in \mathcal{W}(A, X) \), a reweighting function \( f : X \rightarrow \mathcal{W}(A, Y) \) is a function such that for all \( x \in \text{supp}(m) \):
\[
\sum_{y \in Y \mid (x, y) \in R} f(x)(y) = 1 \quad \text{and} \quad f(x)(y) = 0 \quad \text{if} \quad (x, y) \notin R
\]

So, \( f \) reassigns all the weight in \( m \) and \( f \) only redistributes weight to related states according to \( R \). Let \( \mathsf{wf}(m, R) \) be the set of all such functions and \( \mathsf{w}(m, R) = \{ \text{bind}(m, f) \mid f \in \mathsf{wf}(m, R) \} \) be the set of all program configurations obtainable by reassigning the mass at each point according to \( R \).
We will use reweighting functions to transform $\llbracket C \rrbracket (s, h)$ into $\llbracket C \rrbracket (s, h \uplus h')$ (Lemma 4.8), which accordingly dictates that if $\llbracket C \rrbracket (s, h) \vDash \psi$ then $\llbracket C \rrbracket (s, h \uplus h') \vDash \psi \land F$ (Theorem 4.9). Specifically, $\llbracket C \rrbracket (s, h \uplus h')$ will have a larger memory footprint (Section 4.1), the allocated addresses may change as the result of running the program in a larger heap (Section 4.2), and the states of $\llbracket C \rrbracket (s, h)$ that have undefined behavior may lead to defined behavior after augmenting the heap (Section 4.3).

### 4.1 Semantics of the Outcome Separating Conjunction

We now prove semantic properties about $\odot$ that allow us to relate program configurations satisfying $\varphi$ to ones that satisfy $\varphi \odot F$. We express these properties in terms of reweighting functions that relate program states to other states augmented to satisfy a separate symbolic heap $F$.

\[
\text{frame}(F) = \{(1_\varepsilon(s, h), 1_\varepsilon(s, h \uplus h')) | 1_\varepsilon(s, h) \in \text{St}, (s, h') \vDash F \} \cup \{(\text{undef}, \text{undef})\}
\]

Any state $1_\varepsilon(s, h)$ is related to all states $1_\varepsilon(s, h \uplus h')$ such that $(s, h') \vDash F$, which guarantees that if $(s, h) \vDash P$, then $(s, h \uplus h') \vDash P \ast F$. Undefined states are only related to themselves. So, $w(m, \text{frame}(F))$ transforms every state in $\text{supp}(m)$ to also describe $F$. This semantics is captured below.

**Lemma 4.3.** If $m \vDash \varphi$, then for any $m' \in w(m, \text{frame}(F))$, $m' \vDash \varphi \land F$

It is tempting to say that the converse should also hold, but that is not quite right. We took $T \odot F$ to be equal to $T$, therefore if $m \vDash T \odot F$, then we cannot guarantee that all the states in $m$ contain information about $F$. We therefore characterize the semantics only for the states that are not covered by $T$, leaving the other states unconstrained.

**Lemma 4.4.** If $m \vDash \varphi \land F$, then there exist $m_1, m_2$, and $m'_1 \in w(m_1, \text{frame}(F))$ such that $m = m'_1 + m_2$ and $m_1 + m'_2 \vDash \varphi$ for any $m_2$ such that $|m'_2| \leq |m_2|$.

In the lemma above, $m'_1$ represents the nontrivial portion of $m$ and $m_2$ is the portion of $m$ that is covered by $T$. As such, $m'_1$ must be the result of framing $F$ into some $m_1$. Since $m_2$ is covered by $T$, we can replace it with anything smaller than $m_2 - T$ can absorb at least $|m_2|$ worth of mass. These two lemmas provide a semantic basis to reason about what it means for $\varphi \land F$ to hold relative to $\varphi$.

### 4.2 Nominal Heaps

In Section 2, we saw what goes wrong when attempting to use the frame rule with a program that can determine the address of an allocated pointer. If we know that the allocator will always return the next unallocated slot, then running $x := \text{alloc}()$ in the empty heap will always result in $x$ having the address 1. However, running the same program in a larger heap will change the address, meaning that using the frame rule will result in an invalid specification.

Yang and O’Hearn [2002] solve this problem by forcing memory allocation to be nondeterministic. If the allocator could return any address, then the postcondition cannot say anything too specific about which address got returned. Nondeterminism works in unison with the semantics of Hoare Logic, which requires that a single predicate in the postcondition covers all possible end states.

This approach is undesirable in OL for two reasons. First, we do not want to rely on a nondeterministic evaluation model. Second, we want to be able to reason about reachable states and incorrectness, which is not compatible with the latter restriction. For example, if memory allocation is nondeterministic, then the following OSL specification is valid:

\[
\langle \text{ok : emp} \rangle x := \text{alloc}() \bigoplus_{\ell \in \text{Addr}} \langle \text{ok : } x = \ell \rangle
\]

The triple asserts that $x$ is equal to every address in some reachable outcome. Framing in information about another pointer would invalidate one of those outcomes, so the frame rule is unsound.
We instead take the approach of considering heap addresses to be nominal. That is, in the style of nominal logic [Pitts 2003], we will show that the satisfaction of symbolic heaps and outcome assertions is stable under permutation of the heap addresses—the satisfaction relation is equivariant. Practically speaking, this means that symbolic heaps cannot use addresses as constants or do pointer arithmetic. This approach was suggested by Yang and O’Hearn [2002, §4.1], and has been explored by Baktiev [2006], who created a deterministic separation logic with a sound frame rule. Building on that previous work, we extend the idea to outcome assertions, so as to reason about, e.g., probabilistic programs too. We also show that the symbolic heaps of Berdine et al. [2005b] have the equivariance property, so the OSL frame rule is sound using the same basic assertions as were used in previous bi-abduction algorithms [Calcagno et al. 2009, 2011].

Now, let \( \pi \) be a function that permutes the addresses in whatever structure it is applied to via a bijection from Addr to Addr. For example:

\[
\pi((i \in (s, h))) = i \in (\pi(s), \pi(h))
\]

\[
\pi(s) = \pi \circ s \quad \pi(h) = \pi \circ h \circ \pi^{-1}
\]

We formalize the fact that symbolic heaps cannot be too specific about addresses in the following lemma: permuting the addresses in a program state will not affect the truth of a symbolic heap.

**Lemma 4.5 (Equivariance of Symbolic Heaps).** If \( (s, h) \models P \), then \( \pi(s, h) \models P \) for any \( \pi \).

Equivariance must apply to outcome assertions as well. For that, we use reweighting functions since simply permuting the addresses in some configuration \( m \) would not be sufficient—we do not preclude allocation from being nondeterministic, so when a program that allocates memory is evaluated in an individual state \( \sigma \in \text{supp}(m) \), the result could branch into many outcomes, each of which has a different address for the newly allocated pointer and is therefore not simply a permutation of \( m \). As such, the use of reweighting functions permits the mass of each \( \sigma \) to sliced up and redistributed among any combination of states that are all permutations of \( \sigma \).

**Lemma 4.6 (Equivariance).** Let \( \text{Perm} = \{(\sigma, \pi(\sigma)) \mid \sigma \in \text{St}, \pi \text{ is a permutation}\} \). If \( m \models \varphi \), then \( m' \models \varphi \) for any \( m' \in w(m, \text{Perm}) \).

### 4.3 Replacement of Unsafe States

Whereas the semantics of separation logic requires that all states satisfying the precondition are safe (they will not fault), we explicitly omit this requirement in OSL. Efficiently reasoning about incorrectness requires us to only explore a subset of the program paths, and we cannot guarantee that the paths we dropped are safe. To demonstrate this, consider the following specification:

\[
\langle \text{ok} : x \not\mapsto \rangle \text{ free}(x) + C \langle (\text{er} : x \not\mapsto) \oplus \top \rangle
\]

The fact that the left path leads to a memory fault is enough to conclude that the program is incorrect; it would be wasted effort to additionally explore the right path. However, the right path may use other pointers not mentioned in the precondition, meaning that executing \( C \) will lead to undefined. This is fine, since all we said about the right path is \( \top \), which absorbs undefined states.

However, if we used the frame rule to add information about more pointers to the precondition, the result may not be undefined anymore. This is still ok, since no matter what outcome we get from running \( C \), it must be covered by \( \top \). For example, the following (framed) triple is also valid.

\[
\langle \text{ok} : x \not\mapsto * y \mapsto - \rangle \text{ free}(x) + ([y] \leftarrow 1) \langle (\text{er} : x \not\mapsto * y \mapsto -) \oplus \top \rangle
\]

So, the behavior of undefined outcomes can change as the result of running the program in a larger heap. We therefore need to ensure that replacing undefined states will not affect the validity of the postcondition, which is true because \( \top \) is the only assertion that can be satisfied by undefined and—since \( \top \) is always true—it can be satisfied by anything else. To formalize this, we use a relation
Rep ⊆ St × St⊥ that relates undef to any state in St plus ⊥, indicating that previously undefined states may diverge after adding more pointers\(^5\). All other states (ok and er) are related only to themselves. Since ⊥ is not a program state, we additionally define a pruning operation to remove it.

\[
\text{Rep} = \{(\text{undef}, \sigma) \mid \sigma \in \text{St}_⊥\} \cup \{(\sigma, \sigma) \mid \sigma \in \text{St}\} \quad \text{prune}(m)(\sigma) = \begin{cases} m(\sigma) & \text{if } \sigma \neq \bot \\ \emptyset & \text{if } \sigma = \bot \end{cases}
\]

We will also occasionally write prune(S) to mean \{prune(m) \mid m \in S\}. Now we can state the replacement lemma which says that the weight of undefined states can be replaced by anything else without affecting the validity of an outcome assertion.

**Lemma 4.7 (Replacement).** If \(m \not\models \varphi\) and \(m' \in \text{prune}(w(m, \text{Rep}))\), then \(m' \not\models \varphi\).

While dropping the requirement that all start states must lead to defined behavior seemingly makes OSL triples weaker, in reality this requirement is simply shifted into the assertion logic. Including \(\top\) in the postcondition opens the possibility for undefined behavior along some program paths, which is desirable when reasoning about incorrectness since it allows us to only explore part of the program. If the postcondition does not contain \(\top\), then the end configuration must have no undefined states, since \(\top\) is the only assertion that can absorb those states.

### 4.4 Soundness of the Frame Rule

We now have all the ingredients needed to prove the soundness of the frame rule. The first step is to prove the frame property, which describes how the result of running the program on a framed start state \((s, h \uplus h')\) is related to just running it on the unframed state \((s, h)\).

Running the program on \((s, h \uplus h')\) will redistribute the mass of \([C]\) \((s, h)\) according to the three relations that we defined previously. That is, the addresses may be permuted (Perm), information about additional pointers will be added (frame(F)), and the undefined states may be replaced (Rep). As is usual for the frame rule, we also require that the modified program variables are disjoint from the free variables of the symbolic heap \(F\).

**Lemma 4.8 (The Frame Property).** Let \(R = \text{Rep} \circ \text{frame}(F) \circ \text{Perm}\), so \(R \subseteq \Sigma \times \Sigma⊥\). For any program \(C\) and symbolic heap \(F\) such that \(\text{mod}(C) \cap \text{fv}(F) = \emptyset\):

\[
\forall (\mathbb{1}_{\text{ok}}(s, h), \mathbb{1}_{\text{ok}}(s', h')) \in R. \quad [C](s', h') \in \text{prune}(w([C]\ (s, h), R))
\]

Now, given the frame property, we know how the result of running the program on a framed input will relate to running it on an unframed one. In order to establish the soundness of the frame rule, all that remains is to show what happens as we peel away the three layers of the relation \(R\).

**Theorem 4.9 (The Frame Rule).** The following inference is sound.

\[
\frac{\langle \varphi \rangle \ C \ \langle \psi \rangle \quad \text{mod}(C) \cap \text{fv}(F) = \emptyset}{\langle \varphi \circ F \rangle \ C \ \langle \psi \circ F \rangle \ \text{FRAME}}
\]

We briefly sketch the proof here, whereas the full version is in Appendix D.4. Suppose \(m \not\models \varphi \circ F\), then by Lemma 4.4, we can get an \(m' \not\models \varphi\) and by the premise of the rule, we also know that \([C]'(m') \not\models \psi\). Using Lemma 4.8, running the program on \(m\) will give us a result that we can relate to running the program on \(m'\) via the three relations described in the previous sections. All that remains is to peel away those relations using Lemmas 4.3, 4.6 and 4.7 to conclude that \(m \not\models \psi \circ F\).

So Outcome Separation Logic has a sound frame rule, despite relaxing some of the restrictions of separation logic that are intended to ensure the soundness of framing. We achieved this by ensuring\(^5\) e.g., running free(x) ↓ while true do skip in an empty heap leads to undef whereas it will not terminate if x is allocated.
that OSL assertions cannot contain specific details about heap addresses or undefined behavior. Since the OSL frame rule does not rely on nondeterminism, it can be used for deterministic and probabilistic languages too. It also does not require start states to be safe, so OSL can be used to reason about incorrectness without inspecting the entire program. In the next sections, we will see how the OSL theory can be used to build compositional symbolic execution algorithms.

5 TRI-ABDUCTION

The last step before defining symbolic execution algorithms for OSL is to address the matter of parallel composition in programs with choice mechanisms arising from computational effects such as nondeterminism or random sampling. When symbolically executing programs with choice, we must unify the preconditions for the two branches. For example, in the following program that chooses to execute \([x] \leftarrow 1\) or \([y] \leftarrow 2\), we need a precondition that mentions both pointers \(x\) and \(y\), and we need to know what leftover resources to add to the two resulting outcomes.

\[
\begin{align*}
\text{Tri-abduction}, & \quad \text{the parallel composition analogue of bi-abduction, provides us the power to reconcile} \\
& \quad \text{the preconditions of the two program branches. Given} \ P_1 \quad \text{and} \quad P_2, \quad \text{the goal is to find the} \quad \text{anti-frame} \\
& \quad \text{and two leftover frames} \ F_1 \quad \text{and} \quad F_2 \quad \text{that make} \quad P_1 * F_1 \models M \models P_2 * F_2 \quad \text{hold. Using this, we can} \\
& \quad \text{compose parallel program branches according to the inference below.}
\end{align*}
\]

\[
\begin{array}{c}
\frac{\langle P_1 \rangle C_1 \langle Q_1 \rangle}{\langle M \rangle C_1 + a C_2 \langle (Q_1 * F_1) \oplus a (Q_2 * F_2) \rangle} \quad \text{TRI-ABDUCTIVE COMPOSITION}
\end{array}
\]

Remark 2 (Necessity of Tri-Abduction). Though tri-abduction is a central part of symbolic execution using OSL, it would have also been useful in Abductor, which is unable to analyze the program above despite supporting nondeterminism. Abductor operates in two passes; first finding candidate preconditions for each program trace, and then re-evaluating the program with each candidate to check if any are valid for all the program paths [Calcagno et al. 2009]. Since the program above uses disjoint resources in the two branches, no candidate is valid for the entire program. With the help of tri-abduction, we infer more summaries and do so in a single pass.

Remark 3 (Solving Tri-Abduction using Bi-Abduction). Our initial approach to tri-abduction was to simply use bi-abduction: given \(P_1\) and \(P_2\), bi-abduction can give us \(M\) and \(F\) such that \(P_1 * M \models P_2 * F\). Using \(P_1 * M\) as the anti-frame, \(P_1 * M \models (P_1 * M) \models P_2 * F\) is a tri-abduction solution.

However, this approach is inherently asymmetric, with the symbolic heap of the left branch being favored. While it would be possible to also bi-abduce in the opposite direction \((P_2 * ? \models P_1 * ?)\) for symmetry, this still precludes valid solutions. For example, there is no bi-abduction solution for the symbolic heaps \(X \mapsto Y \mapsto \text{Is}(Y, Z)\) and \(\text{Is}(X, Y) \mapsto Y \mapsto Z\) (in either direction), whereas tri-abduction finds the anti-frame \(X \mapsto Y \mapsto Y \mapsto Z\). Tri-abduction is a fundamentally different operation that is precisely designed for parallel composition.

We now describe the procedure for tri-abductive inference. Similar to [Calcagno et al. 2011, Algorithm 3], tri-abduction is done in two stages. First, in Section 5.1, we describe the abduction stage, in which we infer only the anti-frame \(M\) (Algorithm 1). Next, in Section 5.2 we describe how abduction is used as a subroutine to tri-abduce all three parameters \(M\), \(F_1\), and \(F_2\) (Algorithm 2).
Algorithm 1 abduce-par(P,Q)

1: if either Base-Emp, Base-True-L, or Base-True-R apply then
2:   return the M indicated by those rules.
3: else
4:   Try to apply the remaining rules in a top-to-bottom order according to Figure 3
5:   return fail if no such rule applies.
6: let $M' = \text{abduce-par}(\Delta, \Delta')$ based on the premises of the rule.
7: Using this $M'$, return $M$ produced by the current rule.
8: end if

5.1 Abductive Inference

The abductive inference step is performed as a proof search, similar to Calcagno et al. [2011, Algorithm 1], using the proof rules in Figure 3 to infer judgements of the form $P \triangleleft [M] \triangleright Q$, which indicate that $M \models P$ and $M \models Q$. In these judgements, $P$ and $Q$ can be read as the inputs to the algorithm and $M$ is the output. The premise of the inference rule then becomes a recursive call, finding a solution to a smaller abduction query. Some of the rules have side conditions of the form $R \nvdash \text{false}$, which can be checked using the proof system of Berdine et al. [2005b, §4]. Given that each recursive call describes a progressively smaller symbolic heap, we will eventually reach a case with no explicit resources (emp or true), in which a base rule applies.

In cases where the inference rules are numbered, the algorithm applies each of the numbered rules. These cases have inherent symmetry—no particular rule is preferable over the others—so we output multiple solutions. If no inference rules apply, the algorithm returns fail. Algorithm 1 defines abduce-par, which implements the proof search. The inference rules from Figure 3 are applied in the order in which they are shown, with the rules at the top of the figure being preferred over the rules lower down. The rules are divided into groups, which we describe now.

**Base Rules.** The first step is to attempt to apply a base rule in order to terminate the algorithm. Base-Emp applies when both branches describe empty heaps. In this rule, we check that the pure assertions in each branch do not conflict. In Base-True-L and Base-True-Right, we match against the case wherein one of the branches has an arbitrary spacial assertion $\Sigma$ and the other contains the spacial assertion true, indicating that it can absorb more resources not explicitly mentioned, so we are able to move $\Sigma$ into the anti-frame.

**Quantifier Elimination.** The next step is to strip existentials from the inputs $P$ and $Q$ and add them back to the anti-frame $M$ obtained from the recursive call. This is achieved using the EXISTS rule in Figure 3. In bi-abduction, existentials are not stripped from the assertion to the right of the entailment—doing so prevents the algorithm from finding solutions in some cases. For example, $\text{ls}(e, e') \neq e \leftrightarrow \exists X . e \mapsto X \neq e'$ has a solution ($e \neq e'$), but it does not have a solution with the existential removed since nothing can be added to $\text{ls}(e, e')$ to force $e$ to point to a particular $X$. Tri-abduction produces an anti-frame $M$ from scratch, so we are not operating under such constraints, allowing us to strip existentials at an early step in order to simplify further analysis.

It is important to note that in the EXISTS rule, quantified variables in one assertion cannot overlap with the free variables of the other. This ensures that no free variables in $P$ or $Q$ end up existentially quantified in the anti-frame $M$. Without the side condition, the rule is unsound; suppose we want to tri-abduce $\exists X . X = Y$ with $X = 1$, then the EXISTS gives us the anti-frame $\exists X . X = Y \land X = 1$, which is too weak since $\exists X . X = 1 \neq X = 1$. In practice, our symbolic execution algorithm always generates fresh logical variables, so we will not have collisions with our usage of the EXISTS rule.

**Resource Matching.** If a base rule does not apply, then we attempt to match resources from both branches, and then call the algorithm recursively on smaller symbolic heaps with some resources.
we include the cases where the list segments are equal, or where each is a sublist of the other.

If the two branches are $x$ pointed to must be equal too. The next three rules (head of the list, so we move it into
and the other refers to both branches contain the same resource $e$ moved into the returned anti-frame. The $Ls$-Start-L and $Ls$-Start-R rules apply in the case that both branches contain the same resource $e$; however, one includes $e_1$ in a list segment assertion and the other refers to $e_1$ using a points-to predicate. Here, the points-to predicate must be the head of the list, so we move it into $M$ and recurse on the tail of the list.

The MATCH rule applies when both branches use $e_1$ in a points-to predicate, therefore the value pointed to must be equal too. The next three rules ($Ls$-End-) apply when both branches have list segments starting at the same address. Whereas variants of several rules are symmetric, the $Ls$-End-# rules reflect different list configurations. Because we do not want an asymmetric result, we include the cases where the list segments are equal, or where each is a sublist of the other.

As in Calcagno et al. [2009], we do not consider cases where pointers are aliased. For example, if the two branches are $x \mapsto 1$ and $y \mapsto 1$, then it is possible that $x = y$. Precluding this solution
Algorithm 2 triab(P,Q)
1: summaries = ∅
2: for $M \in$ abduce-par($P * true, Q * true$) do
3: Find $F_1$ such that $M \vdash P * F_1$
4: Find $F_2$ such that $M \vdash Q * F_2$
5: Add $(M, F_1, F_2)$ to summaries
6: end for
7: return summaries

helps limit the number of options we consider. Calcagno et al. [2009, Example 3] remark that this loss of precision is not detrimental in practice.

Remark 4. Looking closer at the Ls-End-2 rule, we can also see that this rule only applies in the case that $e_3$ does not appear as a resource in $\Delta$, slightly limiting the number of times both Ls-End-2 and Ls-End-3 will apply concurrently. If $e_3$ does not appear as a resource in $\Delta$, then either Ls-End-2 or Ls-End-3 could potentially apply, so we try to apply both inference rules.

