NP-completeness reductions

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CS 4820, Summer 2020

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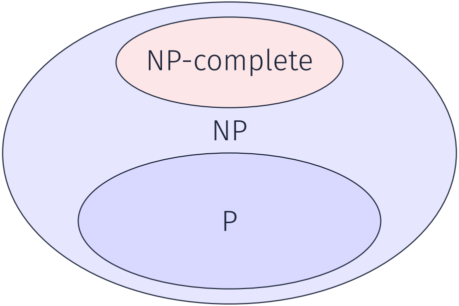

What now?

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One thing is for sure: it's going to be much easier to establish NP-completeness for any further problems. We just need to demonstrate a polynomial-time reduction from CNF-SAT.

NP-completeness architecture

$$
\begin{tikzcd}
\text{Graph Iso.} \arrow[drr] & \text{Ham. Cycle} \arrow[dr] & & \cdots \arrow[dl] & \text{Flow} \geq k \arrow[dll, "\text{poly}"] \\
& & \text{CNF-SAT} \arrow[d, "\text{poly}"] & & \\
& & \text{some other problem} & &
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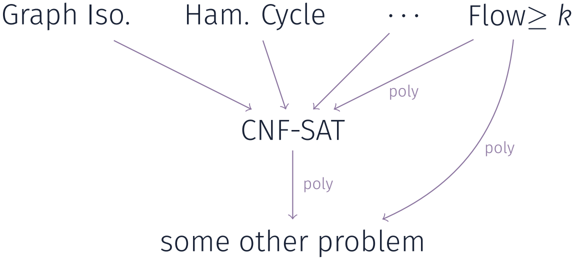

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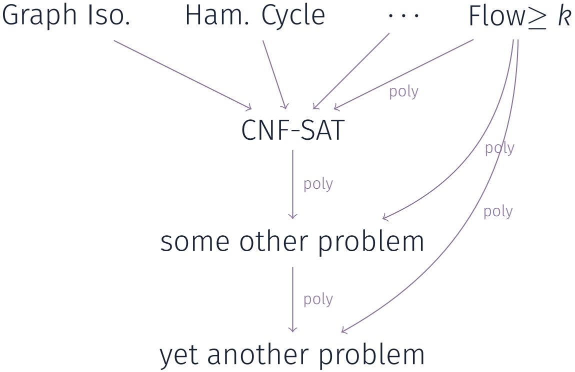

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The 3-CNF-SAT (short: 3-SAT) problem is basically the same thing, but with the stipulation that every clause contains exactly three literals:

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(We impose additional restrictions on the formulae, so this looks... easier? Either way, it turns out to sometimes be convenient for further reductions.)

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  - If the answer should be "no" – that is, φ was a "no" instance of CNF-SAT – then ψ should be a "no" instance of 3-SAT.

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Is this alone sufficient? That is, will we get a correct reduction if we just ensure there is a correspondence between φ and ψ clauses like that?

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Fix: Make source and target clauses satisfiable by the same assignment, to preserve how clauses interact. If we need extra variables, make them not interact with other source clauses' image at all.

From SAT to 3-SAT (3)

Too few literals: Just pad out with fresh variables.

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$$(\ell) \mapsto (\ell \lor y \lor z) \land (\ell \lor \neg y \lor z) \land (\ell \lor y \lor \neg z) \land (\ell \lor \neg y \lor \neg z).$$

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$$\begin{aligned} \textbf{Output:} \quad (\ell_1 \vee \ell_2 \vee z_3) \quad &\wedge \quad (\ell_3 \vee \neg z_3 \vee z_4) \\ &\wedge \quad \ldots \\ &\wedge \quad (\ell_{k-2} \vee \neg z_{k-2} \vee z_{k-1}) \\ \wedge\ (\ell_{k-1} \vee \ell_k \vee \neg z_{k-1}). \end{aligned}$$

Check that this works!

From SAT to 3-SAT (5)

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"$\Rightarrow$": Let i be the first i such that ℓ_i is true. Set $z_j = T$ for all $j \leq i$, $z_j = F$ for all $j > i$. If $i \leq 2$, then all z_j are false, and so the first output clause is satisfied by ℓ_i and the rest are satisfied by the appropriate $\neg z_j$. Otherwise, every clause that contains z_j for $j \leq i$ is satisfied by z_j, the one that contains ℓ_i and $\neg z_i$ is satisfied by ℓ_i, and the remaining ones are satisfied by $\neg z_j$ for the appropriate $j > i$.