Resource Adding. Adding resources that are only present on one side is the last-resort, since it involves checking a potentially expensive side condition of the form $\Pi \land \Sigma * B(e_1, e_2) \not\models false$. The Missing-L and Missing-R rules handle the case wherein one branch refers to resources not present in the other. They are different than the Base-True rules, since they handle cases where both branches refer to resources not explicitly present in the other. For example, Missing-L can solve $x \mapsto X * true \ll [?] \gg y \mapsto Y * true$ even though the Base-True rules do not apply.

If one side of the judgement contains a list segment, but the other side does not contain the spacial assertion true, then there is a possible solution where the list segment is empty. Emp-Ls-L and Emp-Ls-R handle such cases by forcing the list segment to be empty.

5.2 Triabduction Algorithm and Correctness
As we mentioned at the beginning of the section, the tri-abduction algorithm (Algorithm 2) follows a similar structure to that of bi-abduction [Calcagno et al. 2011, Algorithm 3]. We first abduce a set of anti-frames using Algorithm 1 such that $M \models P * true$ and $M \models Q * true$ for each anti-frame $M$. The additional spacial assertion true absorbs extra resources; if $P$ and $Q$ have different memory footprints, then there is no $M$ such that $P \ll [M] \gg Q$, so adding true to both sides of the entailments allows us to produce an $M$ that refers to resources present in only one of the branches.

Next, we use the frame inference procedure from Berdine et al. [2005b, §5] to find symbolic heaps $F_1$ and $F_2$ such that $M \models P \equiv F_1$ and $M \models Q \equiv F_2$. Applying frame inference is necessary because $M$ may mention resources present in $P$, but not $Q$ (or vice versa). The set of solutions returned by Algorithm 2 is semantically valid according to the following correctness result.

Theorem 5.1 (Tri-abduction). If $(M, F_1, F_2) \in$ triab($P, Q$), then $M \models P \equiv F_1$ and $M \models Q \equiv F_2$

We briefly sketch the proof here, whereas the full version is given in Appendix E. The first step is to establish the soundness of the proof system in Figure 3. That is, if $P \ll [M] \gg Q$ is derivable, then $M \models P$ and $M \models Q$. Given that Algorithm 1 operates according to the inference rules, it must be that any $M \in$ abduce-par($P, Q$) also implies $P$ and $Q$. The remainder of the proof follows from the soundness of the proof system in Berdine et al. [2005b, §5].

6 SYMBOLIC EXECUTION
With a sound frame rule and tri-abduction procedure in hand, we are now ready to design symbolic execution algorithms based on OSL. The core algorithm is similar to Abductor [Calcagno et al.
such that $C$

**Theorem 6.1 (Symbolic Execution Soundness).**

Then $\models C \triangleright \triangledown$.

If $\langle ok : P \rangle \in [C]^2 (T)$, then $\models \langle ok : P \rangle C \langle \phi \rangle$

The strategy for the analysis is to accumulate a set of outcomes while moving forward through the program. At each step, every outcome in the current summary must be sequenced with a summary for the next command using bi-abduction. This is achieved using the seq procedure, defined in Figure 4, which takes in an outcome assertion $\phi$, a set of summaries for the next command $C$, and $\bar{x}$, the variables modified by $C$. It computes a set of missing anti-frames $M$ and postconditions $\psi$ such that $\langle \phi \triangleright M \rangle C \langle \psi \rangle$ is a valid specification for $C$.

Sequencing after $T$, $T^\oplus$ and (er : $Q$) has no effect, since $T$ carries no information about the current branch, and in the other two cases the program has diverged or crashed. Sequential composition is implemented in the (ok : $P$) case, where bi-abduction is used to reconcile the current outcome with each summary for the next command. The biab' procedure is similar to AbduceAndAdapt from Calcagno et al. [2011, Fig. 4], in which a renaming step is applied to ensure that the anti-frame $M$ is not phrased in terms of any program variables, therefore meeting the side condition of the frame rule. The details of renaming and why it is required for correctness are explained in Appendix F.

Our algorithm differs from Abductor in the cases with multiple program branches. This is where we use tri-abduction to obtain a precondition that is guaranteed to be valid for all program paths, allowing us to analyze the program in a single pass (unlike Abductor, which must re-evaluate the program using each candidate precondition). After sequencing each of the outcomes in the precondition $\phi_1$ and $\phi_2$ with the next command, we use triab' to obtain the single renamed anti-frame $M$ that is safe for both branches. The soundness property for seq is stated below.

**Lemma 6.2 (Seq).** If $\models \langle ok : P \rangle C \langle \emptyset \rangle$ for all $(P, \emptyset) \in S$ and $\bar{x} = \text{mod}(C)$ and $(M, \psi) \in \text{seq}(\phi, S, \bar{x})$, then $\models \langle \phi \triangleright M \rangle C \langle \psi \rangle$
symbolic execution algorithm. The core symbolic execution algorithm, shown in Figure 5, computes a local symbolic execution which can be augmented using the frame rule to obtain summaries in larger heaps. For example, the semantics for skip is simply the triple \( \langle \text{ok} : \text{emp} \rangle \) skip \( \langle \text{ok} : \text{emp} \rangle \), but this implies that running the program in any heap will yield that same heap in the end.

Executing \( C_1 \triangleright C_2 \) is implemented using seq; we produce the summaries for \( C_1 \) and then sequence them with all the summaries for \( C_2 \). Two summaries are produced for if statements, one where the true branch is taken and one where the false branch is taken. This is similar to the behavior of Abductor and allows us to produce precise specifications for the program behavior without knowing what logical conditions will occur a priori. Similarly, while loops use a least fixed point to unroll the loop, and produce one summary for each possible number of iterations. We will see more options for analyzing loops later on. We analyze choices \( C_1 +_a C_2 \) by computing summaries for each program path and reconciling them with tri-abduction.

The abstract semantics of primitive instructions mostly follow the small axioms of O’Hearn et al. [2001], with failure cases inspired by Incorrectness Separation Logic [Raad et al. 2020]. Each memory operation has three specifications: one in which the pointer is allocated and the operation accordingly succeeds, and two failure cases where the pointer is not allocated or null. Procedure calls rely on pre-computed summaries in a lookup table \( T \), which is a parameter to \( [C]^\# \).

In the remainder of this section, we describe further modifications that can be made to the algorithm in order to improve its performance and ability to analyze various types of programs.

The single-path algorithm

We now present a variant of the previous algorithm that only traverses a single path through the program at a time, and is therefore suitable only for bug-finding. This algorithm is inspired by Pulse [Raad et al. 2020] and Pulse-X [Le et al. 2022], but is based on OSL rather than Incorrectness Logic. We obtain this new algorithm by simply altering the abstract semantics of program choices. Rather than producing a single summary with two outcomes, this version produces two summaries in which the second outcome is replaced by \( T \).

\[
[C_1 +_a C_2]^\# (T) = \{ (P, \varphi \oplus_a T) \mid (P, \varphi) \in [C_1]^\# (T) \} \cup \{ (P, \varphi \oplus_T T) \mid (P, \varphi) \in [C_2]^\# (T) \}
\]
Given this modification, the algorithm remains sound with respect to the same semantics (i.e., Theorem 6.1), but it no longer fits with the spirit of correctness reasoning, since some of the program outcomes are left unspecified. We will see in Section 7.1 how it can be used for efficient bug-finding.

**Bounded unrolling and dropping paths.** As recounted by O’Hearn [2019]; Raad et al. [2020]; Le et al. [2022], the scalability benefits of IL-based analyses comes from their ability to drop disjuncts. Bi-abduction analyses such as Abductor accumulate a disjunction of symbolic heaps, representing the possible end states at each program point. When searching for bugs, it is not necessary to remember all of these possible states; it suffices to only remember the one that represents a bug. The semantics of IL allows strengthening of postconditions, so disjuncts can be soundly dropped.

We take a slightly different view, which nonetheless enables us to drop paths in the same way. We differentiate between program choices that result from logical conditions (i.e., if and while statements) vs computational effects (i.e., nondeterministic or probabilistic choice). In the former cases, we generate multiple summaries in order to precisely keep track of which initial states will result in which outcomes. In the latter case, we use an outcome conjunction rather than a disjunction to join the outcomes. While we cannot drop outcomes per se, we can replace them by ⊤, ensuring that they will not be explored any further according to the definition of seq (Figure 4).

Given our single path algorithm, in which we split each choice (logical and otherwise) into a separate summary, we can drop paths simply by limiting the size of the set $J_C^\#(T)$. For example, while the least fixed point semantics for while loops is clearly uncomputable—it is a set of (possibly) infinite size—we can compute the first $n$ unrollings for some parameter $n$. We can also choose a fixed size for the set $J_C^\#(T)$, after which point we stop generating more summaries. Le et al. [2022] refer to this as depth and width of the analysis, respectively. Since each element of $J_C^\#(T)$ stands alone as a sound summary for the program $C$, then eliminating elements from the set will only preclude possible summaries without affecting the correctness of the existing ones.

Even for correctness analyses, the bounded unrolling solution is reasonable and is used in other symbolic execution systems [Fragoso Santos et al. 2020; Holtzen et al. 2020].

**Loop invariants and partial correctness.** An alternative to bounded unrolling for analyzing loops in correctness applications is to use loop invariants. Loop invariants are not suitable for bug finding, since they only guarantee partial correctness—the postcondition holds if the program terminates, but it may diverge. We can alter the rule for while loops to the following.

$$\{while\ e\ do\ C\}^\#(T) = \{(I, (ok : I \land \neg e) \lor \top^\circ) \mid (I \land e, ok : I) \in C^\#(T)\}$$

The truth of the invariant $I$ is preserved by the loop body, therefore it must remain true if the loop exits. The possibility of nontermination is expressed by the disjunction with $\top^\circ$.

Finding loop invariants is generally undecidable, however techniques from abstract interpretation can be used to find invariants by framing the problem as a fixed point computation over a finite domain, thereby guaranteeing convergence. This is the approach taken in Abductor [Calcagno et al. 2011], which uses the same symbolic heaps as our own, but without outcome conjunctions. In the nondeterministic case, we can convert the outcome conjunctions into disjunctions since $(ok : P) \oplus (ok : Q) \Rightarrow (ok : P \lor Q)$ and therefore we can use the same technique.

The probabilistic case is much more complicated since we need to address the question of almost sure termination. Automated techniques using ranking super-martingales exist [Agrawal et al. 2017], but an exploration of that approach is out of scope for this paper.

**Nondeterministic Allocation.** Many memory bugs in C arise from failing to check whether the address returned by malloc is non-null. This is often modeled using nondeterminism, wherein the semantics of malloc returns either a valid pointer or null, nondeterministically. Since our language is generic over the execution model, we do not have a nondeterministic malloc operation, but
rather only a deterministic alloc operation which is always guaranteed to succeed. We can add $x := \text{malloc}()$ as syntactic sugar for $(x := \text{alloc}()) + (x := \text{null})$, and derive the following semantics:

$$[[x := \text{malloc}()]]^\#(T) = \{(x = X \land \text{emp}, (ok : x = \text{null} \land \text{emp}) \oplus (ok : \exists Y. x \leftrightarrow Y))\}$$

**Reusing Summaries.** Though partial correctness specifications are incompatible with bug-finding, and under-approximate specifications are incompatible with verification, there is still overlap in summaries that can be used for both correctness and incorrectness. Many procedures in a given codebase will not include loops or effects, so their summaries are equally valid for both correctness and incorrectness, and also for use in programs with different interpretations of choice.

In other cases, where a procedure does have multiple outcomes, it is relatively easy to convert a correctness specification into several individual incorrectness ones, since the following implication is sound. We will see this in action in Section 7.1.

$$\langle ok : P \rangle C \langle \psi_1 \oplus_a \psi_2 \rangle \implies \langle ok : P \rangle C \langle \psi_1 \oplus_a T \rangle$$

### 7 CASE STUDIES

We will now demonstrate how the symbolic execution algorithms work by examining two case studies, which show the applicability in both nondeterministic and probabilistic execution models.

#### 7.1 Nondeterministic Vector Reallocation

Our first case study involves a common error in C++ when using the `std::vector` library in which a call to `push_back` may reallocate the vector’s underlying memory buffer, invalidating any pointers to that code that existed before the call. This was also used as a motivating example for Incorrectness Separation Logic [Raad et al. 2020], following their lead, we model the vector as a single pointer and we treat reallocation as nondeterministic. The program is shown below.

```plaintext
main() :
  x ← [v];
push_back(v);
[x] ← 1
```

```plaintext
push_back(v) :
  (y ← [v];
   free(y);
   y := alloc();
   [v] ← y) + skip
```

Before we can analyze the main procedure, we must store $[[\text{push\_back}(v)]]^\#(T)$ in the procedure table. Since `push_back` is a common library function, it makes sense to compute summaries that describe all the outcomes so that we may reuse these summaries for both correctness and incorrectness analyses. The first step in doing so is two compute summaries for the two nondeterministic branches, which are both simple sequential programs.

$$[[y ← [v];
   \text{free}(y);
   y := \text{alloc}();
   [v] ← y]]^\#(T) = \{(v \mapsto A * A \mapsto \_ , ok : \exists B,v \mapsto B * B \mapsto \_ \ast A \psi_1)
   (v \mapsto A * A \psi_2 , er : v \mapsto A * A \psi_2 )
   \ldots\}$$

Now, we can compose the two program branches using tri-abduction. Choosing the first summary for the first branch, we get the following tri-abduction solution.

$$v \mapsto A * A \mapsto \_ \land [v \mapsto A * A \mapsto \_] \lor \text{emp} * [v \mapsto A * A \mapsto \_]$$

So, by framing emp into the first branch and $v \mapsto A * A \mapsto \_ \–$ into the second branch, we get a summary for `push_back` as a whole. This can similarly be done for the other summaries of the first branch, yielding the lookup table below.

$$T = \{(ok : v \mapsto A * A \mapsto \_) push_back(v) \langle (ok : \exists B,v \mapsto B * B \mapsto \_ \ast A \psi_1 ) \oplus (ok : v \mapsto A * A \mapsto \_) \rangle \}
   \{(ok : v \mapsto A * A \psi_2 ) push_back(v) \langle (er : v \mapsto A * A \psi_2 ) \oplus (ok : v \mapsto A * A \psi_2 ) \rangle \}
   \{(ok : v \psi_2 ) push_back(v) \langle (er : v \psi_2 ) \rangle \oplus (ok : v \psi_2 ) \rangle \}$$
The first summary tells us that push_back may reallocate the underlying buffer, in which case the original pointer \( A \) will become deallocated. The next two summaries describe ways in which push_back itself can fail. We will focus on using the first summary to show how main will fail if the buffer gets reallocated. We analyze main in an under-approximate fashion in order to look for bugs. The first step is to compute summaries for the first two commands of main. The load on the first line has three summaries according to Figure 5, we select the first one in which \( v \) is allocated.

\[
\langle \text{ok} : x = X \land v \mapsto Y \rangle \ x \leftarrow [v] \ \langle \text{ok} : x = Y \land v \mapsto Y \rangle \in [x \leftarrow [v]]^\#(T)
\]

The procedure call on the second line requires us to look up summaries in \( T \). We select the first one, but we will use an under-approximate version of it so as only to explore one of the paths

\[
\langle \text{ok} : v \mapsto A \ast A \mapsto - \rangle \ \text{push_back}(v) \ \langle (\text{ok} : \exists B.v \mapsto B \ast B \mapsto - \ast A \not\mapsto) \oplus T \rangle
\]

Now, we use seq to sequentially compose these summaries, which involves bi-abducing the post-condition of \( x \leftarrow [v] \) with the precondition of push_back(\( v \)).

\[
x = Y \land v \mapsto Y \ast [A = Y \ast x \mapsto -] \tau v \mapsto A \ast A \mapsto - \ast [\text{emp}]
\]

So, after renaming, we get the following summary for the composed program:

\[
\langle \text{ok} : v \mapsto x \ast x \mapsto - \rangle \ x \leftarrow [v] \ \text{push_back}(v) \ \langle (\text{ok} : \exists B.v \mapsto B \ast B \mapsto - \ast x \not\mapsto) \oplus T \rangle
\]

Now, observe that the postcondition above is only compatible with one of the summaries in Figure 5 for the last line of the program. Since \( x \) is deallocated in the only specified outcome, the write into \( x \) must fail. Using bi-abduction again, we can construct the following description of the error.

\[
\langle \text{ok} : v \mapsto x \ast x \mapsto - \rangle \ x \leftarrow [v] \ \text{push_back}(v) \ \exists [x] \leftarrow 1 \ ((\text{er} : \exists B.v \mapsto B \ast B \mapsto - \ast x \not\mapsto) \oplus T)
\]

### 7.2 Consensus in Distributed Computing

The microservice architecture—in which many lightweight components communicate via fixed APIs—is becoming increasingly popular in software engineering. While microservices add flexibility and scale well, the fact that they communicate over a network introduces the possibility of failures at many points. Each microservice will typically publish a Service Level Agreement (SLA); a contract with the downstream users conveying, for example, what percentages of service calls will succeed.

In this case study, we will show how OSL can be used to lower bound reliability rates of microservices. We use a basic consensus algorithm, shown below, in which each of three processes broadcasts a value \( v_i \) by storing it in a pointer \( p_i \) and consensus is reached if any two of these processes broadcasted the same value. To model unreliability of the communication channels, the broadcast procedure fails with probability 1%. We would therefore like to know how likely we are to reach consensus, given that two of the processes agree.

```plaintext
main():
p_1 := alloc() \Ann \text{broadcast}(v_1, p_1)\Ann
p_2 := alloc() \Ann \text{broadcast}(v_2, p_2)\Ann
p_3 := alloc() \Ann \text{broadcast}(v_3, p_3)\Ann
v := alloc()\Ann
decide(p_1, p_2, p_3, v)

\text{broadcast}(v, p):
([p] ← v) \Ann \text{error()}
```

We begin by examining the summary table. The broadcast procedure has two outcomes corresponding to whether or not the communication went through. Though there are many summaries
for decide, we show only the one in which that values sent on $p_1$ and $p_2$ are equal.

$$T = \{\langle \text{ok} : p \mapsto \varnothing \cup v = V \rangle \text{broadcast}(v, p) \langle \langle \text{ok} : p \mapsto V \land v = V \rangle \oplus \theta_{0.99} \langle \text{er} : p \mapsto \varnothing \cup v = V \rangle \rangle\} \oplus \langle \text{ok} : p_1 \mapsto V_1 \mapsto p_2 \mapsto V_2 \mapsto p_3 \mapsto V_3 \mapsto v \mapsto \varnothing \cup v = V \rangle \text{decide}(p_1, p_2, p_3, v) \langle \text{ok} : v \mapsto V_1 \mapsto \cdots \rangle\} \oplus \cdots$$

We again use the single-path algorithm to analyze main, but this time we are interested only in the successfully terminating cases. We get the following summaries for each of the first three lines.

$$\langle \text{ok} : v_i = V_i \rangle p_i := \text{alloc()} \uplus \text{broadcast}(v_i, p_i) \langle \langle \text{ok} : v_i = V_i \land p_i \mapsto V_i \rangle \oplus \theta_{0.99} \uplus \theta_{99} \rangle$$

These three summaries can be combined—along with the simplification that $(\varphi \oplus_a \top) \oplus_b \top$ iff $\varphi \oplus_{a,b} \top$—to obtain the following assertion just before the call to decide

$$\langle \text{ok} : v_1 \land v_2 = V_2 \land v_3 = V_3 \land p_1 \mapsto V_1 \land p_2 \mapsto V_2 \land p_3 \mapsto V_3 \rangle \oplus \theta_{0.99} \uplus \theta_{99} \uplus \theta_{9703} \uplus \top$$

Now, we can bi-abduce the first outcome above with the precondition for for decide shown in $T$. This will send the logical condition $V_1 = V_2$ backwards into the precondition, and we will get an overall summary for main telling us that if we run the protocol in a state where $v_1 = v_2$, then we will reach consensus with probability at least 97%.

$$\langle \text{ok} : v_1 = V_1 \land v_2 = V_2 \land v_3 = V_3 \land V_1 = V_2 \rangle \text{main()} \langle \langle \text{ok} : v \mapsto V_1 \mapsto \cdots \rangle \oplus \theta_{0.9703} \rangle$$

8 RELATED WORK

**Pulse and Incorrectness Separation Logic.** As recounted by Raad et al. [2020, §5], Pulse uses under-approximation in four ways in order to achieve scalability:

1. Pulse takes advantage of the IL semantics in order to explore only one path when the program execution branches, and to unroll loops for a bounded number of iterations.
2. Pulse elects to not consider cases in which memory is re-allocated.\(^6\)
3. Pulse uses under-approximate specifications for some library functions.
4. Pulse’s bi-abductive inference assumes that pointers are not aliased unless explicitly stated.

We have shown how (1) is achieved using OSL in the single path algorithm, (2) and (4) are standard assumptions of the bi-abduction procedure of Calcagno et al. [2009, 2011] (which we also use), and (3) is a corollary to (1), since the ability to drop paths opens the possibility for under-approximate procedure summaries. Hence, our single path algorithm can be seen as an alternative model for Pulse, proving that it is sound for bug-finding using the OSL semantics rather than IL.

Pulse does differ from our symbolic execution algorithms in some other ways. For example, Pulse does not support inductive predicates (e.g., list segments) and therefore it has a simplified bi-abduction procedure, which is capable of handling more types of pure assertions. Though the soundness of our frame rule is based on the symbolic heaps of Berdine et al. [2005b], it is possible to add more types of pure assertions than equalities and inequalities (as shown by Baktiev [2006]). This would need to be accompanied by a more powerful bi-abduction procedure, such as that of Brotherston et al. [2017]. It would be interesting whether the inductive predicates offered by OSL could help analyses like Pulse generate more concise summaries, further aiding in their scalability.

**Separation logic with effects.** While standard separation logic relies on nondeterminism for a sound frame rule [Yang and O’Hearn 2002], local reasoning has been extended into other settings. Baktiev [2006] proved that the frame rule is sound in a deterministic language if heap assertions are unaffected by permutation of addresses. Similarly, Tatsuta et al. [2009] created a deterministic separation logic, but the frame rule only applies to programs that do not allocate memory.

---

\(^6\)Re-allocation makes the program semantics non-local, since executions in a larger heap (with more deallocated pointers) add possible end states. Interestingly, re-allocation is local for under-approximate specifications, including in OSL. It would be an interesting to explore an OSL frame rule that is only sound with respect to specifications of the form $\langle \varphi \rangle C (\langle \psi \rangle \oplus \top)$. 

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In addition, there are separation logic variants that combine probabilistic computation and nondeterminism in order to recover a sound frame rule. Tassarotti and Harper [2019] introduced Polaris, a probabilistic variant of concurrent separation logic [O’Hearn 2004], and implemented it in Iris [Jung et al. 2015]. Batz et al. [2019] also created Quantitative Separation Logic, which uses weakest pre-expectation [Morgan et al. 1996] style predicate transformers to derive expected values in probabilistic pointer programs. Our approach differs from these two developments in two ways. First, by opting for a basic execution model with only probabilistic choice (and without nondeterminism), we avoid a significant amount of complexity in the underlying semantics [Jones 1989; Varacca and Winskel 2006]. Second, our approach of relating probabilistic choices with program outcomes is particularly amenable to bi-abductive symbolic execution and generating re-usable procedure summaries, which was not a goal of the aforementioned developments.

Probabilistic Separation Logic (PSL) [Barthe et al. 2019] and subsequent works [Bao et al. 2021, 2022; Li et al. 2023] use an alternative model of separation to characterize probabilistic independence and related probability theoretic properties. Doing so provides a compositional way to reason about probabilistic programs, though this work is orthogonal to our own as it does not deal with heaps.

**Algebraic Program Semantics**

Our algebraic definition of program semantics has similarities to Weighted Programming [Batz et al. 2022], however the goals of our development are different. Whereas we wished to use an algebraic interpretation of choice in order to represent multiple types of (executable) program semantics, the goal of weighted programming is to specify mathematical models and find solutions to optimization problems via static analysis.

Outcome Algebras are reminiscent of Effect Algebras [Foulis and Bennett 1994], which are used to reason about quantum programs. While there are subtle differences between these two definitions, it would be interesting to see if Outcome Algebras are able to capture quantum computation too.

**Unified approaches to correctness and incorrectness.** Maksimović et al. [2022] recently introduced Exact Separation Logic (ESL), which combines the semantics of SL and ISL in order to derive specifications for both correctness and incorrectness within a single program logic and is implemented inside the Gillian symbolic execution engine [Fragoso Santos et al. 2020]. They additionally show how inductive predicates can be compatible with the IL semantics, which is something that has not been demonstrated by Pulse or Pulse-X. Similarly, Bruni et al. [2021, 2023] introduced Local Completeness Logic (LCL), which is based on the semantics of IL, but uses an over-approximate abstract domain to ensure that the under-approximation is never too far away from the strongest post so as to preclude recovering a correctness spec too.

Though the goals of these two developments are similar to our own, we take a different approach; in keeping with the tradition of O’Hearn [2019]; Raad et al. [2020]; Le et al. [2022], we opt to design separate algorithms for correctness and incorrectness, recognizing that fundamental properties of bug-finding allow us trade off a complete view of all program outcomes for increased efficiency. Still, we are able to provide a unification of the metatheory and share summaries for some procedures. Crucially, OSL permits dropping paths just like IL, which is not possible in either ESL or LCL.

## 9 CONCLUSION

Infer—built on separation logic and bi-abduction—is capable of analyzing industrial scale codebases, substantiating the idea that compositionality translates to real-world scalability [Calcagno et al. 2015]. But the deployment of Infer also surfaced that proving the absence of bugs is somewhat of a red herring—software has bugs and sound logical theories are needed to find them [Le et al. 2022].

Incorrectness Logic has shown that it is not only possible to formulate a theory for bug-finding, but it is in fact advantageous from a program analysis view; static analyzers can take certain liberties in searching for bugs that are not valid for correctness verification, such as dropping
program paths for added efficiency. The downside is that the IL semantics is incompatible with correctness analysis, therefore separate implementations and procedure summaries must be used.

With our introduction of Outcome Separation Logic, we seek to get the best of both worlds. As Raad et al. [2020, §6] put it, “aiming for under-approximate results rather than exact ones gives additional flexibility to the analysis designer, just as aiming for over-approximate rather than exact results does for correctness tools.” The fact that OSL supports over-approximation in the traditional sense as well as under-approximation in the sense of Pulse invites the reuse of tools between the two, while still enabling specialized techniques when needed (i.e., loop invariants for correctness, dropping paths for incorrectness). In addition, OSL can be used to reason about deterministic and probabilistic programs whereas separation logic and IL cannot.

Designing OSL was not straightforward; it is not a simple extension of separation logic, but is rather designed from the ground up with new assumptions since the properties that make the standard frame rule sound (nondeterministic allocation, must properties, and safe preconditions) are not suitable for reasoning about incorrectness and effects. The addition of tri-abduction to our symbolic execution algorithms also means that we can analyze more programs with control flow branching compared to Abductor [Calcagno et al. 2009].

There are many opportunities for further developments. On the theoretical side, we plan to augment OSL to support pointer arithmetic based on Array Separation Logic [Brotherston et al. 2017], concurrency (by fusing OSL with Iris [Jung et al. 2015]), and quantum computation. On the applied side, we plan to implement our algorithms in Infer to show the ease with which common infrastructure can be used for both correctness and incorrectness and also provide a testbed for experimenting with heuristics to determine when an analysis should abandon a verification attempt and instead switch to single-path mode to find bugs. Relatedly, we would like to explore connections with abstract interpretation—whereas Ascari et al. [2022] showed that designing abstract domains for IL is infeasible, perhaps OSL is more amenable to abstraction. The power and flexibility of OSL makes it the ideal logical foundation to study these questions.

REFERENCES


We've shown that ⟨All of the following proofs assume any outcome algebra
For any \(a, b\) is defined. So, \(a + b\) = \(a' + b'\) = \(a'' + b''\).
Now, we have:
\[a + b = (a' + a'') + (b' + b'') = (a' + b') + a'' + b''\]
So, \(a' + b'\) must be defined.

LEMMA A.2. For any \(a, b \in A\), \(a \cdot b \leq a\).

PROOF. Since \(sup(A) = \mathbb{1}\), then there must be \(b\) such that \(b + b' = \mathbb{1}\). So, we have:
\[a = a \cdot b = a \cdot (b + b') = ab + ab'\]
We've shown that \(ab + ab' = a\), so by definition, \(ab \leq a\).

COROLLARY A.3. For any \(a, a', b, b' \in A\), if \(a + b\) is defined, then \(aa' + bb'\) is defined.

PROOF. By Lemma A.2, \(aa' \leq a\) and \(bb' \leq b\). By Lemma A.1, \(aa' + bb'\) must be defined since \(a + b\) is defined.

LEMMA A.4. If \(\sum_{i=1}^{n} a_i\) is defined, then there exist \(b_1, \ldots, b_n\) such that \(a_i = (\sum_{j=1}^{i} a_j) \cdot b_i\) for \(1 \leq i \leq n\) and \(\sum_{i=1}^{n} b_i = \mathbb{1}\).
Proof. The proof is by induction on $n$. Suppose $n = 1$, then let $b_1 = 1$, so clearly $\sum_{i=1}^n a_i \cdot b_i = a_1 \cdot 1 = a_1$ and $\sum_{i=1}^n b_i = 1$. Now, suppose that the claim holds for $n$, that is, there exist $b_1, \ldots, b_n$ such that $a_i = (\sum_{j=1}^n a_j) \cdot b_i$ for $1 \leq i \leq n$ and $\sum_{i=1}^n b_i = 1$. We will show that the claim also holds for $n + 1$. First, let $u, v$ be such that:

$$\sum_{i=1}^n a_i = \left( \sum_{i=1}^{n+1} a_i \right) \cdot u \quad \text{and} \quad a_{n+1} = \left( \sum_{i=1}^{n+1} a_i \right) \cdot v \quad \text{and} \quad u + v = 1$$

Now, let $b'_i = u \cdot b_i$ for $1 \leq i \leq n$ and $b'_{n+1} = v$. We know that for each $1 \leq i \leq n$:

$$a_i = \left( \sum_{j=1}^n a_j \right) \cdot b_i = \left( \sum_{j=1}^{n+1} a_j \right) \cdot u \cdot b_i = \left( \sum_{j=1}^{n+1} a_j \right) \cdot b'_i$$

And $a_{n+1} = (\sum_{j=1}^{n+1} a_j) \cdot b'_{n+1}$ by definition. To complete the proof, we show that:

$$\sum_{i=1}^{n+1} b'_i = \sum_{i=1}^n u \cdot b_i + b'_{n+1} = u \cdot \sum_{i=1}^n b_i + v = u \cdot 1 + v = 1$$

\[ \square \]

Lemma A.5. For all $a, b, u, v \in A$, there exists $w \in A$ such that $\sup(a \cdot u, b \cdot v) = \sup(a, b) \cdot w$.

Proof. First, note that $\sup(a \cdot u, b \cdot v) \leq \sup(a, b)$ as a corollary of Lemma A.2. So, by definition of $\leq$, $\sup(a, b) = \sup(a \cdot u, b \cdot v) + c$ for some $c$. Now, by the properties of outcome algebras, there must be a $w$ such that:

$$\sup(a \cdot u, b \cdot v) = (\sup(a \cdot u, b \cdot v) + c) \cdot w = \sup(a, b) \cdot w$$

\[ \square \]

B. PROGRAM SEMANTICS

Definition B.1 (Scott Continuity). Consider a semiring $\langle A, +, \cdot, 0, 1 \rangle$ with partial order $\leq$. A function (or partial function) $f: A \to A$ is Scott continuous if for any directed set $D \subseteq A$ (where all pairs of elements in $D$ have a supremum), $\sup_{a \in D} f(a) = f(\sup D)$. The semiring is Scott continuous if $+$ and $\cdot$ are Scott continuous in both arguments [Karner 2004].

Lemma B.2 (Totality of Bind). The bind function defined in Definition 3.8 is a total function (this is not immediate, since it uses partial addition).

Proof. First, we note that $\sum_{a \in \text{supp}(m)} m(a)$ must be defined by the definition of $W$. Now, by the semiring laws, we get:

$$\sum_{a \in \text{supp}(m)} m(a) = \sum_{a \in \text{supp}(m)} m(a) \cdot 1$$

Since $\sup(A) = 1$ and since $f(a)(b) \in A$, then there must be some value $u_{a,b}$ such that $f(a)(b) + u_{a,b} = 1$. Then:

$$= \sum_{a \in \text{supp}(m)} m(a) \cdot (f(a)(b) + u_{a,b})$$
And by the semiring laws:

\[
\begin{align*}
&= \sum_{a \in \text{supp}(m)} m(a) \cdot f(a)(b) + m(a) \cdot u_{a,b} \\
&= \sum_{a \in \text{supp}(m)} m(a) \cdot f(a)(b) + \sum_{a \in \text{supp}(m)} m(a) \cdot u_{a,b}
\end{align*}
\]

So, clearly \( \sum_{a \in \text{supp}(m)} m(a) \cdot f(a)(b) \) must be defined, and therefore \( \text{bind}(m, f) = \sum_{a \in \text{supp}(m)} m(a) \cdot f(a)(b) \) is defined. \( \square \)

**Theorem B.3 (Fixed Point Existence).** The function \( F_{(C,e)} \) defined above has a least fixed point.

**Proof.** It will suffice to show that \( F_{(C,e)} \) is Scott continuous, at which point, we can apply the Kleene fixed point theorem to conclude that the least fixed point exists. First, we define the pointwise order \( f_1 \sqsubseteq f_2 \) iff \( f_1(s, h) \sqsubseteq f_2(s, h) \) for all \( (s, h) \), where \( f_1(s, h) \sqsubseteq f_2(s, h) \) iff there exists \( m \) such that \( f_1(s, h) + m = f_2(s, h) \). Now, we will show that the monad bind is Scott continuous with respect to that order. Let \( D \) be a directed set.

\[
\sup_{f \in D} \text{bind}(m, f) = \sup_{f \in D} \sum_{s \in \text{supp}(m)} m(s) \cdot \begin{cases} f(a) & \text{if } s = 1_R(a) \\ \text{unit}_M(s) & \text{if } s = 1_L(-) \end{cases}
\]

Now, by continuity of the semiring, suprema distribute over sums and products.

\[
\begin{align*}
&= \sum_{s \in \text{supp}(m)} m(s) \cdot \begin{cases} \sup_{f \in D} f(a) & \text{if } s = 1_R(a) \\ \text{unit}_M(s) & \text{if } s = 1_L(-) \end{cases} \\
&= \sum_{s \in \text{supp}(m)} m(s) \cdot \begin{cases} \text{(sup } D\text{)}(a) & \text{if } s = 1_R(a) \\ \text{unit}_M(s) & \text{if } s = 1_L(-) \end{cases} \\
&= \text{bind}(m, \text{sup } D)
\end{align*}
\]

Finally, we show that \( F_{(C,e)} \) is Scott continuous with respect to the order defined above.

\[
\sup_{f \in D} F_{(C,e)}(f) = \lambda(s, h). \sup_{f \in D} F_{(C,e)}(f)(s, h)
\]

\[
\begin{align*}
&= \lambda(s, h). \sup_{f \in D} \begin{cases} \text{bind}(\llbracket C \rrbracket (s, h), f) & \text{if } \llbracket e \rrbracket (s) = \text{true} \\
\text{unit}(s, h) & \text{if } \llbracket e \rrbracket (s) = \text{false} \end{cases} \\
&= \lambda(s, h). \sup_{f \in D} \begin{cases} \text{bind}(\llbracket C \rrbracket (s, h), f) & \text{if } \llbracket e \rrbracket (s) = \text{true} \\
\text{sup } \text{unit}(s, h) & \text{if } \llbracket e \rrbracket (s) = \text{false} \end{cases} \\
&= \lambda(s, h). \begin{cases} \text{bind}(\llbracket C \rrbracket (s, h), \text{sup } D) & \text{if } \llbracket e \rrbracket (s) = \text{true} \\
\text{unit}(s, h) & \text{if } \llbracket e \rrbracket (s) = \text{false} \end{cases} \\
&= F_{(C,e)}(\text{sup } D)
\end{align*}
\]

**Theorem B.4 (Totality of Program Semantics).** If all expressions used in guards are Boolean valued, then the semantics \( \llbracket C \rrbracket \) is a total function.

**Proof.** By induction on the structure of \( C \). The cases for skip and all atoms are trivial.

\( \triangleright C = C_1 ; C_2 \). By the induction hypothesis, we assume \( \llbracket C_1 \rrbracket \) and \( \llbracket C_2 \rrbracket \) are total, and by Lemma B.2 we know that bind is total, therefore \( \text{bind}(\llbracket C_1 \rrbracket (s, h), \llbracket C_2 \rrbracket) \) is total.
C PROPERTIES OF REWEIGHTING FUNCTIONS

Note that in this section, all uses of the monad operations bind and unit are assumed to be the ones from Definition 3.8.

Lemma C.1. Reweighting functions respect the monoid operations and scalar multiplication:

1. \( w(0, R) = \{0\} \)
2. \( w(m_1 + m_2, R) = \{m'_1 + m'_2 \mid m'_1 \in w(m_1, R), m'_2 \in w(m_2, R)\} \)
3. \( w(a \cdot m, R) = \{a \cdot m' \mid m' \in w(m, R)\} \)

Proof. For the monoid unit:

\[
w(\emptyset, R) = \{\text{bind}(0, f) \mid f \in \text{wf}(\emptyset, R)\} = \{0 \mid f \in \text{wf}(\emptyset, R)\} = \{0\}
\]

For the monoid composition:

\[
w(m_1 + m_2, R) = \{\text{bind}(m_1 + m_2, f) \mid f \in \text{wf}(m_1 + m_2, R)\}
= \{\text{bind}(m_1, f) + \text{bind}(m_2, f) \mid f \in \text{wf}(m_1 + m_2, R)\}
\]

Now, suppose \( m \in \{\text{bind}(m_1, f) + \text{bind}(m_2, f) \mid f \in \text{wf}(m_1 + m_2, R)\} \), so there is some \( f \) such that \( m = \text{bind}(m_1, f) + \text{bind}(m_2, f) \). Clearly, \( \text{bind}(m_1, f) \in \text{wf}(m_1, R) \) and \( \text{bind}(m_2, f) \in \text{wf}(m_2, R) \), so \( m \in \{m'_1 + m'_2 \mid m'_1 \in \text{wf}(m_1, R), m'_2 \in \text{wf}(m_2, R)\} \), and therefore \( \text{wf}(m_1 + m_2, R) \subseteq \{m'_1 + m'_2 \mid m'_1 \in \text{wf}(m_1, R), m'_2 \in \text{wf}(m_2, R)\} \).

Now suppose that \( m \in \{m'_1 + m'_2 \mid m'_1 \in \text{wf}(m_1, R), m'_2 \in \text{wf}(m_2, R)\} \), so there are reweighting functions \( f_1 \) and \( f_2 \) such that \( m = \text{bind}(m_1, f_1) + \text{bind}(m_2, f_2) \). Now, for all \( a \in A \), we know that there must be a \( u_1 \) and \( u_2 \) satisfying:

\[
m_1(a) = (m_1(a) + m_2(a)) \cdot u_1
\]
\[
m_2(a) = (m_1(a) + m_2(a)) \cdot u_2
\]

And also \( u_1 + u_2 = 1 \). Let \( f \) be defined as follows:

\[
f(a) = u_1 \cdot f_1(a) + u_2 \cdot f_2(a)
\]

Clearly \( f \) is a reweighting function since by construction \( f(a)(b) = \emptyset \) for all \( (a, b) \notin R \) and \(|f(a)| = u_1 \cdot |f_1(a)| + u_2 \cdot |f_2(a)| = u_1 \cdot 1 + u_2 \cdot 1 = u_1 + u_2 = 1 \). Now, we have:

\[
m_1(a) \cdot f_1(a) + m_2(a) \cdot f_2(a) = (m_1(a) + m_2(a)) \cdot u_1 \cdot f_1(a) + (m_1(a) + m_2(a)) \cdot u_2 \cdot f_2(a)
\]
\[
= (m_1(a) + m_2(a)) \cdot (u_1 \cdot f_1(a) + u_2 \cdot f_2(a))
\]
\[
= (m_1(a) + m_2(a)) \cdot f(a)
\]

□
So, that means that:

\[
\text{bind}(m_1, f_1) + \text{bind}(m_2, f_2) = \sum_{a \in \text{supp}(m_1)} m_1(a) \cdot f_1(a) + \sum_{a \in \text{supp}(m_2)} m_2(a) \cdot f_2(a)
\]

\[
= \sum_{a \in \text{supp}(m_1) \cup \text{supp}(m_2)} m_1(a) \cdot f_1(a) + m_2(a) \cdot f_1(a)
\]

\[
= \sum_{a \in \text{supp}(m_1) \cup \text{supp}(m_2)} (m_1(a) + m_2(a)) \cdot f(a)
\]

\[
= \text{bind}(m_1 + m_2, f)
\]

So, \( m \in w(m_1 + m_2, R) \) and therefore \( \{m_1' + m_2' \mid m_1' \in w(m_1, R), m_2' \in w(m_2, R)\} \subseteq w(m_1 + m_2, R) \), so \( w(m_1 + m_2, R) = \{m_1' + m_2' \mid m_1' \in w(m_1, R), m_2' \in w(m_2, R)\} \).