From SAT to 3-SAT (6)

"$\Leftarrow$": Claim that some ℓ_i is true; then the input clause is satisfied by that literal. Suppose not. Then all of the ℓ_i are false, and so the output clauses simplify to

$$(z_3) \wedge (\neg z_3 \vee z_4) \wedge \ldots \wedge (\neg z_{k-2} \vee z_{k-1}) \wedge (\neg z_{k-1}).$$

This is not satisfiable (note that $(\neg x \vee y)$ is $x \rightarrow y$). $\square$

We created a polynomial number of clauses for each original clause with only straightforward computations, and can check that we have a satisfying assignment to the resulting CNF iff we have one to the original CNF.

Hence done, and 3-SAT is NP-complete. □

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This approach of taking discrete parts of the source problem and converting them to parts of the target problem, which interact with each other in a controlled manner, is common. We call the parts we construct in the target problem gadgets.

Intermission

Let's take a short break. **Exercise:** Try converting

$$(\neg x \vee \neg y) \wedge (\neg x \vee a) \wedge (\neg x \vee y \vee \neg a \vee \neg b \vee c) \wedge (x \vee \neg c) \wedge (x \vee c)$$

to 3-SAT, and find two distinct satisfying assignments.

The Independent Set problem

Given an undirected graph $G = (V, E)$, an independent set is a set of vertices $I \subseteq V$ such that no two vertices in I share an edge: $\nexists v, v' \in I.\, v \neq v' \wedge \{v, v'\} \in E$.

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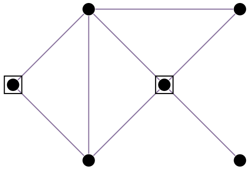

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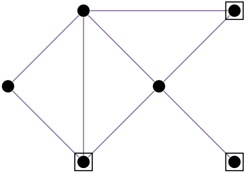

The Independent Set decision problem asks the following: given a graph $G = (V, E)$ and an integer k, is there an independent set of at least k vertices?

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Here is a not-quite-rigorous intuitive framework that I have personally found useful.

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This solution consists of many smaller components. For instance, for INDEPENDENT SET, the solution is an independent set, which amounts to a choice of whether we put in each vertex of the graph.

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This solution consists of many smaller components. For instance, for Independent Set, the solution is an independent set, which amounts to a choice of whether we put in each vertex of the graph.

We can imagine a process of picking these components one by one, as if we ran a greedy algorithm. Every time we pick something, some future choices are ruled out: e.g. including vertices adjacent to the vertex we just picked.

If at some point we run out of choices, we get stuck and have to backtrack: undo some number of choices, and try adding a different component instead.

Otherwise, at some point we've chosen everything there is to choose, and found a certificate that we have a "yes" instance.

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(The search tree we get in this fashion looks a lot like an NTM execution cone, and is in general of exponential size.)

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Then, an assignment to a variable is ruled out if we can immediately see it won't lead to a solution: e.g. we just made some clause unsatisfiable by picking the opposite of all of its literals.

How to proceed in general? (5)

Alternatively, we could think of it as a process of satisfying clauses one-by-one. Every clause needs to be satisfied by some literal in it. So first we say that we satisfy $(x_1 \lor \neg x_3 \lor x_7)$ by setting x_3 to F, then we satisfy $(x_3 \lor x_4)$ by setting x_4 to T, ...

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Then, a way of satisfying a clause is ruled out if we previously did something that contradicts it: e.g. we already satisfied another clause by setting a variable so that this literal is rendered false.

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Both views are valid and useful, and give rise to reductions that we will encounter.

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For INDEPENDENT SET, we will use the second one.

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Either way, once we have decided on a search tree structure for both the source and the target problem, we can build gadgets that simulate the structure of a source choice using target choices. Sometimes, we have to plug multiple of them together.

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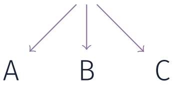
For instance, imagine the source problem lets you choose between three different options:



On the other hand, the target problem has choices that lead to two options (which don't interact with anything else), choices where your hand is forced (you only get one option), and choices that always lead to you getting stuck:

How to proceed in general? (9)

So can model the choice structure of the source problem with three choices of the target problem:

$$\begin{tikzcd}
 & & & {} \arrow[dll] \arrow[drr] & & & \\
 & A \arrow[dl] \arrow[dr] & & & & B \arrow[dl] \arrow[dr] & \\
A' \arrow[d] & & B' \arrow[d] & & A'' \arrow[d] & & B'' \arrow[d] \\
C_1 & & C_2 & & C_3 & & \color{red}{X}
\end{tikzcd}$$

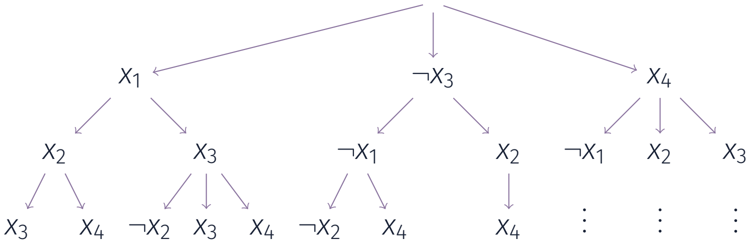
Towards NP-completeness of Independent Set

As said earlier, we will use the clause-satisfying view of 3-SAT.