Now, for scalar multiplication:

\[
w(a \cdot m, R) = \{\text{bind}(a \cdot m, f) \mid f \in w(a \cdot m, R)\}
\]

\[
= \{a \cdot \text{bind}(m, f) \mid f \in w(a \cdot m, R)\}
\]

Note that \( \text{wf}(a \cdot m, R) = w(m, R) \) since \( \text{supp}(a \cdot m) = \text{supp}(m) \).

\[
= \{a \cdot m' \mid m' \in w(m, R)\}
\]

\[\square\]

**Lemma C.2.** If \((a, b) \in R\), then \( \text{unit}(b) \in w(\text{unit}(a), R) \).

**Proof.** First, let \( f \) be defined as follows:

\[
f(a') = \begin{cases} 
\text{unit}(b) & \text{if } a = a' \\
0 & \text{otherwise}
\end{cases}
\]

Clearly \( f \) is a reweighting function for \( \text{unit}(a) \) and \( R \). Now, we have the following:

\[
\text{unit}(b) = f(a) = \text{bind}(\text{unit}(a), f) \in w(\text{unit}(a), R)
\]

\[\square\]

**Lemma C.3.** Let \( R \subseteq A \times B \) and \( S \subseteq B \times C \).

\[
\bigcup_{m' \in w(m, R)} w(m', S) = w(m, S \circ R)
\]

**Proof.**

\((\Rightarrow)\) Suppose \( m'' \in \bigcup_{m' \in w(m, R)} w(m', S) \), so for some \( f \) that is a reweighting function for \( m \) and \( R \), \( m'' = \text{bind}(m, f) \) and for some \( g \) that is a reweighting function for \( m' \) and \( S \), \( m'' = \text{bind}(m', g) \). Now, we get:

\[
m'' = \text{bind}(m', g) = \text{bind}(\text{bind}(m, f), g) = \text{bind}(m, \lambda a.\text{bind}(f(a), g))
\]

To complete the proof, we just need to show that \( \lambda a.\text{bind}(f(a), g) \) is a reweighting function for \( S \circ R \). That, for all \( a \in \text{supp}(m) \), \( \sum_{c \in C} (a, c) \in S \circ R \) \( \text{bind}(f(a), g)(c) = 1 \) and \( \text{bind}(f(a), g)(c) = 0 \) if \( (a, c) \notin S \circ R \). We show this as follows:

\[
\sum_{c \in C} \text{bind}(f(a), g)(c) = \sum_{c \in C} \sum_{(a, c) \in S \circ R} f(a)(b) \cdot g(b)(c)
\]

If \( (a, b) \notin R \), then \( f(a)(b) = 0 \), so we can change the sum to be the following.

\[
= \sum_{c \in C} \sum_{(a, c) \in S \circ R} f(a)(b) \cdot g(b)(c)
\]
Unless \( b \) is such that \( (b, c) \in S \), then \( g(b)(c) = 0 \), so the only values of \( c \) that will make the expression nonzero are those related to \( b \).

\[
\sum_{b \in B | (a, b) \in R} \sum_{c \in C | (b, c) \in S} f(a)(b) \cdot g(b)(c) = \sum_{b \in B | (a, b) \in R} \sum_{c \in C | (b, c) \in S} f(a)(b) \cdot g(b)(c)
\]

Since \( f \) and \( g \) are reweighting functions, \( \sum_{c \in C | (b, c) \in S} g(b)(c) = 1 \) and \( \sum_{b \in B | (a, b) \in R} f(a)(b) = 1 \).

\[
= \sum_{b \in B | (a, b) \in R} f(a)(b) \cdot 1 = 1
\]

Now, suppose that \((a, c) \notin S \circ R\). That means that there is no \( b \) such that \((a, b) \in R\) and \((b, c) \in S\). So, for every \( b \) in the sum below, either \((a, b) \notin R\), in which case \( f(a)(b) = 0 \), or \((b, c) \notin S\), in which case \( g(b)(c) = 0 \).

\[
\text{bind}(f(a), g)(c) = \sum_{b \in \text{supp}(f(a))} f(a)(b) \cdot g(b)(c) = 0
\]

\((\Leftarrow)\) Suppose \( m' \in w(m, S \circ R)\). That means that there is some reweighting function \( h \) such that \( m' = \text{bind}(m, h)\). We will first show how to construct two reweighting functions \( f \) and \( g \) for \( R \) and \( S \) respectively, such that \( h(a) = \text{bind}(f(a), g) \). Fix an \( a \in A \). For every \( c \in \text{supp}(h(a)) \), there must be at least one \( b \) such that \((a, b) \in R\) and \((b, c) \in S\), since \((a, c) \notin S \circ R\). Now, we fix a relation \( S' \subseteq S \subseteq B \times C \) in which for every \( c \in \text{supp}(h(a)) \), there is exactly one \( b \) such that \((b, c) \in S'\) and \((a, b) \in R\) (but each \( b \) could be related to multiple \( cs \)). We define \( f \) as follows:

\[
f(a)(b) = \sum_{c \in C | (b, c) \in S'} h(a)(c)
\]

We now argue that \( f \) meets the criteria of a reweighting function. By definition, for any \( b \) such that \((b, c) \in S'\), then \((a, b) \in R\), and since those are the only points where the sum is nonempty, \( f(a)(b) = 0 \) when \((a, b) \notin R\). We also have:

\[
\sum_{b \in B | (a, b) \in R} f(a)(b) = \sum_{b \in B | (a, b) \in R} \sum_{c \in C | (b, c) \in S'} h(a)(c)
\]

Recall that \( S' \) is constructed such that each \( c \) is related to exactly one \( b \) such that \((a, b) \in R\), so the nested sum collapses into a single one.

\[
= \sum_{c \in C | (a, c) \in S \circ R} h(a)(c) = 1
\]

Now, since \( \sum_{c \in \text{supp}(h(a))} h(a)(c) \) is defined, then \( \sum_{c \in C | (b, c) \in S'} h(a)(c) \) is defined for each \( b \), so by Lemma A.4, we know that for each \( c \) there is a \( u_{b,c} \) such that \( h(a)(c) = (\sum_{c' \in C | (b, c') \in S'} h(a)(c')) \cdot u_{b,c} \) and \( \sum_{c \in C | (b, c) \in S'} u_{b,c} = 1 \). Now, we define \( g \) as follows, which is clearly a reweighting function by construction.

\[
g(b)(c) = \begin{cases} u_{b,c} & \text{if } (b, c) \in S' \\ 0 & \text{otherwise} \end{cases}
\]
Finally, we show that $\text{bind}(f(a), g) = h(a)$ as follows:

$$\text{bind}(f(a), g)(c) = \sum_{b \in \text{supp}(f(a))} f(a)(b) \cdot g(b)(c)$$

$$= \sum_{b \in \text{supp}(f(a))} \left( \sum_{c' \in C|(b,c') \in S'} h(a)(c') \right) \cdot \left\{ \begin{array}{ll} u_{b,c} & \text{if } (b,c) \in S' \\ \emptyset & \text{otherwise} \end{array} \right.$$ 

By design, for any given $c$, there is only one $b$ for which $(b,c) \in S'$, so we can simplify the expression to:

$$= \left( \sum_{c' \in C|(b,c') \in S'} h(a)(c') \right) \cdot u_{b,c}$$

$$= h(a)(c)$$

$\square$

**D OUTCOME SEPARATION LOGIC**

**Lemma D.1 (Normalization).** For any $m$, there exists $m'$ such that $|m'| = 1$ and $m = |m| \cdot m'$.

**Proof.** By Lemma A.4, there must be $(b_s)_{s \in \text{supp}(m)}$ such that $m(s) = (\sum_{t \in \text{supp}(m)} m(t)) \cdot b_s$ and $\sum_{s \in \text{supp}(m)} b_s = 1$. Now, let $m'$ be defined as follows:

$$m'(s) = \left\{ \begin{array}{ll} b_s & \text{if } s \in \text{supp}(m) \\ \emptyset & \text{if } s \notin \text{supp}(m) \end{array} \right.$$ 

So, clearly $|m'| = \sum_{s \in \text{supp}(m)} b_s = 1$. For every $s$, we also have $m(s) = (\sum_{t \in \text{supp}(m)} m(t)) \cdot b_s = |m| \cdot b_s = |m| \cdot m'(s)$, so $m = |m| \cdot m'$.

$\square$

**Lemma D.2 (Splitting).** If $|m'| \leq a \cdot |m_1| + \bar{a} \cdot |m_2|$, then there exist $m'_1$ and $m'_2$ such that $|m'_1| \leq |m_1|$ and $|m'_2| \leq |m_2|$ and $m' = a \cdot m'_1 + \bar{a} \cdot m'_2$.

**Proof.** Since $a \cdot |m_1| + \bar{a} \cdot |m_2| \geq |m'|$, then there must be some $b$ such that $a \cdot |m_1| + \bar{a} \cdot |m_2| = |m'| + b$, and therefore there is also an $a'$ such that:

$$|m'| = (|m'| + b) \cdot a' = (a \cdot |m_1| + \bar{a} \cdot |m_2|) \cdot a'$$

By Lemma D.1, we know there must be an $m''$ such that $|m''| = 1$ and $m' = |m'| \cdot m''$. Now, let $m'_1 = |m_1| \cdot a' \cdot m''$ and $m'_2 = |m_2| \cdot a' \cdot m''$. So, now we have:

$$m' = |m'| \cdot m''$$

$$= (a \cdot |m_1| + \bar{a} \cdot |m_2|) \cdot a' \cdot m''$$

$$= a \cdot |m_1| \cdot a' \cdot m'' + \bar{a} \cdot |m_2| \cdot a' \cdot m''$$

$$= a \cdot m'_1 + \bar{a} \cdot m'_2$$

And:

$$|m'_1| = ||m_1| \cdot a' \cdot m''|$$

$$= |m_1| \cdot a' \cdot |m''|$$

$$= |m_1| \cdot a \cdot 1$$

$$\leq |m_1|$$

By a symmetric argument, $|m'_2| \leq |m_2|$.

$\square$
D.1 The Outcome Separating Conjunction

**Lemma 4.3.** If \( m \models \varphi \), then for any \( m' \in w(m, \text{frame}(F)) \), \( m' \models \varphi \odot F \)

**Proof.** By induction on the structure of \( \varphi \).

- \( \varphi = \top \). Since \( \varphi \odot F = \top \), then clearly \( m' \models \varphi \odot F \).
- \( \varphi = \top^\circ \). This means that \( m = \emptyset \) and therefore \( m' = \emptyset \) as well. Since \( \varphi \odot F = \top^\circ \), then clearly \( m' \models \varphi \odot F \).
- \( \varphi = \varphi_1 \lor \varphi_2 \). We know \( m \models \varphi_1 \) or \( m \models \varphi_2 \). Without loss of generality, suppose that \( m \models \varphi_1 \).
- By the induction hypothesis, we know that \( m' \models \varphi_1 \odot F \). We can therefore weaken this to conclude that \( m' \models (\varphi_1 \lor \varphi_2) \odot F \). The case where \( m \models \varphi_2 \) is symmetrical.
- \( \varphi = \varphi_1 \oplus \varphi_2 \). We know that \( m_1 \models \varphi_1 \) and \( m_2 \models \varphi_2 \) for some \( m_1 \) and \( m_2 \) such that \( m = a \cdot m_1 + \bar{a} \cdot m_2 \). Now taking any \( m' \in w(m, \text{frame}(F)) \), we know that:

\[
\begin{align*}
w(m, \text{frame}(F)) &= w(a \cdot m_1 + \bar{a} \cdot m_2, \text{frame}(F)) \\
&= \{m'_1 + m'_2 \mid m'_1 \in w(a \cdot m_1, \text{frame}(F)), m'_2 \in w(\bar{a} \cdot m_2, \text{frame}(F))\} \\
&= \{a \cdot m'_1 + \bar{a} \cdot m'_2 \mid m'_1 \in w(m_1, \text{frame}(F)), m'_2 \in w(m_2, \text{frame}(F))\}
\end{align*}
\]

So there must be \( m'_1 \in w(m_1, \text{frame}(F)) \) and \( m'_2 \in w(m_2, \text{frame}(F)) \) such that \( m' = a \cdot m'_1 + \bar{a} \cdot m'_2 \). By the induction hypothesis, \( m'_1 \models \varphi_1 \odot F \) and \( m'_2 \models \varphi_2 \odot F \), so \( m' \models (\varphi_1 \oplus \varphi_2) \odot F \).

- \( \varphi = \epsilon : P \). We know that \( |m| = 1 \) and every \( \sigma \in \text{supp}(m) \) has the form \( \langle s, h \rangle \) such that \( (s, h) \models P \). Now, take \( m' \in w(m, \text{frame}(F)) \), so \( m' \models \text{bind}(m, f) \) for some \( f \) that is a reweighting function for \( m \) and \( \text{frame}(F) \). By the first property of reweighting functions, \( f \) cannot change the mass of \( m \), so \( |m'| = 1 \). Additionally, for every element in \( \text{supp}(m') \), there must be an element in \( \text{supp}(m) \) related by \( \text{frame}(F) \), so each element of \( m' \) has the form \( \langle \epsilon, s \rangle \) such that \( (s, \epsilon) \models P \) and \( (s, s) \models F \), and so clearly \( (s, h \uplus h') \models P \models F \), and therefore also \( m' \models (\epsilon : P) \odot F \).

\[\square\]

**Lemma 4.4.** If \( m \models \varphi \odot F \), then there exist \( m_1, m_2, \) and \( m'_1 \in w(m_1, \text{frame}(F)) \) such that \( m = m'_1 + m_2 \) and \( m_1 + m'_2 \models \varphi \) for any \( m'_2 \) such that \( |m'_2| \leq |m_2| \).

**Proof.** By induction on the structure of \( \varphi \).

- \( \varphi = \top \). Suppose \( m \models \top \odot F \). Let \( m_1 = m'_1 = 0 \) and \( m_2 = m \). Clearly \( 0 \in w(\emptyset, \text{frame}(F)) \) and \( m_1 + m_2 = 0 + m = m \). Now take any \( m'_2 \) such that \( |m'_2| \leq |m_2| = |m| \). Clearly \( m'_2 \models \top \), since everything satisfies \( \top \).
- \( \varphi = \top^\circ \). Suppose \( m \models \top^\circ \odot F \), so \( m = 0 \). Let \( m_1 = m'_1 = m_2 = 0 \). Clearly \( 0 \in w(\emptyset, \text{frame}(F)) \) and \( m_1 + m_2 = 0 + 0 = m \). Now take any \( m'_2 \) such that \( |m'_2| \leq |m_2| = |m| \). We know that \( |m'_2| = 0 \) by anti-symmetry of \( \leq \), so \( \sum_{\sigma \in \text{supp}(m'_2)} m'_2(\sigma) = 0 \). Now, take any \( \sigma \in \text{supp}(m'_2) \), we know that \( m'_2(\sigma) + \sum_{\tau \neq \sigma} m'_2(\tau) = 0 \), so \( m'_2(\sigma) \leq 0 \), and by anti-symmetry we can conclude that \( m'_2(\sigma) = 0 \) at every point, so \( m'_2 = 0 \) and thus \( m'_2 \models \top^\circ \).
- \( \varphi = \varphi_1 \lor \varphi_2 \). We know \( m \models \varphi_1 \) or \( m \models \varphi_2 \). Without loss of generality, suppose that \( m \models \varphi_1 \).
- By the induction hypothesis, there are \( m_1, m_2, \) and \( m'_1 \in w(m_1, \text{frame}(F)) \) such that \( m = m'_1 + m_2 \) and \( m_1 + m'_2 \models \varphi_1 \) for any \( m'_2 \) such that \( |m'_2| \leq |m_2| \). We can weaken this claim to conclude that \( m_1 + m'_2 \models \varphi_1 \lor \varphi_2 \) as well.
- \( \varphi = \varphi_1 \oplus \varphi_2 \). We know that \( u_1 \models \varphi_1 \) and \( u_2 \models \varphi_2 \) for some \( u_1 \) and \( u_2 \) such that \( a \cdot u_1 + \bar{a} \cdot u_2 = m \).
- By the induction hypothesis, there is \( s_1, s_2, \) and \( s'_1 \in w(s_1, \text{frame}(F)) \) such that \( u_1 = s'_1 + s_1 \) and \( s_1 + s'_2 \models \varphi_1 \) for any \( s'_2 \) such that \( |s'_2| \leq |s_2| \) and there is \( t_1, t_2, \) and \( t'_1 \in w(t_1, \text{frame}(F)) \) such that \( u_2 = t'_1 + t_2 \) and \( t_1 + t'_2 \models \varphi_2 \) for any \( t'_2 \) such that \( |t'_2| \leq |t_2| \).
Now, let \( m_1 = a \cdot s_1 + \overline{a} \cdot t_1, m_2 = a \cdot s_2 + \overline{a} \cdot t_2, \) and \( m'_1 = a \cdot s'_1 + \overline{a} \cdot t'_1. \) We first show that \( m'_1 \in w(m_1, \text{frame}(F)): \)

\[
m'_1 = a \cdot s'_1 + \overline{a} \cdot t'_1 \\
\in \{ a \cdot s'_1 + \overline{a} \cdot t'_1 \mid s'_1 \in w(s_1, \text{frame}(F)), t'_1 \in w(t_1, \text{frame}(F)) \}
\]

Also, we have:

\[
m = a \cdot u_1 + \overline{a} \cdot u_2 \\
= a \cdot (s'_1 + s_2) + \overline{a} \cdot (t'_1 + t_2) \\
= (a \cdot s'_1 + \overline{a} \cdot t'_1) + (a \cdot s_2 + \overline{a} \cdot t_2) \\
= m'_1 + m_2
\]

Now, take any \( m'_2 \) such that \( |m'_2| \leq |m_2|, \) in other words, \( |m'_2| \leq a \cdot |s_2| + \overline{a} \cdot |t_2|. \) By Lemma D.2, there must be \( s'_2 \) and \( t'_2 \) such that \( |s'_2| \leq |s_2| \) and \( |t'_2| \leq |t_2| \) and \( m'_2 = a \cdot s'_2 + \overline{a} \cdot t'_2. \)

From the induction hypotheses, we know that \( s_1 + s'_2 \models \varphi_1 \) and \( t_1 + t'_2 \models \varphi_2. \) Finally, since \( a \cdot (s_1 + s'_2) + \overline{a} \cdot (t_1 + t'_2) = (a \cdot s_1 + \overline{a} \cdot t_1) + (a \cdot s'_2 + \overline{a} \cdot t'_2) = m_1 + m'_2, \) we get that \( m_1 + m'_2 \models \varphi_1 \oplus_\pi \varphi_2. \)

Now, we must show that \( m \in w(m_1, \text{frame}(F)). \) This is clearly the case since by construction, at every point \( m_1 \) is the sum of points from \( m \) according to the relation \( S \subseteq \text{frame}(F), \) and all the mass of \( m \) is accounted for. It also must be the case that \( m_1 \models e : P \) since by assumption, each element in the domain of \( m_1 \) satisfies \( P. \) Since \( m_2 = \emptyset, \) then it follows that \( m_1 + m'_2 \models e : P \) for any \( m'_2 \) such that \( |m'_2| \leq |m_2| = 0. \)

\( \square \)

### D.2 Nominal Heaps

**Lemma D.3 (Equivariance of Expressions).** For any \( e \in \text{Exp}, s \in S, \) and permutation \( \pi: \)

\[
[e] (\pi(s)) = \pi([e] (s))
\]

**Proof.** By induction on the expression \( e. \)

\( \triangleright e = x. \)

\[
[x] (\pi(s)) = \pi(s)(x) = (\pi \circ s)(x) = \pi(s(x)) = \pi([x] (s))
\]

\( \triangleright e = X. \) Same as the previous case.

\( \triangleright e = k. \) Note that \( k \notin \text{Addr}, \) so \( \pi(k) = k, \) and therefore \( [k] (\pi(s)) = k = \pi(k) \)

\( \triangleright e_1 = e_2. \)

\[
[e_1 = e_2] (\pi(s)) = \begin{cases} 
\text{true} & \text{if } [e_1] (\pi(s)) = [e_2] (\pi(s)) \\
\text{false} & \text{if } [e_1] (\pi(s)) \neq [e_2] (\pi(s)) 
\end{cases}
\]
By the induction hypothesis:
\[
\begin{cases}
\text{true} & \text{if } \pi([e_1] (s)) = \pi([e_2] (s)) \\
\text{false} & \text{if } \pi([e_1] (s)) \neq \pi([e_2] (s))
\end{cases}
\]

Since \( \pi \) is a bijection, \( \pi(x) = \pi(y) \) iff \( x = y \), so:
\[
\begin{cases}
\text{true} & \text{if } [e_1] (s) = [e_2] (s) \\
\text{false} & \text{if } [e_1] (s) \neq [e_2] (s)
\end{cases}
\]

\[= [e_1 = e_2] (s)\]

Finally, \( \pi \) has no effect on Boolean values.

\[\pi([e_1 = e_2] (s))\]

\(e = \neg e'\). By the induction hypothesis, \([e'] (\pi(s)) = \pi([e'] (s))\). Clearly, logically negating both will preserve equality.

\[\square\]

**Lemma D.4 (Equivariance of Pure Assertions).** If \( s \vDash \Pi \), then \( \pi(s) \vDash \Pi \).

**Proof.** By induction on \( \Pi \).

- \( \Pi = \text{true} \). Trivial since \( \pi(s) \vDash \text{true} \) always holds.
- \( \Pi = \Pi_1 \land \Pi_2 \). We know that \( s \vDash \Pi_1 \) and \( s \nvdash \Pi_2 \). By the induction hypothesis, \( \pi(s) \vDash \Pi_1 \) and \( \pi(s) \nvDash \Pi_2 \), so \( \pi(s) \vDash \Pi_1 \land \Pi_2 \).
- \( \Pi = e \). Suppose \( s \nvDash e \), so that means that \([e] (s) = \text{true} \). By Lemma D.3, we also know that \([e] (\pi(s)) = \pi([e] (s)) = \pi(\text{true}) = \text{true} \), so \( \pi(s) \vDash e \).

\[\square\]

**Lemma D.5 (Equivariance of Spacial Assertions).** If \((s, h) \vDash \Sigma \), then \( \pi(s, h) \vDash \Sigma \).

**Proof.** By induction on \( \Sigma \).

- \( \Sigma = \text{true} \). Trivial since \( \pi(s, h) \vDash \text{true} \) always holds.
- \( \Sigma = \text{emp} \). We know that \( h = \emptyset \), therefore \( \pi(h) = \emptyset \) too, so \( \pi(s, h) \vDash \text{emp} \).
- \( \Sigma = \Sigma_1 \lor \Sigma_2 \). We know that \((s, h_1) \vDash \Sigma_1 \) and \((s, h_2) \nvDash \Sigma_2 \) such that \( h = h_1 \lor h_2 \). By the induction hypothesis, \( \pi(s, h_1) \vDash \Sigma_1 \) and \( \pi(s, h_2) \nvDash \Sigma_2 \). Since permuting distributes over \( \lor \), this means that \( \pi(h) = \pi(h_1 \lor h_2) = \pi(h_1) \lor \pi(h_2) \), and so \( \pi(s, h) \vDash \Sigma_1 \lor \Sigma_2 \).
- \( \Sigma = e_1 \mapsto e_2 \). We know that \( h \) is a singleton heap where the address \([e_1] (s)\) points to the value \([e_2] (s)\). Now:

\[
\pi(h) = \pi \circ h \circ \pi^{-1}
\]

\[= \pi \circ (\lambda \ell. [e_2] (s)) \text{ if } \ell = [e_1] (s) \circ \pi^{-1}
\]

\[= \lambda \ell. \pi([e_2] (s)) \text{ if } \pi^{-1}(\ell) = [e_1] (s)
\]

Since permuting cannot affect equalities:

\[= \lambda \ell. \pi([e_2] (s)) \text{ if } \pi(\pi^{-1}(\ell)) = \pi([e_1] (s))
\]

By Lemma D.3:

\[= \lambda \ell. [e_2] (\pi(s)) \text{ if } \ell = [e_1] (\pi(s))
\]

So \( \pi(h) \) is a singleton heap where the address \([e_1] (\pi(s))\) points to the value \([e_2] (\pi(s))\), and therefore \( \pi(s, h) \vDash e_1 \mapsto e_2 \).
\( \Sigma = \text{ls}(e_1, e_2) \). The proof is by induction on the length of the list segment. Suppose the list segment has length zero, then it must be the case that \((s, h) \models e_1 = e_2 \land \text{emp} \). We know that this assertion is invariant to renaming from the previous cases and Lemmas 4.5 and D.4. Now, suppose that the claim holds for a list of length \(n\), and the current list has length \(n+1\). That means that \((s, h) \not\models \exists X. e_1 \leftrightarrow X \ast \text{ls}(X, e_2) \). By the previous cases and the induction hypothesis, the claim holds.

\[ \Box \]

**Lemma 4.5 (Equivariance of Symbolic Heaps).** If \((s, h) \models P\), then \(\pi(s, h) \models P\) for any \(\pi\).

**Proof.** By definition, \(P\) must have the form \(P = \exists X. \Pi \land \Sigma\), so if \((s, h) \models \exists X. \Pi \land \Sigma\), then we know that \((s', h) \models \Pi\) and \((s', h) \models \Sigma\) where \(s' = s[X \mapsto \vec{o}]\) for some \(\vec{o}\). Using Lemmas D.4 and D.5, we get that \(\pi(s', h) \models \Pi\) and \(\pi(s', h) \models \Sigma\). Now, we can see that \(\pi(s') = \pi(s[X \mapsto \vec{o}]) = \pi(s)[\vec{X} \mapsto \pi(\vec{o})]\), so \(\pi(s, h) \models \exists X. \Pi \land \Sigma\).

\[ \Box \]

**Lemma 4.6 (Equivariance).** Let \(\text{Perm} = \{(\sigma, \pi(\sigma)) \mid \sigma \in \text{St}, \pi \text{ is a permutation}\}\). If \(m \models \varphi\), then \(m' \not\models \varphi\) for any \(m' \in w(m, \text{Perm})\). This gives us:

\[
m' \in w(m, \text{Perm})
\]

\[
= w(a \cdot m_1 + \bar{a} \cdot m_2, \text{Perm})
\]

\[
= \{a \cdot m'_1 + \bar{a} \cdot m'_2 \mid m'_1 \in w(m_1, \text{Perm}), m'_2 \in w(m_2, \text{Perm})\}
\]

So \(m' = a \cdot m'_1 + \bar{a} \cdot m'_2\) for some \(m'_1 \in w(m_1, \text{Perm})\) and \(m'_2 \in w(m_2, \text{Perm})\). By the induction hypotheses, we get that \(m'_1 \not\models \varphi_1\) and \(m'_2 \not\models \varphi_2\). Now, putting these together, we get \(m' \not\models \varphi_1 \oplus \varphi_2\).

\[
\varphi = (\varepsilon : P)\). We know that \(|m| = 1\) and the entire support of \(m\) has the form \(1_e(s, h)\) such that \((s, h) \not\models P\). Now, take some \(m' \in w(m, \text{Perm})\). By definition, every element \(\sigma' \in \text{supp}(m')\) must have a corresponding \(\sigma \in \text{supp}(m)\) such that \(\sigma' = \pi(\sigma)\) for some permutation \(\pi\). Now, since sigma has the form \(1_e(s, h)\), then \(\sigma'\) must have the form \(\pi(1_e(s, h)) = 1_e(\pi(s, h))\), and since we know that \((s, h) \not\models P\), then by Lemma 4.5, \(\pi(s, h) \not\models P\) too. Since this applies to all elements of \(m'\) and since \(|m'| = |m| = 1\), then \(m' \not\models (\varepsilon : P)\).

\[ \Box \]

**D.3 Replacement of Unsafe States**

First, let us define some notation for expressing the safe and unsafe portions of a distribution.

\[
safe(m)(\sigma) = \begin{cases} m(\sigma) & \text{if } \sigma \neq \text{undef} \\ 0 & \text{if } \sigma = \text{undef} \end{cases}
\]

\[
\text{unsafe}(m)(\sigma) = \begin{cases} 0 & \text{if } \sigma \neq \text{undef} \\ m(\text{undef}) & \text{if } \sigma = \text{undef} \end{cases}
\]

So \(\text{safe}(m)\) contains all the ok and er states, whereas \(\text{unsafe}(m)\) contains the undefined ones, and clearly \(m = \text{safe}(m) + \text{unsafe}(m)\).
Lemma D.6 (Replacement). If $m \models \varphi$, then $\text{safe}(m) + m' \not\models \varphi$ for any $m'$ such that $|m'| \leq |\text{unsafe}(m)|$.

Proof. By induction on the assertion $\varphi$.

- $\varphi = \top$. Since $|m'| \leq |\text{unsafe}(m)|$, then the sum $\text{safe}(m) + m'$ is defined and it trivially satisfies $\top$.
- $\varphi = \top \land \varphi_2$. Without loss of generality, suppose $m \models \varphi_1$. Now, take any $m'$ such that $|m'| \leq |\text{unsafe}(m)|$. By the induction hypothesis, $\text{safe}(m) + m' \not\models \varphi_1$. We can weaken this to conclude that $\text{safe}(m) + m' \not\models \varphi_1 \lor \varphi_2$.
- $\varphi = \varphi_1 \lor \varphi_2$. We know that $m_1 \models \varphi_1$ and $m_2 \models \varphi_2$ for some $m_1$ and $m_2$ such that $a \cdot m_1 + \overline{a} \cdot m_2 = m$. Now, take any $m'$ such that $|m'| \leq |\text{unsafe}(m)| = a \cdot |\text{unsafe}(m_1)| + \overline{a} \cdot |\text{unsafe}(m_2)|$. By Lemma D.2 there must be $m_1'$ and $m_2'$ such that $|m_1'| \leq |\text{unsafe}(m_1)|$ and $|m_2'| \leq |\text{unsafe}(m_2)|$ and $m' = a \cdot m_1 + \overline{a} \cdot m_2$. By the induction hypothesis, $\text{safe}(m_1) + m_1' \models \varphi_1$ and $\text{safe}(m_2) + m_2' \models \varphi_2$.

Now, we have:

$$a \cdot (\text{safe}(m_1) + m_1') + \overline{a} \cdot (\text{safe}(m_2) + m_2') = (a \cdot \text{safe}(m_1) + \overline{a} \cdot \text{safe}(m_2)) + (a \cdot m_1' + \overline{a} \cdot m_2') = \text{safe}(m) + m'$$

So, $\text{safe}(m) + m' \not\models \varphi_1 \lor \varphi_2$.
- $\varphi = (\epsilon : P)$. By definition, $\text{undef} \not\in \text{supp}(m)$, so $\text{safe}(m) = m$ and $\text{unsafe}(m) = \emptyset$, and so $m' = \emptyset$ as well. So, $\text{safe}(m) + m' = m + \emptyset = m$, which satisfies $\epsilon : P$ by assumption.

\[\square\]

Lemma 4.7 (Replacement). If $m \not\models \varphi$ and $m' \in \text{prune}(w(m, \text{Rep}))$, then $m' \not\models \varphi$.

Proof. First, we know that:

$$m' \in \text{prune}(w(m, \text{Rep}))$$

$$= \text{prune}(w(\text{safe}(m) + \text{unsafe}(m), \text{Rep}))$$

$$= \text{prune}((m_1 + m_2 \mid m_1 \in w(\text{safe}(m), \text{Rep}), m_2 \in w(\text{unsafe}(m), \text{Rep})))$$

Now, $m_1 = \text{safe}(m)$ since safe states are only related to themselves according to $\text{Rep}$. In addition, safe states are not related to $\bot$, so the prune can apply just to the unsafe part.

$$= \{\text{safe}(m) + m'' \mid m'' \in \text{prune}(w(\text{unsafe}(m), \text{Rep}))\}$$

So, $m' = \text{safe}(m) + m''$ for some $m'' \in \text{prune}(w(\text{unsafe}(m), \text{Rep}))$. It must be the case that $|m''| \leq |\text{unsafe}(m)|$ since reweighting functions preserve mass, but the prune operation could remove mass. We can therefore apply Lemma D.6 to conclude that $m' \not\models \varphi$. \[\square\]

D.4 The Frame Rule

For all the proofs in this section, we let $R = \text{Rep} \circ \text{frame}(F) \circ \text{Perm}$.

Lemma D.7 (Sequencing). If $m' \in \text{prune}(w(m, R))$ and $f(s', h') \in \text{prune}(w(f(s, h), R))$ for all $(\mathcal{I}_\text{ok}(s, h), \mathcal{I}_\text{ok}(s', h')) \in R$, then $\text{bind}(m', f) \in \text{prune}(w(\text{bind}(m, f), R))$.

Proof. First, let $f'$ be defined as follows:

$$f' (\sigma) = \begin{cases} 
  f(s, h) & \text{if } \sigma = \mathcal{I}_\text{ok}(s, h) \\
  \text{unit}_w(\sigma) & \text{otherwise} 
\end{cases}$$

Now, we argue that if $(\sigma, \tau) \in R$, then $\text{prune}(f'(\tau)) \in \text{prune}(w(f'(\sigma), R))$. We do so by case analysis.
\( \sigma = 1_{ok}(s, h) \) and \( \tau = 1_{ok}(s', h') \). By definition, \( f'(\tau) = f(s', h') \) and \( f'(\sigma) = f(s, h) \) such that \( (1_{ok}(s, h), 1_{ok}(s', h')) \in R \). We know that \( f(s', h') \in \text{prune}(w(f(s, h), R)) \) and so \( f'(\tau) \in \text{prune}(w(f'(\sigma), R)) \). Now, since \( f'(\tau) = f(s', h') \), then \( f'(\tau) \) cannot contain \( \bot \) and so \( \text{prune}(f'(\tau)) = f'(\tau) \) and therefore \( \text{prune}(f'(\tau)) \in \text{prune}(w(f'(\sigma), R)) \).

\( \sigma = \text{undef} \) and \( \tau = 1_{ok}(s', h') \). First note that \( f'(\sigma) = \text{unit}_w(\text{undef}) \), which means that \( \text{prune}(w(f'(\sigma), R)) \) contains all \( m \), since \( R \) relates \text{undef} to any state (or \( \bot \)) and then all the \( \bot \) elements will be pruned out. It is therefore trivial that \( \text{prune}(f'(\tau)) = f'(\tau) \in \text{prune}(w(f'(\sigma), R)) \).

In this final case, \( \tau \) cannot have the form \( 1_{ok} \), since only ok and \text{undef} states are related to ok states according to \( R \), and we have already handled both of those cases. This means that \( \sigma \) must also not be an ok state, since ok states are only related to other ok states. Therefore, \( f'(\sigma) = \text{unit}_w(\sigma) \) and \( f'(\tau) = \text{unit}_w(\tau) \). So, \( w(f'(\sigma), R) = w(\text{unit}_w(\sigma), R) \), and we know that \( \text{unit}_w(\tau) \in w(\text{unit}_w(\sigma), R) \) by Lemma C.2 since \((\sigma, \tau) \in R \). Applying prune to both sides will preserve the set containment.

Now, we have:

\[
\text{bind}(m', f) = \text{bind}_w \left( m', \lambda \sigma. \begin{cases} f(s', h') & \text{if } \sigma = 1_{ok}(s', h') \\ \text{unit}_w(\sigma) & \text{otherwise} \end{cases} \right) 
= \text{bind}_w(m', f')
\]

Since \( m' \in \text{prune}(w(m, R)) \), then there must be some reweighting function \( g \) such that \( m' = \text{prune}(\text{bind}_w(m, g)) \).

\[
= \text{bind}_w(\text{prune}(\text{bind}_w(m, g)), f')
\]

Note that \( f' \) will leave the \( \bot \) states untouched, so we can move the prune operation to after \( f' \).

\[
= \text{bind}_w(\text{bind}_w(m, g), \text{prune} \circ f') \\
= \text{bind}_w(m, \lambda \sigma. \text{bind}_w(g(\sigma), \text{prune} \circ f')) \\
= \sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot \sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau) \cdot \text{prune}(f'(\tau))
\]

We already showed that for all \((\sigma, \tau) \in R, \text{prune}(f'(\tau)) \in w(f'(\sigma), R) \), and in the above equation, \((\sigma, \tau) \) must be in \( R \), since \( \tau \in \text{supp}(g(\sigma)) \) and \( g \) is a reweighting function for \( R \).

\[
eq \{ \sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot \sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau) \cdot m_{\sigma, \tau} \mid (\sigma, \tau) \in R, m_{\sigma, \tau} \in \text{prune}(w(f'(\sigma), R)) \}
\]

We can move the prune operation outside, since the other operations are just computing linear combinations of the \( m_{\sigma, \tau} \)s

\[
= \text{prune}(\{ \sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot \sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau) \cdot m_{\sigma, \tau} \mid (\sigma, \tau) \in R, m_{\sigma, \tau} \in w(f'(\sigma), R) \})
\]

By Lemma C.1:

\[
= \text{prune}(w(\sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot \sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau) \cdot f'(\sigma), R)) \\
= \text{prune}(w(\sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot (\sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau)) \cdot f'(\sigma), R))
\]
Since \( g \) is a reweighting function, then \( \sum_{\tau \in \text{supp}(g(\sigma))} g(\sigma)(\tau) = 1. \)

\[
\begin{align*}
&= \text{prune}(w(\sum_{\sigma \in \text{supp}(m)} m(\sigma) \cdot f'(\sigma), R)) \\
&= \text{prune}(w(\text{bind}_W(m, f'), R)) \\
&= \text{prune}(w(\text{bind}(m, f'), R))
\end{align*}
\]

\[\Box\]

**Lemma D.8.** If \((\mathbb{1}_{ok}(s, h), \mathbb{1}_{ok}(s', h')) \in R\), then there is some \( h'' \) and permutation \( \pi \) such that \( s' = \pi(s) \) and \( h' = \pi(h) \cup h'' \) and \((\pi(s), h'') \in F\).

**Proof.** First note that \( R = \text{Rep} \circ \text{frame}(F) \circ \text{Perm} \), so there must be \( \sigma \) and \( \sigma' \) such that \( (\mathbb{1}_{ok}(s, h), \sigma) \in \text{Perm} \) and \((\sigma, \sigma') \in \text{frame}(F)\), and \((\sigma', \mathbb{1}_{ok}(s', h')) \in \text{Rep}\). By the definition of \( \text{Perm} \), \( \sigma = \pi(\mathbb{1}_{ok}(s, h)) = \mathbb{1}_{ok}(\pi(s), \pi(h)) \) for some permutation \( \pi \). By the definition of \( \text{frame}(F) \), \( \sigma' = \mathbb{1}_{ok}(\pi(s), \pi(h) \cup h'') \) such that \((\pi(s), h'') \in F\). Finally, since we just showed that \( \sigma' \) is not an unde\( \text{f} \) state, then it can only be related to itself according to \( \text{Rep} \), and so \( \mathbb{1}_{ok}(\pi(s), \pi(h) \cup h'') = \mathbb{1}_{ok}(s', h') \), therefore \( s' = \pi(s) \) and \( h' = \pi(h) \cup h'' \).

\[\Box\]

**Lemma D.9.** If \((\mathbb{1}_{ok}(s_1, h_1), \mathbb{1}_{ok}(s_2, h_2)) \in R\) whenever \( h(\llbracket e \rrbracket(s)) \in \text{Val} \), then

\[
\text{update}(\pi(s), \pi(h) \cup h', \llbracket e \rrbracket(s), s_2, h_2) \in w(\text{update}(s, h, \llbracket e \rrbracket(s), s_1, h_1), R)
\]

**Proof.** Let \( \ell = \llbracket e \rrbracket(s) \). By Lemma D.3, \( \llbracket e \rrbracket(s) = \pi(\llbracket e \rrbracket(s)) = \pi(\ell) \). We complete the proof by case analysis:

1. \( h(\ell) \in \text{Val} \). Since \( \pi(h)(\pi(\ell)) = (\pi \circ h \circ \pi^{-1})(\pi(\ell)) = \pi(h(\ell)) \), then \( \pi(h)(\pi(\ell)) \) must also be a value and then so must \( (\pi(h) \cup h')(\pi(\ell)) \). So, we just need to show that \( \text{unit}(s_2, h_2) \in w(\text{unit}(s_1, h_1), R) \), which follows from Lemma C.2.
2. \( h(\ell) = \bot \). By a similar argument to the previous case, it must be that \( (\pi(h) \cup h')(\pi(\ell)) = \bot \) too. So, we just need to show that \( \text{error}(\pi(s), \pi(h) \cup h', \pi(\ell)) \) is \( \bot \) as well. Note that ok and er states are treated identically by \( R \), so \((\mathbb{1}_{error}(\pi(s), \pi(h) \cup h'), \mathbb{1}_{error}(s, h)) \in R \) and the claim follows by Lemma C.2.
3. \( \ell \notin \text{dom}(h) \). So, \( \text{update}(s, h, \llbracket e \rrbracket(s), s_1, h_1) = \text{unit}_W(\text{undef}) \), and since \( R \) related \text{undef} to all states, \( w(\text{update}(s, h, \llbracket e \rrbracket(s), s_1, h_1), R) \) contains all program configurations, which means that \( \text{update}(\pi(s), \pi(h) \cup h', \llbracket e \rrbracket(s), s_2, h_2) \) is contained in it trivially.

\[\Box\]

**Lemma D.10.** If \((\sigma, \tau) \in R\) for every \( \sigma \in \text{supp}(m_1) \) and \( \tau \in \text{supp}(m_2) \) and \( |m_1| = |m_2| \), then \( m_2 \in w(m_1, R) \).

**Proof.** Using Lemma D.1, let \( m'_2 \) be a program configuration such that \( m_2 = |m_2| \cdot m'_2 \) and \( |m'_2| = 1 \). Let \( f \) be defined as follows \( f(\sigma) = m'_2 \). Clearly \( f \) is a reweighting function for \( m_1 \) and \( R \), since \( |m'_2| = 1 \) and if \( \sigma \in \text{supp}(m_1) \), then \((\sigma, \tau) \in R\) for every \( \tau \in \text{supp}(m'_2) \). So, it will suffice to show
that \( m_2 = \text{bind}_W(m_1, f) \), which we do as follows:

\[
\text{bind}_W(m_1, f) = \sum_{\sigma \in \text{supp}(m_1)} m_1(\sigma) \cdot f(\sigma)
\]

\[
= \sum_{\sigma \in \text{supp}(m_1)} m_1(\sigma) \cdot m'_2
\]

\[
= (\sum_{\sigma \in \text{supp}(m_1)} m_1(\sigma)) \cdot m'_2
\]

\[
= |m_1| \cdot m'_2 = |m_2| \cdot m'_2 = m_2
\]

\[
\square
\]

**Lemma 4.8 (The Frame Property).** Let \( R = \text{Rep} \circ \text{frame}(F) \circ \text{Perm}, \) so \( R \subseteq \Sigma \times \Sigma_\bot \). For any program \( C \) and symbolic heap \( F \) such that \( \text{mod}(C) \cap \text{fv}(F) = \emptyset \):

\[
\forall (l_\text{ok}(s, h), l_\text{ok}(s', h')) \in R. \quad [C] (s', h') \in \text{prune}(w([C] (s, h), R))
\]

**Proof.** By induction on the structure of the program \( C \).

- \( C = \text{skip} \). So, \([C] (s', h') = \text{unit}(s', h') = \text{unit}_W(l_\text{ok}(s', h'))\). Also, \( \text{prune}(w([C] (s, h), R)) = w(\text{unit}_W(l_\text{ok}(s, h)), R) \) (the prune does nothing since ok states are not related to \( \bot \)). The claim follows from Lemma C.2.

- \( C = C_1 \circ C_2 \). We know that \([C] (s', h') = [C_1 \circ C_2] (s', h') = \text{bind}([C_1] (s', h'), [C_2])\). By the induction hypothesis, we can conclude that \([C_1] (s', h') \in \text{prune}(w([C_1] (s, h), R))\). Therefore by the induction hypothesis and Lemma D.7, we can conclude that \( \text{bind}([C_1] (s', h'), [C_2]) \in \text{prune}(w(\text{bind}([C_1] (s, h), [C_2]), R)) \).

- \( C = C_1 +_a C_2 \). We know that \([C_1 +_a C_2] (s', h') = a \cdot [C_1] (s', h') + \bar{a} \cdot [C_2] (s', h')\). By the induction hypothesis, \([C_i] (s', h') \in \text{prune}(w([C_i] (s, h), R))\) for each \( i \in \{1, 2\} \). Therefore \( a \cdot [C_1] (s', h') \in \text{prune}(w(a \cdot [C_1] (s, h), R)) \) and \( \bar{a} \cdot [C_2] (s', h') \in \text{prune}(w(\bar{a} \cdot [C_2] (s, h), R)) \), and so

\[
[C_1 +_a C_2] (s', h') = a \cdot [C_1] (s', h') + \bar{a} \cdot [C_2] (s', h')
\]

\[
\in \text{prune}(w(a \cdot [C_1] (s, h) + \bar{a} \cdot [C_2] (s, h), R))
\]

\[
= \text{prune}(w([C_1 +_a C_2] (s, h), R))
\]

We now move on to the cases involving expressions and primitive instructions. By Lemma D.8, \( s' = \pi(s) \) and \( h' = \pi(h) \cup h'' \) such that \( (\pi(s), \pi(h) \cup h'') \equiv F \), so we will use that fact in the following cases. Additionally, many of the cases have a single outcome, so by Lemma C.2 it suffices to show that \( (\sigma, \tau) \in R \) where \([C] (s, h) = \text{unit}_W(\sigma)\) and \([C] (s', h') = \text{unit}_W(\tau)\) in those cases.