Example: input formula

$$(x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee x_3 \vee x_4).$$

Search tree if we try to satisfy clauses left to right:



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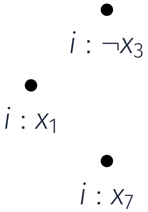

Replicating the choice structure (2)

Once we have already chosen to satisfy the ith clause using, say, x_1, we no longer need or want to choose another literal to satisfy the same clause.

In INDEPENDENT SET, we can make a choice (of vertex) block a subsequent choice (of vertex) by introducing edges between them, so we add edges between all the vertices of a clause:

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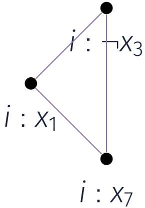

Also, if we choose to satisfy a clause by, say, $\neg x_3$, then we can't satisfy any other clauses using x_3 anymore: after all, in our assignment, x_3 would be set to F.

This is another straightforward case of two choices being mutually exclusive, which we can realise using an edge. So we add edges between any two contradictory literals:

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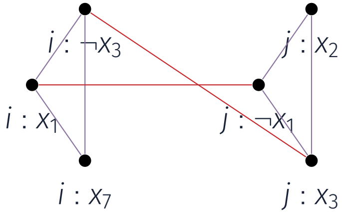
$$\underbrace{(\neg x_1 \vee x_2 \vee x_3)}_{j\text{th clause}}$$

$\mapsto$



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Conclude: Independent Set is NP-complete. □

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The GRAPH 3-COLOURING decision problem asks: given a graph, can it be vertex 3-coloured?

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Graph Colouring: Preliminaries (1)

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Suppose our three colours are with red, green and blue.

What colours could we give the unpainted (white) vertex here?

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What colours could we give the unpainted (white) vertex here?

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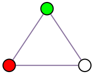

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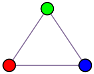
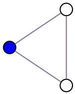


What colours could we assign to the two blank vertices here?



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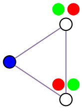

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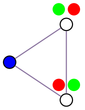

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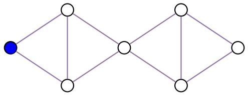

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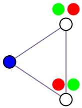

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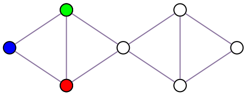

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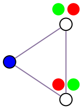

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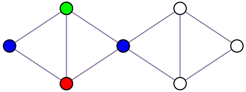

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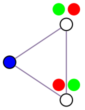

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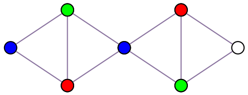

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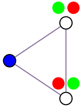

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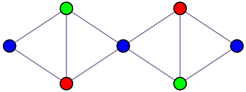

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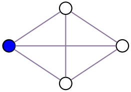

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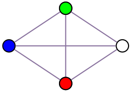

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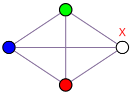

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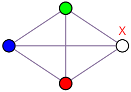

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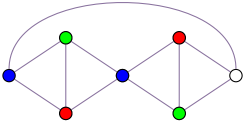
Doesn't have to be a complete graph:



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Natural framing of what we did while building intuition: repeatedly pick a colour for some vertex, getting stuck when a vertex can't be consistently coloured.

Choice structure of 3-colouring

First idea: In our view of 3-SAT, we pick either T or F for each variable.

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No: 3-colouring is invariant under permuting colours. That is, if we take a valid colouring and globally swap green and red, we still have a valid colouring.

But this is not true for T and F in 3-SAT assignments! (Consider $(x_1 \vee x_2 \vee x_3)$ with all true.)

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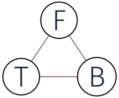

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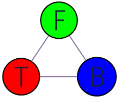

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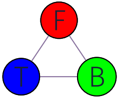

Let's do a test run of how we want the implicit solution to 3-SAT to be picked.

$$(x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3 \vee x_4).$$

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- $x_3 \mapsto T$. Stuck on clause 2!

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- Getting stuck when we happened to rule out all ways of making some clause true.