- \( C = \text{if } e \text{ then } C_1 \text{ else } C_2 \). Since \( s' = \pi(s) \), then \([e] (s') = [e] (\pi(s)) = \pi([e] (s)) = [e] (s)\) where the last step is valid since we assume \( e \) is boolean valued and therefore the result of evaluating it will not be changed by permuting addresses. This means that both programs will take the same path. Without loss of generality, suppose \( [e] (s) = \text{true} \). So, that means that \([C] (s, h) = [C_1] (s, h)\) and \([C] (s', h') = [C_2] (s', h')\). We complete the proof by applying the induction hypothesis to conclude that:

\[
[C_1] (s', h') \in \text{prune}(w([C_1] (s, h), R))
\]

The case where \([e] (s) = \text{false} \) is symmetric.
\( C = \text{while } e \text{ do } C'. \) First, we will show that \( F_{\langle C', e \rangle}(s', h')^n(\perp)(s', h') \in \text{prune}(w(F_{\langle C', e \rangle}^n(\perp)(s, h), R)) \) for any \((\mathbb{I}_{\text{ok}}(s, h), \mathbb{I}_{\text{ok}}(s', h')) \in R.\) The proof is by induction on \( n. \) If \( n = 0, \) then the claim holds using Lemma C.1:

\[
F_{\langle C', e \rangle}^0(s', h')(\perp)(s', h') = \perp(s', h') = \emptyset \in \text{prune}(w(\emptyset, R)) = \text{prune}(w(F_{\langle C', e \rangle}^0(\perp)(s, h), R))
\]

Now suppose the claim holds for \( n \) and suppose the claim holds for \( n + 1:
\[ F_{\langle C', e \rangle}^{n+1}(\perp)(s', h') = \begin{cases} \text{bind}(\mathbb{C}'(\perp)(s', h'), F_{\langle C', e \rangle}(s', h')(\perp)) & \text{if } [e](s', h') = \text{true} \\ \text{unit}(s', h') & \text{if } [e](s', h') = \text{false} \end{cases} \]

If \([e](s', h') = \text{true}, \) then the claim holds by Lemma D.7 and the induction hypothesis. In the second case, it holds by Lemma C.2.

Now, we complete the proof as follows:
\[
\text{prune}(w(\sup_{n \in \mathbb{N}} F_{\langle C', e \rangle}^n(\perp)(s, h), R))
\]

Using the claim we just proved, along with the fact that the \( F_{\langle C', e \rangle}^n(\perp)(s', h') \) terms form a chain, we can conclude that the supremum of the chain upholds the property as well and so:
\[
\text{for any } \pi \supseteq \mathbb{C}' \text{ we know that:}
\]

\[\mathbb{C}'(s, h) = \text{unit}(s[x \mapsto [e](s)], h)\]

\[\mathbb{C}'(\pi(s), \pi(h) \cup h'') = \text{unit}(\pi(s)[x \mapsto [e](s)], \pi(h) \cup h'')\]

By Lemma D.3, \([e](\pi(s)) = \pi([e](s)), \) therefore \( \pi(s)[x \mapsto [e](s)] = \pi(s)[x \mapsto [e](s)], \pi(h)) \in \text{Perm}. \)

Now, since \( x \in \text{mod}(\mathbb{C}), \) then \( x \notin \text{fv}(F), \) so updating \( x \) in \( \pi(s) \) will not affect the truth of \( F, \) therefore \( \pi(s)[x \mapsto [e](s)], h)) \in \text{frame}(F). \)

Putting these together along with the fact that Rep is reflexive gives us that \((\mathbb{I}_{\text{ok}}(s[x \mapsto [e](s)], h), \mathbb{I}_{\text{ok}}(\pi(s[x \mapsto [e](s)]), h) \cup h')) \in R. \)

\( C = (x := e). \) We know the following:

\[\mathbb{C}'(s, h) = \text{bind}_{\mathbb{W}}(\text{alloc}(\text{dom}(h)), \lambda \ell. \text{unit}(s[x \mapsto \ell], h[\ell \mapsto 0]))\]

\[\mathbb{C}'(\pi(s), \pi(h) \cup h'') = \text{bind}_{\mathbb{W}}(\text{alloc}(\text{dom}(\pi(h) \cup h'')), \lambda \ell. \text{unit}(\pi(s)[x \mapsto \ell], (\pi(h) \cup h''))[\ell \mapsto 0]))\]

Since \( |[\mathbb{C}](s, h)| = |[\mathbb{C}](\pi(s), \pi(h) \cup h'')| = 1, \) we can use Lemma D.10 to complete the proof as long as all the elements in both supports are related. Take \( \sigma \in \text{supp}([\mathbb{C}](s, h)) \) and \( \tau \in \text{supp}([\mathbb{C}](\pi(s), \pi(h) \cup h''), \) so \( \sigma = \mathbb{I}_{\text{ok}}(s[x \mapsto \ell_1], h[\ell_1 \mapsto 0]) \) and \( \tau = \mathbb{I}_{\text{ok}}(\pi(s)[x \mapsto \ell_2], (\pi(h) \cup h'')[\ell_2 \mapsto 0]) \) for some \( \ell_1 \) and \( \ell_2 \) such that \( \ell_1 \notin \text{dom}(h) \cup \text{im}(s) \cup \text{im}(h) \) and \( \ell_2 \notin \text{dom}(\pi(h) \cup h'') \cup \text{im}(\pi(s)) \cup \text{im}(\pi(h) \cup h'')). \) Now, let \( \pi' \) be defined as follows:

\[
\pi'(\ell) = \begin{cases} \ell_2 & \text{if } \ell = \ell_1 \\ \pi(\ell_1) & \text{if } \ell = \pi^{-1}(\ell_2) \\ \pi(\ell) & \text{otherwise} \end{cases}
\]

Clearly \( \pi' \) is also a valid permutation since it just swaps the values of \( \ell_1 \) and \( \pi^{-1}(\ell_2) \) in \( \pi. \) Note that since \( \ell_2 \notin \text{dom}(\pi(h) \cup h'') \cup \text{im}(\pi(s)) \cup \text{im}(\pi(h) \cup h''), \) then \( \pi^{-1}(\ell_2) \notin \text{dom}(\pi(h) \cup h'') \cup \text{im}(\pi(s)) \cup \text{im}(\pi(h) \cup h''). \)
Now, we have:

\[ 1_{\text{ok}}(\pi(s)[x \mapsto \ell_2], (\pi(h) \cup h''')[\ell_2 \mapsto 0]) = 1_{\text{ok}}(\pi'(s)[x \mapsto \ell_2], (\pi'(h) \cup h'')[\ell_2 \mapsto 0]) \]

Since \( \ell_2 \not\in \text{dom}(h'') \):

\[ = 1_{\text{ok}}(\pi'(s)[x \mapsto \ell_2], \pi'(h)[\ell_2 \mapsto 0] \cup h'') \]

\[ = 1_{\text{ok}}(\pi'(s)[x \mapsto \pi'(\ell_1)], \pi'(h)[\pi'(\ell_1) \mapsto 0] \cup h'') \]

\[ = 1_{\text{ok}}(\pi'(s)[x \mapsto \ell_1]), \pi'(h[\ell_1 \mapsto 0]) \cup h'' \]

Now, clearly \((1_{\text{ok}}(s[x \mapsto \ell_1], h[\ell_1 \mapsto 0]), 1_{\text{ok}}(\pi(s[x \mapsto \ell_1]), \pi(h[\ell_1 \mapsto 0]))) \in \text{Perm}. \)

Also, since \( x \in \text{mod}(C) \), then updating \( x \) in \( \pi(s) \) will not affect the truth of \( F \), so \((\pi(s)[x \mapsto \ell_2], h''') \vDash F \) and therefore \((1_{\text{ok}}(\pi'(s[x \mapsto \ell_1]), \pi'(h[\ell_1 \mapsto 0])), 1_{\text{ok}}(\pi'(s[x \mapsto \ell_1]), \pi'(h[\ell_1 \mapsto 0]) \cup h'')) \in \text{frame}(F) \), so we are done.

\( \triangleright \) \( C = \text{free}(e) \). Using Lemma D.9, we only need to show that if \( h([e](s)) \in \text{Val} \), then:

\[ (1_{\text{ok}}(s, h([e](s) \mapsto \bot], 1_{\text{ok}}(\pi(s), (\pi(h) \cup h'')([e](\pi(s)) \mapsto \bot))) \in R \]

If \( h([e](s)) \in \text{Val} \), then \([e](\pi(s)) \in \text{dom}(\pi(h)) \), and since \( h'' \) is disjoint from \( \pi(h) \), then \([e](\pi(s)) \notin \text{dom}(h'') \), so:

\[ (\pi(h) \cup h'')([e](\pi(s)) \mapsto \bot] = \pi(h)([e](\pi(s)) \mapsto \bot] \cup h'' \]

By Lemma D.3

\[ = \pi(h)([\pi([e_1](s)) \mapsto \bot] \cup h'' \]

\[ = \pi(h)([\pi([e_1](s) \mapsto \bot] \cup h'' \]

So, clearly \((1_{\text{ok}}(s, h([e_1](s) \mapsto \bot], 1_{\text{ok}}(\pi(s), \pi(h([e_1](s) \mapsto \bot])) \in \text{Perm} \) and also \((1_{\text{ok}}(\pi(s), \pi(h([e_1](s) \mapsto \bot])) \in \text{frame}(F) \), so we are done.

\( \triangleright \) \( C = ([e_1] \leftarrow e_2) \). Using Lemma D.9, we only need to show that if \( h([e_1](s)) \in \text{Val} \), then:

\[ (1_{\text{ok}}(s, h([e_1](s) \mapsto [e_2](s))), 1_{\text{ok}}(\pi(s), (\pi(h) \cup h'')([e_1](\pi(s)) \mapsto [e_2](\pi(s)))) \in R \]

If \( h([e_1](s)) \in \text{Val} \), then \([e_1](\pi(s)) \in \text{dom}(\pi(h)) \), and since \( h'' \) is disjoint from \( \pi(h) \), then \([e_1](\pi(s)) \notin \text{dom}(h'') \), so:

\[ (\pi(h) \cup h'')([e_1](\pi(s)) \mapsto [e_2](\pi(s)) = \pi(h)([e_1](\pi(s)) \mapsto [e_2](\pi(s)) \cup h'' \]

By Lemma D.3

\[ = \pi(h)([\pi([e_1](s)) \mapsto [e_2](s))] \cup h'' \]

\[ = \pi(h)([\pi([e_1](s) \mapsto [e_2](s))] \cup h'' \]

So, clearly \((1_{\text{ok}}(s, h([e_1](s) \mapsto [e_2](s))), 1_{\text{ok}}(\pi(s), \pi(h([e_1](s) \mapsto [e_2](s)))) \in \text{Perm} \) and \((1_{\text{ok}}(\pi(s), \pi(h([e_1](s) \mapsto [e_2](s)))) \in \text{frame}(F) \), so we are done.

\( \triangleright \) \( C = (x \leftarrow [e]) \). Using Lemma D.9, we only need to show that if \( h([e](s)) \in \text{Val} \), then:

\[ (1_{\text{ok}}(s \mapsto h([e](s)), h), 1_{\text{ok}}(\pi(s)[x \mapsto (\pi(h) \cup h''')([e](\pi(s))])) \in R \]

\[ \text{The fact that } \ell_2 \not\in \text{im}(\pi(s)) \cup \text{im}(\pi(h) \cup h'') \text{ must hold was overlooked in the proof of Baktiev [2006, Thm. 3.1], but without this requirement we cannot guarantee that } \pi'(s) = \pi(s) \text{ and } \pi'(h) = \pi(h). \]
If $h(\mathcal{E}(s)) \in \text{Val}$, then $\mathcal{E}\pi(s) \in \text{dom}(\pi(h))$, and since $h''$ is disjoint from $\pi(h)$, then $\mathcal{E}\pi(s) \not\in \text{dom}(h'')$, so:

$$
\pi(s)[x \mapsto (\pi(h) \cup h'')(\mathcal{E}\pi(s))] = \pi(s)[x \mapsto \pi(h)(\mathcal{E}\pi(s))]
$$

$$
= \pi(s)[x \mapsto \pi(h)(\mathcal{E}\pi(s))]
$$

$$
= \pi(s)[x \mapsto \pi(h)(\mathcal{E}\pi(s))]
$$

$$
= \pi(s)[x \mapsto h(\mathcal{E}\pi(s))]
$$

So, clearly $(\mathcal{I}_{ok}(s[x \mapsto h(\mathcal{E}\pi(s))]), h, \mathcal{I}_{ok}(\pi(s[x \mapsto h(\mathcal{E}\pi(s))]), \pi(h))) \in \text{Perm}$ and also $(\mathcal{I}_{ok}(\pi(s[x \mapsto h(\mathcal{E}\pi(s))]), \pi(h)), \mathcal{I}_{ok}(\pi(s[x \mapsto h(\mathcal{E}\pi(s))]), \pi(h) \cup h'')) \in \text{frame}(F)$, so we are done.

$\triangleright C = \text{error}()$. By Lemma C.2, it suffices to show that $(\mathcal{I}_{ev}(s, h), \mathcal{I}_{ev}(\pi(s), \pi(h) \cup h'')) \in R$, which follows from the fact that $(\mathcal{I}_{ok}(s, h), \mathcal{I}_{ok}(\pi(s), (\pi(h) \cup h'')) \in R$ since $R$ treats ok and er states in the same way.

$\triangleright C = f(\mathcal{E})$. Let $C'$ be the body of $f$. By the same argument used in the assignment case, $(\mathcal{I}_{ok}(s[x \mapsto \mathcal{E}(s)], h), \mathcal{I}_{ok}(\pi(s[x \mapsto \mathcal{E}(s)]), \pi(h) \cup h'')) \in R$. So, by the induction hypothesis:

$$
\mathcal{E}\pi(s[x \mapsto \mathcal{E}(s)]), \pi(h) \cup h'' \in w([C']\pi(s[x \mapsto \mathcal{E}(s)], h, R)
$$

\[\square\]

**Theorem 4.9 (The Frame Rule).** The following inference is sound.

$$
\begin{array}{ccc}
\langle \phi \rangle & C & \langle \psi \rangle \\
\text{mod}(C) \cap \text{fv}(F) = \emptyset & \text{FRAME} & \langle \phi \circ F \rangle & C & \langle \psi \circ F \rangle
\end{array}
$$

**Proof.** Suppose $m \models \phi \circ F$. Then by Lemma 4.4 we know that there are $m_1, m_2, \text{ and } m_1' \in w(m_1, \text{frame}(F))$ such that $m = m_1' + m_2$ and $m_1 + m_2' \not\models \phi$ for any $m_2'$ such that $|m_2'| \leq |m_2|$. So that means that $m_1 + |m_2| \cdot \text{unit}(\text{undef}) \not\models \phi$. Now, we know that:

$$
[C]'(m_1 + |m_2| \cdot \text{unit}(\text{undef})) = [C]'(m_1) + |m_2| \cdot \text{unit}(\text{undef})
$$

So, therefore $[C]'(m_1) + |m_2| \cdot \text{unit}(\text{undef}) \not\models \psi$ since $\models \langle \phi \rangle C \langle \psi \rangle$. Now, observe that $m_1' \in w(m_1, \text{frame}(F)) \subseteq \text{prune}(w(m_1, R))$ since both additional components of $R$ are reflexive. We can also conclude that $m_2 \in \text{prune}(w(|m_2| \cdot \text{unit}(\text{undef}), R))$ since $R$ permits undef states to be remapped to anything. Therefore, combining these we get that:

$$
m = m_1' + m_2
$$

$$
\in \{u + v \mid u \in \text{prune}(w(m_1, R), v \in \text{prune}(w(|m_2| \cdot \text{unit}(\text{undef}), R))\}
$$

$$
= \text{prune}(w(m_1 + |m_2| \cdot \text{unit}(\text{undef}), R))
$$

So, using Lemmas 4.8 and D.7, we know that:

$$
[C]'(m) \in \text{prune}(w([C]'(m_1 + |m_2| \cdot \text{unit}(\text{undef})), R))
$$

$$
= \text{prune}(w([C]'(m_1 + |m_2| \cdot \text{unit}(\text{undef})), \text{Rep} \circ \text{frame}(F) \circ \text{Perm}))
$$

And by Lemma C.3:

$$
= \text{prune} \left( \bigcup_{m' \in w([C]'(m_1 + |m_2| \cdot \text{unit}(\text{undef})), \text{Perm})} m' \in w(m', \text{frame}(F)) \right)
$$

All that remains now is to peel away the layers in the above expression. More concretely, we know that there is some $m' \in w([C]'(m_1 + |m_2| \cdot \text{unit}(\text{undef}), \text{Perm})$ and $m'' \in w(m', \text{frame}(F))$ such
that \([C]^{\uparrow}(m) \in \text{prune}(w(m'', \text{Rep}))\). By Lemma 4.6, \(m' \models \psi\), and by Lemma 4.3 \(m'' \models \psi \oplus F\). Finally, by Lemma 4.7, \([C]^\uparrow(m) \models \psi \oplus F\).

\[\square\]

E TRI-ABDUCTION

**Lemma E.1.** If \(P \not< [M] \triangleright Q\) is derivable, then \(M \not< P\) and \(M \not< Q\)

**Proof.** The proof is by induction on the derivation of \(P \not< [M] \triangleright Q\).

1. **Base-Emp.** We need to show that \(\Pi \wedge \Pi' \wedge \text{emp} \not< [M] \triangleright \Pi \wedge \text{emp} \not< [M] \triangleright \Pi' \wedge \text{emp}\), both hold by the semantics of logical conjunction.

2. **Base-True-L.** We need to show that \(\Pi \wedge \Pi' \wedge \Sigma \not< [M] \triangleright \Pi \wedge \text{true} \not< [M] \triangleright \Pi' \wedge \Sigma\), both hold by the semantics of logical conjunction.

3. **Base-True-R.** This case is symmetric to Case 2.

4. **Exists.** Here, we know that \(M \not< \Delta\) and \(M \not< \Delta'\). We also know that \(\overline{X}\) is not free in \(\Delta'\) with \(\overline{Y}\) removed and vice versa. Now let:

\[\overline{Z} = \overline{X} \cap \overline{Y} \quad \overline{X}' = \overline{X} \setminus \overline{Z} \quad \overline{Y}' = \overline{Y} \setminus \overline{Z}\]

That is, \(\overline{Z}\) is the variables occurring in both \(\overline{X}\) and \(\overline{Y}\), \(\overline{X}'\) is the variables only occurring in \(\overline{X}\) and \(\overline{Y}'\) is the variables only occurring in \(\overline{Y}\). This means that \(\overline{X}'\), \(\overline{Y}'\), and \(\overline{Z}\) are disjoint.

We first show that \(\exists X.\exists Y.M \not< \exists X.\exists Y.M\). Suppose \((s, h) \not< \exists X.\exists Y.M\). Applying the generic semantics, we know that \((s', h) \not< M\) where \(s' = s[\overline{X}' \mapsto \overline{v}_1][\overline{Y}' \mapsto \overline{v}_2][\overline{Z} \mapsto v_3]\) for some \(\overline{v}_1, \overline{v}_2\), and \(v_3\). Given that we know \(M \not< \Delta\), we now have \((s', h) \not< \Delta\) and therefore \((s[Y' \mapsto \overline{v}_2], h) \not< \exists X.\exists Y.\Delta\) (since \(\overline{X} = \overline{X}' \cup \overline{Z}\)). Now, given that \(\overline{Y} \cap (fv(\Delta) \setminus \overline{X}) = \emptyset\), we know that none of the variables in \(\overline{Y}'\) are free in \(\Delta\), and so we can remove them from the state to conclude that \((s, h) \not< \exists X.\exists Y.\Delta\).

It can also be shown that \(\exists X.\exists Y.M \not< \exists X.\exists Y.\Delta\) by a symmetric argument.

5. **Ls-Start-L.** Here, we know \(M \not< \Delta * \text{ls}(e_3, e_2)\) and \(M \not< \Delta'\). We now want to show \(M * e_1 \mapsto e_3 \models \Delta * \text{ls}(e_1, e_2)\) and \(M * e_1 \mapsto e_3 \models \Delta' * e_1 \mapsto e_3\).

Suppose that \((s, h) \not< M * e_1 \mapsto e_3\), and so \((s, h_1) \not< M\) and \((s, h_2) \not< e_1 \mapsto e_3\) for some \(h_1\) and \(h_2\) such that \(h = h_1 \uplus h_2\).

Since \(M \not< \Delta * \text{ls}(e_3, e_2)\), we get that \((s, h_1) \not< \Delta * \text{ls}(e_3, e_2)\) and recombining, we get that \((s, h) \not< \Delta * \text{ls}(e_3, e_2)\) \(e_1 \mapsto e_3\). Now, \(\text{ls}(e_3, e_2) * e_1 \mapsto e_3 \models \exists X.e_1 \mapsto X * \text{ls}(X, e_2)\) and \(\exists X.e_1 \mapsto X * \text{ls}(X, e_2) \models \text{ls}(e_1, e_2)\), so we get that \((s, h) \not< \Delta * \text{ls}(e_1, e_2)\) and therefore \(M * e_1 \mapsto e_3 \models \Delta * \text{ls}(e_1, e_2)\).

We also know that \(M \not< \Delta'\) which means that because \((s, h_1) \not< M, (s, h_1) \not< \Delta'\). This means that \((s, h_1 \uplus h_2) \not< \Delta' * e_1 \mapsto e_3\) and \(h = h_1 \uplus h_2\), so we now have \((s, h) \not< \Delta' * e_1 \mapsto e_3\), therefore \(M * e_1 \mapsto e_3 \models \Delta' * e_1 \mapsto e_3\).

6. **Ls-Start-R.** This case is symmetric to Case 5.

7. **Match.** Here, we know that \(M \not< \Delta \wedge e_2 = e_3\) and \(M \not< \Delta' \wedge e_2 = e_3\). We want to show that \(M * e_1 \mapsto e_2 \models \Delta * e_1 \mapsto e_2\) and \(M * e_1 \mapsto e_2 \models \Delta' * e_1 \mapsto e_2\).

Let us first show \(M * e_1 \mapsto e_2 \models \Delta * e_1 \mapsto e_2\). Because we know that \(M \not< \Delta\) and \(e_1 \mapsto e_2 \models e_1 \mapsto e_2\), it follows that \(M * e_1 \mapsto e_2 \not< \Delta * e_1 \mapsto e_2\).

Let us now show \(M * e_1 \mapsto e_2 \not< \Delta' * e_1 \mapsto e_2\). We know \(M \not< \Delta' \wedge e_2 = e_3\), now suppose that \((s, h) \not< M * e_1 \mapsto e_2\), so \((s, h) \not< \Delta' \wedge e_2 = e_3\) as well. This means that \(e_2 = e_3\) in state \(s\), and therefore it must also be the case that \((s, h) \not< e_1 \mapsto e_3\). Given what else we know, we conclude that \((s, h) \not< \Delta' * e_1 \mapsto e_3\).

8. **Ls-End-1.** We want to show \(M * \text{ls}(e_1, e_2) \not< \Delta * \text{ls}(e_1, e_2)\) and \(M * \text{ls}(e_1, e_2) \not< \Delta' * \text{ls}(e_1, e_2)\).

Here, we know \(M \not< \Delta \wedge e_2 = e_3\) which, by the generic semantics, means \(M \not< \Delta\) and \(M \not< e_2 = e_3\).
We first define the renaming procedure in Algorithm 3, which is identical to that of Calcagno et al. If we know that \( M \vdash \Delta \), we get that \( M \ast (e_1, e_2) \vdash \Delta \ast ls(e_1, e_2) \).

Let us show \( M \ast ls(e_1, e_2) \vdash \Delta \ast ls(e_1, e_2) \). Given that \( M \vdash \Delta \), we get that \( M \ast ls(e_1, e_2) \vdash \Delta \ast ls(e_1, e_2) \).

Let us now show \( M \ast ls(e_1, e_2) \vdash \Delta \ast ls(e_1, e_3) \). Given that \( M \vdash \Delta \land e_2 = e_3 \), we get that \( M \ast ls(e_1, e_2) \vdash \Delta \ast ls(e_1, e_2) \land e_2 = e_3 \). We can also clearly see that \( ls(e_1, e_2) \land e_2 = e_3 \not\equiv ls(e_1, e_3) \), and so \( M \ast ls(e_1, e_2) \vdash \Delta \ast ls(e_1, e_3) \).

(9) **Ls-End-2.** Here, we know that \( M \vdash \Delta \ast ls(e_1, e_2) \) and \( M \vdash \Delta \). We want to show that \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_2) \) and \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_3) \).

Let us first show \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_2) \). First, we know that \( M \vdash \Delta \ast ls(e_1, e_2) \), and so \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_3) \ast ls(e_1, e_3) \). Clearly, it is also the case that \( ls(e_1, e_2) \ast ls(e_1, e_3) \not\equiv ls(e_1, e_3) \), and so combining these facts we get \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_2) \).

Let us now show \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_3) \). We know that \( M \vdash \Delta \), so clearly \( M \ast ls(e_1, e_3) \vdash \Delta \ast ls(e_1, e_3) \).

(10) **Ls-End-3.** This case is symmetric to Case 10.

(11) **Missing-L.** Here, we know that \( M \vdash \Delta \) and \( M \vdash \Pi \land (\Sigma \ast true) \). This means we also know that \( M \vdash \Pi \) and \( M \equiv \Sigma \ast true \) by semantic definition.

Let us first show that \( M \ast B(e_1, e_2) \vdash \Delta \ast B(e_1, e_2) \). Suppose that \( (s, h) \vdash M \ast B(e_1, e_2) \). We know that \( (s, h_1) \vdash M \) and \( (s, h_2) \vdash B(e_1, e_2) \) for some \( h_1 \) and \( h_2 \) such that \( h = h_1 \uplus h_2 \). Since \( M \vdash \Delta \), then \( (s, h_1) \vdash \Delta \). This means that \( (s, h) \vdash \Delta \ast B(e_1, e_2) \), therefore \( M \ast B(e_1, e_2) \vdash \Delta \ast B(e_1, e_2) \).

Let us now show that \( M \ast B(e_1, e_2) \vdash \Pi \ast (\Sigma \ast true) \). Here, we know that \( M \vdash \Pi \ast (\Sigma \ast true) \) and trivially \( B(e_1, e_2) \vdash true \). This means \( M \ast B(e_1, e_2) \vdash \Pi \ast (\Sigma \ast true) \ast true \); therefore, \( M \ast B(e_1, e_2) \vdash \Pi \ast (\Sigma \ast true) \).

(12) **Missing-R.** This case is symmetric to Case 11.

(13) **Emp-Ls-L.** We know that \( M \vdash \Delta \land e_1 \equiv e_2 \) and \( M \vdash \Delta \land e_1 \equiv e_2 \). This means that \( M \vdash \Delta \), so the right side of the tri-abductive judgement is taken care of.

Let us now show \( M \vdash \Delta \ast ls(e_1, e_2) \). We first establish that \( e_1 \equiv e_2 \land emp \equiv ls(e_1, e_2) \) by definition, since \( ls(e_1, e_2) \equiv (emp \land e_1 \equiv e_2) \lor \exists X. e_1 \leftrightarrow X \ast ls(X, e_2) \).

Now, we know that \( M \vdash \Delta \land e_1 \equiv e_2 \), which also means that \( M \vdash (\Delta \ast emp) \land e_1 \equiv e_2 \), or equivalently, \( M \vdash (\Delta \land e_1 \equiv e_2) \ast (emp \land e_1 \equiv e_2) \). Using \( e_1 \equiv e_2 \land emp \equiv ls(e_1, e_2) \), we get \( M \vdash (\Delta \land e_1 \equiv e_2) \ast ls(e_1, e_2) \), and by weakening we get \( M \vdash \Delta \ast ls(e_1, e_2) \).

(14) **Emp-Ls-R.** This case is symmetric to Case 13.

\[ \square \]

**Theorem 5.1 (Tri-abduction).** If \((M, F_1, F_2) \in triab(p, Q)\), then \( M \equiv P \ast F_1 \) and \( M \equiv Q \ast F_2 \)

**Proof.** In our tri-abduction algorithm, we call abduce-par on \( P \ast true \) and \( Q \ast true \), so we know based on Lemma E.1 that if \( P \ast true \not\equiv [M \Rightarrow Q \ast true \) is derivable, then \( M \vdash P \ast true \) and \( M \vdash Q \ast true \) since abduce-par operates by applying the inference in Figure 3. The procedure for finding \( F_1 \) and \( F_2 \) follows that of Berdine et al. [2005, §5] and so \( M \equiv P \ast F_1 \) and \( M \equiv Q \ast F_2 \) by Berdine et al. [2005b, Theorem 7].

\[ \square \]

F SYMBOLIC EXECUTION

F.1 Renaming

We first define the renaming procedure in Algorithm 3, which is identical to that of Calcagno et al. [2011, Fig. 4] except that we additionally require \( \vec{e} \) to be disjoint from \( \vec{x} \). Renaming produces a new anti-frame \( M_0 \) which is similar to \( M \) except that it is guaranteed not to mention any program variables and so it trivially meets the side condition of the frame rule. It additionally provides a vector of expressions \( \vec{e} \) to be substituted for the free variables in the postcondition \( Y \) so as to match \( M_0 \).
We also weaken the right hand side by replacing \( \bar{e} \) with fresh existentially quantified variables in \( x \).

Now, we can existentially quantify both sides of the entailment. Since logical variables are fresh, \( \bar{e} \) is disjoint from \( \bar{Y} \) and \( x \), and \( \text{Var} \) such that \( \Delta \ast M' \vdash \Delta \ast M[\bar{e}/\bar{Y}] \).

**return** \((\bar{e}, \bar{Y}, M')\)

### Algorithm 3 rename\((\Delta, M, Q, Q, \bar{X}, \bar{x})\)

Let \( \bar{Y} \) be the free logical variables of \( Q \) and all the assertions in \( Q \).

Pick \( \bar{e} \) disjoint from \( \bar{Y} \) and \( \bar{x} \) such that \( \Delta \ast M \equiv \bar{e} = \bar{Y} \).

Pick \( M' \) disjoint from \( \bar{X}, \bar{Y}, \) and \( \text{Var} \) such that \( \Delta \ast M' \equiv \Delta \ast M[\bar{e}/\bar{Y}] \).

**return** \((\bar{e}, \bar{Y}, M')\)

Now, we recall the definitions of the following two procedures:

\[
\begin{align*}
\text{biab}'(\exists \bar{Z}. \Delta, Q, \psi, \bar{x}) &= \{ (M', (\psi \circ \exists \bar{X} \bar{F}[\bar{X}/\bar{x}]))[\bar{Y}]/\bar{Y}] \} \\
| (M, F) \in \text{biab}(\Delta, Q) \\
(\bar{e}, \bar{Y}, M') &= \text{rename}(\Delta, M, Q, \{\psi\}, \bar{Z})
\end{align*}
\]

\[
\begin{align*}
\text{triab}'(P_1, P_2, \psi_1, \psi_2, \bar{x}) &= \{ (M', (\psi_1 \circ \exists \bar{X} \bar{F}_1[\bar{X}/\bar{x}])[\bar{Y}]/\bar{Y}], \\
(\psi_2 \circ \exists \bar{X} \bar{F}_2[\bar{X}/\bar{x}]))[\bar{Y}]/\bar{Y}] ) \\
| (M, F_1, F_2) \in \text{triab}(P_1, P_2) \\
(\bar{e}, \bar{Y}, M') &= \text{rename}(\text{emp}, M, \{\psi_1, \psi_2\}, \emptyset)
\end{align*}
\]

The biab’ procedure is similar to AbduceAndAdapt from Calcagno et al. [2011, Fig. 4]. Since the bi-abduction procedure does not support existentially quantified assertions on the left hand side, the existentials must be stripped and then re-added later (as is also done in Calcagno et al. [2011, Algorithm 4]). The renaming step ensures that the anti-frame \( M' \) is safe to use with the frame rule.

We capture the motivation behind biab’ using the following correctness lemma, stating that biab’ produces a suitable frame and anti-frame so as to adapt a specification \( \equiv \text{ok} : Q \) to use a different precondition \( P \).

**Lemma F.1.** For all \((M, \psi') \in \text{biab}'(P, Q, \psi, \bar{x})\), if \( \equiv \text{ok} : Q \) C \( \langle \psi \rangle \) and \( \bar{x} = \text{mod}(C) \), then

\[
\equiv \text{ok} : P \ast M \) C \langle \psi' \rangle
\]

**Proof.** By definition, any element of \( \text{biab}'(P, Q, \psi, \bar{x}) \) (where \( P = \exists \bar{Z}. \Delta \) must have the form \((M', (\psi \circ \exists \bar{Z} \bar{X} \bar{F}[\bar{X}/\bar{x}])\)\)]\(\bar{Y}))/\bar{Y}])\) where \( (\bar{e}, \bar{Y}, M') = \text{rename}(\Delta, M, Q, \{\psi\}, \bar{Z}) \) and \((M, F) \in \text{biab}(\Delta, Q).\)

By the definition of rename, we know that:

\[
\Delta \ast M' \equiv \Delta \ast M[\bar{e}/\bar{Y}] \]

Since \( M' \) is assumed to be disjoint from \( \bar{Z} \), then \( \exists \bar{Z}. M' \) if \( M' \), so we can existentially quantify both sides to obtain:

\[
P \ast M' \equiv \exists \bar{Z}. \Delta \ast M[\bar{e}/\bar{Y}] \tag{1}
\]

In addition, \((M, F) \in \text{biab}(\Delta, Q), \) so we also know that:

\[
\Delta \ast M \equiv Q \ast F \tag{2}
\]

In Figure 5, we assumed all the logical variables used are fresh, so \( \Delta \) must be disjoint from \( \bar{Y} \) (the free variables of \( Q \) and \( \psi \)), and therefore \( \Delta[\bar{e}/\bar{Y}] = \Delta \), so substituting both sides, we get:

\[
\Delta \ast M[\bar{e}/\bar{Y}] \equiv (Q \ast F)[\bar{e}/\bar{Y}] \tag{3}
\]

We also weaken the right hand side by replacing \( \bar{x} \) with fresh existentially quantified variables in \( F \).

\[
\Delta \ast M[\bar{e}/\bar{Y}] \equiv (Q \ast \exists \bar{X} \bar{F}[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}] \tag{4}
\]

Now, we can existentially quantify both sides of the entailment. Since logical variables are fresh, \( \bar{Z} \) is disjoint from \( Q \).

\[
\exists \bar{Z}. \Delta \ast M[\bar{e}/\bar{Y}] \equiv (Q \ast \exists \bar{Z} \bar{X} \bar{F}[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}] \tag{5}
\]

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And finally, we combine Equations (1) and (2) to get:

\[ P \ast M' \models (Q \ast \exists \vec{X}. F[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}] \tag{3} \]

Now, given that \( \langle \text{ok} : Q \rangle C \langle \psi \rangle \), we can use the frame rule to get:

\[ \vdash \langle \text{ok} : Q \ast \exists \vec{X}. F[\vec{X}/\vec{x}] \rangle C (\psi \ast \exists \vec{X}. F[\vec{X}/\vec{x}]) \]

This is clearly valid, since \( \vec{x} = \text{mod}(C) \) has been removed from the assertion that we are framing in, therefore satisfying \( \text{mod}(C) \cap \text{fv}(\exists \vec{X}. F[\vec{X}/\vec{x}]) = \emptyset \). We can also substitute \( \vec{e} \) for \( \vec{Y} \) in the pre- and postconditions since we assumed that \( \vec{e} \) is disjoint from the program variables \( \vec{x} \), and therefore \( \vec{e} \) remains constant after executing \( C \).

\[ \vdash \langle \text{ok} : (Q \ast \exists \vec{X}. F[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}] \rangle C ((\psi \ast \exists \vec{X}. F[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}]) \]

Finally, using the rule of consequence with Equation (3), we strengthen the precondition to get:

\[ \vdash \langle \text{ok} : P \ast M' \rangle C ((\psi \ast \exists \vec{X}. F[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}]) \]

The triab’ procedure is similar, but it is fundamentally based on tri-abduction and is accordingly used for parallel composition instead of sequential composition. We include two separate proofs corresponding to the two ways in which tri-abduction is using during symbolic execution. The first (Lemma F.2) pertains to merging the anti-frames obtained by continuing to evaluate a single program \( C \) after the control flow has already branched whereas the second (Lemma F.3) deals with merging the preconditions from two different program program branches, \( C_1 \) and \( C_2 \).

**Lemma F.2.** If \((M, \psi_1', \psi_2') \in \text{triab}'(M_1, M_2, \psi_1, \psi_2, \vec{x})\) and \(\vdash \langle \psi_1 \ast M_1 \rangle C \langle \psi_1 \rangle \) and \(\vdash \langle \psi_2 \ast M_2 \rangle C \langle \psi_2 \rangle \) and \(\vec{x} = \text{mod}(C)\), then \(\vdash \langle \psi_1 \ast M \rangle C \langle \psi_1' \rangle \) and \(\vdash \langle \psi_2 \ast M \rangle C \langle \psi_2' \rangle \).

**Proof.** By definition, any element of \(\text{triab}'(M_1, M_2, \psi_1, \psi_2, \vec{x})\) will have the form:

\[ \left( M', (\psi_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}], (\psi_2 \ast \exists \vec{X}. F_2[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}] \right) \]

Where \( (\vec{e}, \vec{Y}, M') = \text{rename}(\text{emp}, M, \{\psi_1, \psi_2\}, \emptyset, \vec{x}) \) and \((M, F_1, F_2) \in \text{triab}(P_1, P_2)\). From rename, we know that \( M' \models M[\vec{e}/\vec{Y}] \) and from tri-abduction, we know that \( M \models M_i \ast F_i \) for \( i = 1, 2 \), so \( M' \models (M_1 \ast F_1)[\vec{e}/\vec{Y}] \). We can weaken this by replacing \( \vec{x} \) in \( F_1 \) with fresh existentially quantified variables to obtain \( M' \models (M_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}] \). By assumption, we know that \(\vdash \langle \psi_1 \ast M_i \rangle C \langle \psi_i \rangle \) for \( i = 1, 2 \). So, using the frame rule, we get:

\[ \vdash \langle \psi_1 \ast (M_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}]) \rangle C \langle \psi_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}] \rangle \]

This is valid since \( \exists \vec{X}. F_1[\vec{X}/\vec{x}] \) is disjoint from \( \vec{x} \) (the modified program variables) by construction. We also assumed in rename that \( \vec{e} \) is disjoint from \( \vec{x} \), so we can substitute \( \vec{e} \) for \( \vec{Y} \) to get:

\[ \vdash \langle (\psi_1 \ast (M_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}]))[\vec{e}/\vec{Y}] \rangle C \langle (\psi_1 \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}])i[\vec{e}/\vec{Y}] \rangle \]

Note that \( \psi_1[\vec{e}/\vec{Y}] = \psi_i \), since the logical variables \( \vec{Y} \) are generated freshly, independent of \( \psi_i \), as was mentioned in Figure 5. So, using the rule of consequence we get:

\[ \vdash \langle \psi_i \ast M \rangle C \langle (\psi_i \ast \exists \vec{X}. F_1[\vec{X}/\vec{x}])[\vec{e}/\vec{Y}] \rangle \]

**Lemma F.3.** If \((M, \psi_1', \psi_2') \in \text{triab}'(P_1, P_2, \psi_1, \psi_2, \vec{x})\) and \(\vdash \langle \text{ok} : P_1 \rangle C_1 \langle \psi_1 \rangle \) and \(\vdash \langle \text{ok} : P_2 \rangle C_2 \langle \psi_2 \rangle \) and \(\vec{x} = \text{mod}(C_1, C_2)\), then \(\vdash \langle \text{ok} : M \rangle C_1 \langle \psi_1' \rangle \) and \(\vdash \langle \text{ok} : M \rangle C_2 \langle \psi_2' \rangle \).
Proof. By definition, any element of $\text{tria}(P_1, P_2, \psi_1, \psi_2, \bar{x})$ will have the form:

$$\left(M', (\psi_1 \circ \exists \bar{x}.F_1[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}], (\psi_2 \circ \exists \bar{x}.F_2[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}]\right)$$

Where $(\bar{e}, \bar{Y}, M') = \text{rename}(\text{emp}, M, \{\psi_1, \psi_2\}, \emptyset, \bar{x})$ and $(M, F_1, F_2) \in \text{tria}(P_1, P_2)$. From rename, we know that $M' = M[\bar{e}/\bar{Y}]$ and from tri-abduction, we know that $M \models P_i \circ F_i$ for $i = 1, 2$, so $M' \models (P_i \circ F_i)[\bar{e}/\bar{Y}]$. We can weaken this by replacing $\bar{x}$ in $F_i$ with fresh existentially quantified variables to obtain $M' \models (P_i \circ F_i)[\bar{e}/\bar{Y}]$.

By the frame rule, we know that $\models (\text{ok} : P_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}]) C_i (\psi_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}])$ since $\exists \bar{x}.F_i[\bar{X}/\bar{x}]$ must be disjoint from the modified program variables $\bar{x}$. By substituting into both the pre and postconditions, we get $\models (\text{ok} : P_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}]) C_i ((\psi_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}])$ (this is valid since rename guarantees that $\bar{e}$ is disjoint from $\bar{x}$). Finally, we complete the proof by applying the rule of consequence with $M' \models (P_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}]$ to obtain:

$$\models (\text{ok} : M') C_i ((\psi_i \circ \exists \bar{x}.F_i[\bar{X}/\bar{x}])[\bar{e}/\bar{Y}])$$

$\square$

F.2 Sequencing Proof

Lemma 6.2 (Seq). If $\models (\text{ok} : P) C \langle \theta \rangle$ for all $(P, \theta) \in S$ and $\bar{x} = \text{mod}(C)$ and $(M, \psi) \in \text{seq}(\varphi, S, \bar{x})$, then $\models (\varphi \circ M) C \langle \psi \rangle$

Proof. By induction on the structure of $\varphi$.

- $\varphi = T$. We need to show that $\models (T \circ \text{emp}) C \langle T \rangle$ holds. This triple is clearly valid since any triple with the postcondition $T$ is trivially true.
- $\varphi = T^\emptyset$. We need to show that $\models (T^\emptyset \circ \text{emp}) C \langle T^\emptyset \rangle$ holds. This triple is clearly valid since only $\emptyset$ satisfies the precondition and $[C](\emptyset) = \emptyset$.
- $\varphi = \varphi_1 \triangleright \varphi_2$ where $\triangleright \in \{\lor, \triangleright_a\}$. We need to show:

$$\models (\langle \psi_1 \triangleright \psi_2 \rangle \circ M) C \langle \psi_1 \triangleright \psi_2 \rangle$$

Where $(M, \psi_1', \psi_2') \in \text{tria}(M_1, M_2, \psi_1, \psi_2, \bar{x})$ and $(M_i, \psi_i) \in \text{seq}(\varphi_i, S, \bar{x})$ for each $i \in \{1, 2\}$. By the induction hypothesis, we know that $\models (\varphi_1 \circ M_1) C \langle \psi_1 \rangle$, so by Lemma F.2 we get that $\models (\varphi_1 \circ M) C \langle \psi_1 \rangle$. We now complete the proof separately for the two logical operators:

- $\varphi = \varphi_1 \lor \varphi_2$. Suppose that $m \equiv (\varphi_1 \lor \varphi_2) \circ M$, so $m \equiv \varphi_1 \circ M$ for some $i \in \{1, 2\}$. Since $\models (\varphi_1 \circ M) C \langle \psi_1 \rangle$, we know that $[C]^\emptyset(m) \models \psi_1$, and we can weaken this to conclude that $[C]^\emptyset(m) \models \psi_1'$. We also know that $[C]^\emptyset(m) = [C]^\emptyset(a \cdot m_1 + \overline{a} \cdot m_2) = a \cdot [C]^\emptyset(m_1) + \overline{a} \cdot [C]^\emptyset(m_2)$, and $[C]^\emptyset(m) \equiv \psi'_1 \circ \varphi_2$. We also know that $[C]^\emptyset(m) = \psi_1' \circ \varphi_2$.