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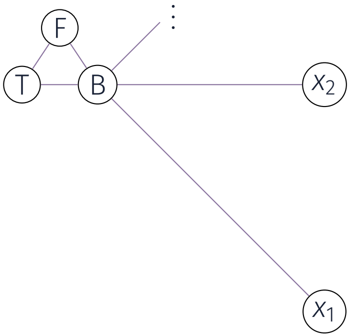
The first one ("variable gadget") is simple enough:



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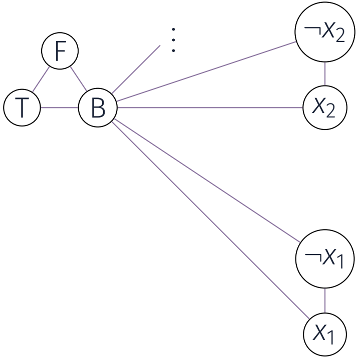

Each x_i can't get colour B, so must be T or F.

Add $\neg x_i$ vertex for convenience.

Replicating the choice structure (3)

What about the second one?

Need colouring to get stuck when we can no longer satisfy a clause.

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When does colouring get stuck?

As we saw before: when a vertex needs to be coloured but already has vertices in all three colours next to itself.

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That is, e.g. for $(x_1 \vee \neg x_3 \vee x_4)$, the vertices we labelled x_1, $\neg x_3$ and x_4 have all been coloured the same as F.

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That is, e.g. for $(x_1 \vee \neg x_3 \vee x_4)$, the vertices we labelled x_1, $\neg x_3$ and x_4 have all been coloured the same as F.

Want a gadget that forces three vertices to have different colours exactly when that is the case.

Replicating the choice structure (4)

Idea: Use the triangle construction to create a vertex that is forced to have colour T iff the first literal has colour F, and another that is forced to have colour B iff the third literal has colour F. Use the second literal as is.

Then connect these three vertices to a new vertex associated with the clause.

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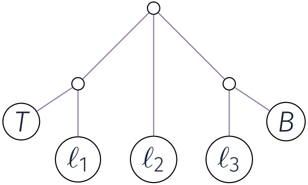

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Indeed, since we coloured the top of the clause gadget, the vertices it was connected to were not all different colours. If ℓ_2 was coloured as *T*, then done. Otherwise, ℓ_2 was coloured as *F*. So either one of the other two was also coloured as *F*, or the other two are coloured the same.

Proving the reduction correct (2)

One of the other two coloured as F: then either ℓ_1 or ℓ_3 can't have been coloured as F, so done.

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The other two are coloured the same: they can't both be T or B (since one is adjacent to T, and the other to B), so must both be F. So both ℓ_1 and ℓ_2 must in fact have been coloured T.

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In particular, when ℓ_3 is T, the colouring of the node above it is still forced to a single choice (F). Circumstances then conspire so that we can assign the other two literals in such a way that once again we get three different colours next to the peak.

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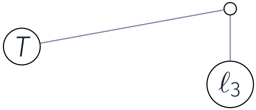

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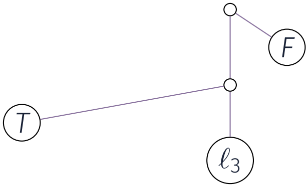
$(\ell_1 \vee \ell_2 \vee \ell_3) \quad \mapsto$ 

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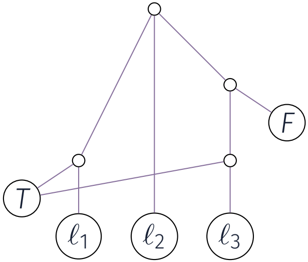

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Every clause is satisfied, so at least one of its incoming literal vertices must be coloured *T*.

Whatever ℓ_3 is, we can colour the vertex above ℓ_3 that is connected to *F* as *T*: either it's *F*, and the vertices above it must be *B* and then *T*, or it's *T*, the vertex above it can be *B*, and the vertex connected to *F* and the peak must then be *T*.

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If ℓ_2 is *T*, we are then done, as two vertices adjacent to the peak are the same colour and so we can pick a colour for the peak and missing vertex in the gadget.

Proving the reduction correct, for real (2)

So suppose ℓ_2 is F. Then, either ℓ_1 must be T, in which case we can colour the vertex above it F as well; or ℓ_1 is F and ℓ_3 is T, in which case we can colour the vertex above ℓ_1 B, and the ones above ℓ_3 with F and B in order, and so the peak can be coloured T.

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In this fashion, we proceed to colour all clauses, and thus have a 3-colouring. □

Clearly, everything is polynomial. So GRAPH 3-COLOURING is NP-complete.