- $\varphi = \varphi_1 \triangleright_a \varphi_2$. Suppose that $m \equiv (\varphi_1 \triangleright_a \varphi_2) \circ M$, and so there are $m_1$ and $m_2$ such that $m = a \cdot m_1 + \overline{a} \cdot m_2$ and $m_i \equiv \varphi_i \circ M$ for each $i$. Since $\models (\varphi_i \circ M) C \langle \psi_i \rangle$, we know that $[C]^\emptyset(m_1) \models \psi_1'$, $[C]^\emptyset(m_2) \models \psi_2'$, and we also know that $[C]^\emptyset(m) = [C]^\emptyset(a \cdot m_1 + \overline{a} \cdot m_2) = a \cdot [C]^\emptyset(m_1) + \overline{a} \cdot [C]^\emptyset(m_2)$, and $[C]^\emptyset(m) \equiv \psi_1' \circ \varphi_2$. We also know that $[C]^\emptyset(m) = \psi_1' \circ \varphi_2$.

- $\varphi = \text{ok} : P$. We need to show that $\models (\text{ok} : P \circ M) C \langle \psi \rangle$ where $(M, \psi') \in \text{biab}(P, Q, \psi, \bar{x})$ and $(Q, \psi) \in S$. By assumption, we know that $\models (\text{ok} : Q) C \langle \psi \rangle$. The remainder of the proof follows directly from Lemma F.1.

- $\varphi = \text{er} : Q$. We need to show that $\models (\text{er} : Q) C \langle \text{er} : Q \rangle$. This trivially holds since any $m$ satisfying $\text{er} : Q$ must consist only of $1_{\text{er}}(s, h)$ states, and so $[C]^\emptyset(m) = m$.

$\square$
F.3 Symbolic Execution Proofs

**Lemma F.4.** Let:

\[
\begin{align*}
\mathcal{L}(S) = \& \left\{ (e \land \text{emp}, \text{ok} : \neg e \land \text{emp}) \right\} \\
\& \left\{ (M_1 \ast M_2 \wedge e, \psi) \mid (M_1, \varphi) \in \text{seq}(e : e \land \text{emp}, [C]^\#(T), \text{mod}(C)), (M_2, \psi) \in \text{seq}(\varphi, S, \text{mod}(C)) \right\}
\end{align*}
\]

For any \( n \in \mathbb{N} \) and \((P, \varphi) \in \mathcal{L}(\emptyset), \models \langle \text{ok} : P \rangle \) while \( e \) do \( C \langle \varphi \rangle \).

**Proof.** By induction on \( n \). Suppose \( n = 0 \), then \( f^0(\emptyset) = \emptyset \), so the claim vacuously holds. Now, suppose the claim holds for \( n \), we will show it holds for \( n + 1 \). First, observe that:

\[
\begin{align*}
f^{n+1}(\emptyset) &= f(f^n(\emptyset)) \\
&= \left\{ (e \land \text{emp}, \text{ok} : \neg e \land \text{emp}) \right\} \\
&\cup \left\{ (M_1 \ast M_2 \wedge e, \psi) \mid (M_1, \varphi) \in \text{seq}(e : e \land \text{emp}, [C]^\#(T), \text{mod}(C)), (M_2, \psi) \in \text{seq}(\varphi, f^n(\emptyset), \text{mod}(C)) \right\}
\end{align*}
\]

So any \((P, \varphi) \in \mathcal{L}(\emptyset)\) comes from one of the two sets in the above union. Suppose it is in the first, so we need to show that \(\models \langle \text{ok} : \neg e \land \text{emp} \rangle \) while \( e \) do \( C \langle \text{ok} : \neg e \land \text{emp} \rangle \). This is clearly true, since the loop does not execute in states where \( \neg e \) holds and therefore the whole command is equivalent to skip.

Now suppose we are in the second case, so the element has the form \( (M_1 \ast M_2 \wedge e, \psi) \) where \( (M_1, \varphi) \in \text{seq}(e : e \land \text{emp}, [C]^\#(T), \text{mod}(C)) \) and \( (M_2, \psi) \in \text{seq}(\varphi, f^n(\emptyset), \text{mod}(C)) \). By Lemma D.7, we know that \(\models \langle \text{ok} : M_1 \wedge e \rangle C \langle \varphi \rangle \) and by Lemma D.7 and the induction hypothesis, we get \(\models \langle \varphi \uplus M_2 \rangle \) while \( e \) do \( C \langle \varphi \rangle \). Using the frame rule, we also get \(\models \langle \text{ok} : M_1 \ast M_2 \wedge e \rangle C \langle \varphi \uplus M_2 \rangle \), and so we can sequence the previous specifications to get \(\models \langle \text{ok} : M_1 \ast M_2 \wedge e \rangle C \langle \varphi \rangle \) while \( e \) do \( C \langle \varphi \rangle \). Now, since the precondition stipulates that \( e \) is true, the loop must run for at least one iteration, so for any \( m \models \langle \text{ok} : M_1 \ast M_2 \wedge e \rangle C \langle \varphi \rangle \) while \( e \) do \( C \langle \varphi \rangle \), and so \(\models \langle \text{ok} : M_1 \ast M_2 \wedge e \rangle \) while \( e \) do \( C \langle \varphi \rangle \).

**Lemma F.5.** If for every \((s, h) \models \text{P}, \exists s' \text{ and } t' \text{ such that } [C](s, h) = \text{unit}_W(1_e(s', t')) \) and \((s', h') \models \text{Q}, \exists \langle \text{ok} : P \rangle C \langle \epsilon : Q \rangle \).

**Proof.** Suppose that \( m \models \text{ok} : P \). That means that \( |m| = 1 \) and all elements of supp\( (m) \) have the form \( 1_{ok}(s, h) \) where \( (s, h) \not\models \text{P} \). By assumption, we know that \( [C](s, h) = \text{unit}_W(1_e(s', t')) \) such that \( (s', t') \models \text{Q} \). This means that every element of \([C](m)\) must have the form \( 1_{\epsilon}(s', t') \) and also \( |[C](m)| = 1 \) since each \( C \) does not change the mass of the distribution, so \( [C](m) \models \epsilon : Q \).

**Theorem 6.1 (Symbolic Execution Soundness).** If \((P, \varphi) \in [C]^\#(T), \models \langle \text{ok} : P \rangle C \langle \varphi \rangle \).

**Proof.** By induction on the structure of the program \( C \).

- \( C = \text{skip} \). We need to show that \(\models \langle \text{ok} : \text{emp} \rangle \text{skip} \langle \text{ok} : \text{emp} \rangle \), which is trivially true.
- \( C = C_1 \uplus C_2 \). By definition, any element of \([C_1 \uplus C_2]^\#(T)\) must have the form \((P \ast M, \psi)\) where \((P, \varphi) \in [C_1]^\#(T) \) and \((M, \psi) \in \text{seq}(\varphi, [C_2]^\#(T), \text{mod}(C_2))\). By the induction hypothesis, we know that \(\models \langle \text{ok} : P \rangle C_1 \langle \varphi \rangle \) and by Lemma D.7 we know that \(\models \langle \varphi \uplus M \rangle C_2 \langle \psi \rangle \). Using the frame rule, we get that \(\models \langle \text{ok} : P \uplus M \rangle C_1 \langle \varphi \uplus M \rangle \) (given the renaming step used in seq, \( M \) contains no program variables, so it must obey the side condition of the frame rule). Finally, we can join the two specifications to conclude that \(\models \langle \text{ok} : P \uplus M \rangle C_1 \uplus C_2 \langle \psi \rangle \).
- \( C = \text{if } e \text{ then } C_1 \text{ else } C_2 \). Any element of \([\text{if } e \text{ then } C_1 \text{ else } C_2]^\#(T)\) must either have the form \((P \wedge e, \varphi)\) where \((P, \varphi) \in \text{seq}(e : e \land \text{emp}, [C_1]^\#(T), \text{mod}(C_1)) \) or \((P \wedge \neg e, \varphi)\) where \((P, \varphi) \in \)
seq(\text{ok} : \neg e \land \text{emp}, [C_2]^T (T), \text{mod}(C_2))$. Suppose the former, then by the induction hypothesis and Lemma D.7, we know that \( \models \langle \text{ok} : P \land e \rangle C_1 \langle \psi \rangle \). Since \( e \) must be true for all states satisfying the precondition, the result of running \( C_1 \) is the same as running if \( e \) then \( C_1 \) else \( C_2 \) (the true branch will always be taken), so \( \models \langle \text{ok} : P \land e \rangle \) if \( e \) then \( C_1 \) else \( C_2 \langle \psi \rangle \). The case for the false branch is symmetric.

\( C = \text{while } e \text{ do } C \). By the Kleene fixed point theorem, \( [\text{while } e \text{ do } C]^T = \bigcup_{n \in \mathbb{N}} f^n(\emptyset) \) where \( f(S) \) is defined as in Lemma F.4. So, any \( (P, \varphi) \in [\text{while } e \text{ do } C]^T \) must also be an element of \( f^n(\emptyset) \) for some \( n \). We complete the proof by applying Lemma F.4.

\( C = C_1 +_a C_2 \). Any element of \( [C_1 +_a C_2]^T \) must have the form \( (M, (\psi'_1 \oplus_a \psi'_2)) \) where \( (M, \psi'_1, \psi'_2) \in \text{tria} \cdot (M_1, \psi_1, \psi_2, \text{mod}(C_1, C_2)) \) and \( (M, \psi_1) \in [C_1]^T \) and \( (M_2, \psi_2) \in [C_2]^T \). By the induction hypothesis, we know that \( \models \langle \text{ok} : M_1 \rangle C_1 \langle \psi_1 \rangle \) for \( i = 1, 2 \). By Lemma F.3 we know that \( \models \langle \text{ok} : M \rangle C_i \langle \psi'_i \rangle \). Now, we show that \( \models \langle \text{ok} : M \rangle C_1 +_a C_2 \langle \psi'_1 \oplus_a \psi'_2 \rangle \) by definition, \( [C_1 +_a C_2]^T (m) = a \cdot [C_1]^T (m) + \overline{a} \cdot [C_2]^T (m) \). Now, using what we obtained from Lemma F.3, we know that since \( m \models M \), \( [C_1]^T (m) \models \psi'_1 \) for each \( i \in \{1, 2\} \). Combining these two, we get that \( a \cdot [C_1]^T (m) + \overline{a} \cdot [C_2]^T (m) \models \psi'_1 \oplus_a \psi'_2 \).

The remaining cases are for primitive instructions, most of which are pure, meaning that each program state maps to a single outcome according to the program semantics. In these cases, it suffices to show that if \( (P, \varphi) \in [c]^T \), then \( [c] (s, h) \models \varphi \) for all \( (s, h) \models P \) by Lemma F.5.

\( C = (x := e) \). Suppose that \( (s, h) \models \text{ok} : x = X \land \text{emp} \), so \( s(x) = s(X) \) and \( h = \emptyset \). Now, \( [x := e] (s, h) = \text{unit}(s[x \mapsto e] (s), h) \), so let \( s' = s[x \mapsto [e] (s)] \). Clearly, \([e] (s) = [e[X/x]] (s) \) since \( s(x) = s(X) \). It must also be the case that \( [e[X/x]] (s) = [e[X/x]] (s') \) since \( s \) and \( s' \) differ only in the values of \( x \), and \( x \) does not appear in \( e[X/x] \). So, \( s'(x) = [e] (s) = [e[X/x]] (s) = [e[X/x]] (s') \), and therefore \( (s', h) \models \text{ok} : x = e[X/x] \land \text{emp} \). The remainder of the proof follows by Lemma F.5.

\( C = (x := \text{alloc}()) \). We need to show that \( \models \langle \text{ok} : \text{emp} \land x = X \rangle x := \text{alloc} () \langle \text{ok} : \exists Y \cdot x \mapsto Y \rangle \). Suppose that \( m \models \text{ok} \land x = X \), so each state in \( m \) has the form \( \Delta_{\text{ok}, s, h} \) where \( (s, h) \models \text{emp} \land x = X \), so \( s(x) = s(X) \) and \( h = \emptyset \). We know that \( [x := \text{alloc}] (s, h) = \text{bind}_{\text{W}}(\text{alloc}(s, h), \lambda t. \text{unit}(s[x \mapsto t], h[t \mapsto \emptyset])) \) where \( t \) does not appear in \( s \) or \( h \). Let \( s' = s[x \mapsto t] \) and \( h' = h[t \mapsto \emptyset] \). Clearly, \( h'(s') = h(t) = 0 \), so \( (s', h') \models \exists Y \cdot x \mapsto Y \). Since this is true for all end states, and since alloc does not alter the total mass of the distribution, then \( [x := \text{alloc}()]^T (m) \models \text{ok} : \exists Y \cdot x \mapsto Y \).

\( C = \text{free}(x) \). There are three cases since specifications for free can take on multiple forms. In the first case, we need to show that \( \models \langle \text{ok} : e \mapsto X \rangle \text{free}(e) \langle \text{ok} : e \not\mapsto \rangle \). Suppose \( (s, h) \models e \mapsto X \), so \( h([e] (s)) = s(X) \) and \( h \models \text{free}(e) (s, h) = \text{unit}(s, h[[e] (s) \mapsto \bot]) \), and since \( (s, h[[e] (s) \mapsto \bot]) \not\models e \not\mapsto \), the claim follows by Lemma F.5.

In the other cases, we need to show that:

\[ \langle \text{ok} : e \not\mapsto \rangle \text{free}(e) \langle \text{er} : e \not\mapsto \rangle \quad \text{and} \quad \langle \text{ok} : e = \text{null} \rangle \text{free}(e) \langle \text{er} : e = \text{null} \rangle \]

Suppose \( (s, h) \models e \not\mapsto \) and so \( h([e] (s)) = \bot \). Clearly, \( \text{free}(e) (s, h) = \text{error}(s, h) \), so by Lemma F.5 the claim holds. The case where \( e = \text{null} \) is nearly identical.

\( C = [e_1] \leftarrow e_2 \). There are three cases, first we must show that \( \models \langle \text{ok} : e_1 \mapsto X \rangle [e_1] \leftarrow e_2 \langle \text{ok} : e_1 \mapsto e_2 \rangle \). Suppose \( (s, h) \models e_1 \mapsto X \), so \( h([e_1] (s)) = s(X) \) and therefore that memory address is allocated since \( s(X) \in \text{Val} \). This means that \( [e_1] \leftarrow e_2 \langle s, h[[e_1] (s) \mapsto [e_2] (s)) \rangle \) and clearly \( (s, h[[e_1] (s) \mapsto [e_2] (s)) \not\models e_1 \mapsto e_2 \) by definition, so the claim holds by Lemma F.5.
In the remaining case, we must show that \( \models \langle \text{ok} : e_1 \neq \rangle \langle e_1 \rangle \leftarrow e_2 \langle \text{er} : e_1 \neq \rangle \) and \( \models \langle \text{ok} : e_1 = \text{null} \rangle \langle e_1 \rangle \leftarrow e_2 \langle \text{er} : e_1 = \text{null} \rangle \). The proof is similar to the second case for free(\( e \)).

\( C = x \leftarrow [e] \). There are three cases, first we must show that \( \models \langle \text{ok} : x = X \land e \mapsto Y \rangle x \leftarrow [e] \langle \text{ok} : x = Y \land e[X/x] \mapsto Y \rangle \). Suppose that \( (s, h) \models x = X \land e \mapsto Y \), so \( s(x) = s(X) \) and \( h([e] (s)) = s(Y) \). We also know that \([x \leftarrow [e]] (s, h) = \text{unit}(s[x \mapsto h([e] (s))]) \), let \( s' = s[x \mapsto h([e] (s))] \), as showed in the case \( x := e \), \([e] (s) = [e[X/x]] (s) = [e[X/x]] (s') \).

So, this means that \( h([e[X/x]] (s')) = h([e] (s)) = s(Y) \) and \( s'(x) = [e] (s) = s(Y) \), so clearly \( (s', h) \models x = X \land e[X/x] \mapsto Y \) and the claim follows from Lemma F.5.

In the remaining case, we need to show that \( \models \langle \text{ok} : e \neq \rangle x \leftarrow [e] \langle \text{er} : e \neq \rangle \) and \( \models \langle \text{ok} : e = \text{null} \rangle x \leftarrow [e] \langle \text{er} : e = \text{null} \rangle \). The proof is similar to the second case for free(\( e \)).

\( C = \text{f} \langle \text{e} \rangle \). Any element of \([\text{f} \langle \text{e} \rangle]^T \langle \text{T} \rangle \) must have the form \((P \land \text{x} = \text{X.X}) \) where \( (P, \varphi) \in \langle \text{seq} (\text{x} = \text{e}[\langle \text{X.X} \rangle, \langle \text{f} \langle \text{X.X} \rangle, \langle \text{mod} (\text{f}) \rangle) \rangle). \) By Lemma D.7, we get:

\[ \models \langle \text{ok} : P \land \text{x} = \text{e}[\langle \text{X.X} \rangle, \langle \text{f} \langle \text{X.X} \rangle, \langle \text{mod} (\text{f}) \rangle) \rangle \langle \varphi \rangle \]

Now, we need to show that \( \models \langle \text{ok} : P \land \text{x} = \text{X.X} \rangle \langle \text{f} \langle \text{e} \rangle \rangle \langle \varphi \rangle \). Suppose that \( m \models \text{ok} : P \land \text{X.X} \). Let \( m' \) be obtained by taking every state \( \hat{1}_{ok} (s, h) \in \text{supp} (m) \) and modifying it to be \( \hat{1}_{ok} (s, \text{X.X} \mapsto \text{X.X}) \). By a similar argument to the \( x := e \) case, we know that \( m' \models \text{ok} : \text{X.X} = \text{X.X} \). We know \( P \) is disjoint from the program variables by the definition of \( \text{seq} \), so \( m' \models \text{ok} : P \) as well, since \( m \models \text{ok} : P \) and the only difference between \( m \) and \( m' \) is updates to the program variables. Let \( C \) be the body of \( f \) and note that \([\text{f} \langle \text{X.X} \rangle]^T \langle \text{m} \rangle \) is the initial modification of the program state is just updating variable values to themselves.

We know from \( \models \langle \text{ok} : P \land \text{x} = \text{e}[\langle \text{X.X} \rangle, \langle \text{f} \langle \text{X.X} \rangle, \langle \text{mod} (\text{f}) \rangle) \rangle \langle \text{f} \langle \text{X.X} \rangle \rangle \langle \varphi \rangle \) that \([\text{C}]^T \langle \text{m} \rangle \models \varphi \) and by definition, \([\text{f} \langle \text{X.X} \rangle]^T \langle \text{m} \rangle \models \varphi \). In addition, we show that the two refinements for single-path computation and loops invariant

sound too:

- Single Path. Any element of \([C_1 +_a C_2]^T \langle \text{T} \rangle \) has one of two forms. In the first case, we need to show that \( \models \langle \text{ok} : P \rangle \langle C_1 +_a C_2 \rangle \langle \varphi \oplus_a \text{T} \rangle \) given that \( \models \langle \text{ok} : P \rangle \langle C_1 \rangle \langle \varphi \rangle \). Suppose that \( m \models \text{ok} : P \). By our assumption, we know that \( [C_1]^T \langle \text{m} \rangle \models \varphi \). Now, \( [C_1 +_a C_2]^T \langle \text{m} \rangle = a \cdot [C_1]^T \langle \text{m} \rangle + \overline{a} \cdot [C_2]^T \langle \text{m} \rangle \) and clearly \( [C_2]^T \langle \text{m} \rangle \models \text{T} \), so \( [C_1 +_a C_2]^T \langle \text{m} \rangle \models \varphi \oplus_a \text{T} \). The second case is symmetrical.

- Loop invariants. We need to show that \( \models \langle \text{ok} : I \rangle \) while \( e \) do \( C \langle (\text{ok} : I \land \neg e) \lor \text{T}^{\oplus} \rangle \) given that \( \models \langle \text{ok} : I \land e \rangle \langle C \langle \text{ok} : I \rangle \rangle \). Note that this case is only valid for deterministic or nondeterministic programs (not probabilistic ones). Suppose \( m \models \text{ok} : I \), so every state in \( \text{supp} (m) \) has the form \( \hat{1}_{ok} (s, h) \) where \( (s, h) \models I \). By assumption, we know that every execution of the loop body will preserve the truth of \( I \), so either all the states in \( \langle \\text{while} \; e \; \text{do} \; C \rangle (s, h) \) must satisfy \( I \land \neg e \) or there are no terminating states. In other words, \( \langle \\text{while} \; e \; \text{do} \; C \rangle (s, h) \models \langle \text{ok} : I \land \neg e \rangle \lor \text{T}^{\oplus} \).

In the deterministic case, we are done since there can only be a single start state. In the nondeterministic case, each start state \( (s, h) \) leads to a set of end states satisfying the \( \langle \text{ok} : I \land \neg e \rangle \lor \text{T}^{\oplus} \), then the union of all these states will also satisfy \( \langle \text{ok} : I \land \neg e \rangle \lor \text{T}^{\oplus} \).

□