Reasoning Inside The Box
Deduction in Herbrand Logics

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Abstract

Herbrand structures are a subclass of standard first-order structures commonly used in logic and automated reasoning due to their strong definitional character. This paper is devoted to the logics induced by them: Herbrand and semi-Herbrand logics, with and without equality. The rich expressiveness of these logics entails that there is no adequate effective proof system for them. We therefore introduce infinitary proof systems for Herbrand logics, and prove their completeness. Natural and sound finitary approximations of the infinitary systems are also presented.

1 Introduction

The standard semantic approach to defining logical consequence is based on structures. A formula φ is said to be a consequence of a set of formulas T if φ is true in *every* structure in which T is true. The quantification in this definition ranges over arbitrary structures, whose domains are unrestricted. For applications of logic in computer science in general, and automated reasoning in particular, bounding this quantification has proven to be fruitful. (Finite model theory [14] is a case in point.)

A recent work [19], titled "The Herbrand Manifesto: Thinking Inside the Box", suggests focusing on Herbrand structures. These structures, in which every element has a unique name, are a salient component in completeness proofs for first-order logic. They are also widely used in various subfields of artificial intelligence, such as: automated reasoning, where they are used for automated theorem proving [7, 11]; deductive databases and logic programming, where they provide semantics for logic programs [1, 25]; and logic education [18], where they simplify the semantics of first-order logic. The logic (consequence relation) induced by Herbrand structures has both computational and pedagogical advantages over classical first-order logic. Indeed, the fact that in such structures each element is uniquely definable by some closed term of the language enables a convenient way of performing symbolic computations. Taking an even stronger notion of definability by fixing the domain of Herbrand structures to precisely the set of closed terms offers a significant simplification of the induced semantics.

In this paper we study Herbrand structures and equip the logics induced by them with Gentzen-style proof systems. Accordingly, the title of the current paper paraphrases that of [19], by replacing "thinking" with the more computational notion of formal "reasoning". A crucial step in the development of the proof theory of the logics induced by Herbrand structures

is to identify the different components comprising them: (i) every element of the domain has a name in the formal language; and (ii) every such name is unique. Structures that satisfy only (i) are called semi-Herbrand structures, and those who satisfy both are called Herbrand structures. We also consider the addition of equality to the language, and thus obtain four variants of the logic, which we call "Herbrand logics". Herbrand logics are "super-classical", in the sense that classical first-order logic is strictly contained in them (as consequence relations).

The modular approach undertaken here allows for the development of proof systems for each of these logics. But what could be expected from such proof systems? The rich expressive power of Herbrand logics entails that they are inherently incomplete, in the sense that one cannot hope for sound, effective proof systems that are complete for them. Therefore, we introduce *infinitary* sequent calculi, that are sound and complete for the various Herbrand logics. The key feature of these infinitary systems is a generalization of the ω-rule (see, e.g., [32]). The fact that this well-known, natural derivation rule suffices for a deductive characterization of Herbrand structures is non-trivial, as evident by the construction employed in the completeness proofs. Finitary and sound approximations of the infinitary systems are also introduced. Those are obtained by replacing the generalized ω-rule with a generalization of its finitary counterpart, namely the standard induction rule for Peano Arithmetics [20].

The rest of this paper is organized as follows: In Section 2 variants of Herbrand structures are defined, and the semantics they induce are investigated. In Section 3 we introduce sound and complete infinitary proof systems for Herbrand logics. Section 4 presents finitary approximations of these systems. Further exploration of Herbrand structures in the context of other useful logics is initiated in Section 5. We then conclude with Section 6, where other directions for further research are suggested.

2   Herbrand Logics

In [19] it was suggested that Herbrand structures should be studied on their own merit, rather than solely as an instrumental tool in AI and logic-related research. As it turns out, this framework offers a congenial environment for practicing logic, has strong expressive power, and is also useful from computational and educational points of view. This section is devoted to the various logics Herbrand structures induce.

2.1   Preliminaries

A first-order language $\mathcal{L}$ consists of a set of variables, the standard logical connectives $\wedge, \vee, \supset$ and $\neg$, the logical quantifiers $\forall$ and $\exists$, and a signature that consists of a set $func(\mathcal{L})$ of function symbols, and a set $pred(\mathcal{L})$ of predicate symbols. We use x, y, z, w (possibly with subscripts) to denote variables of $\mathcal{L}$. We also denote by $consts(\mathcal{L})$ the set of 0-ary function symbols (constants), and by $func^+(\mathcal{L})$ the set $func(\mathcal{L}) \setminus consts(\mathcal{L})$. A term (formula) is called closed if no free variable occurs in it, and is called open otherwise. The set of closed terms of $\mathcal{L}$ is denoted by $cl(\mathcal{L})$, and the set of its closed atomic formulas is denoted by $catoms(\mathcal{L})$.

A first-order structure for $\mathcal{L}$ is a pair $\langle D, I \rangle$, such that D is a non-empty set and I is an interpretation function, assigning an n-ary function (relation) over D to every n-ary function (predicate) symbol of $\mathcal{L}$. We adopt the substitutional approach [24,34], and define the semantics without using valuations. The language $\mathcal{L}(M)$ associated with a first-order structure M is obtained from $\mathcal{L}$ by the addition of a constant $\overline{a}$ for each $a \in D$, interpreted as a. Satisfaction of closed atomic formulas and their boolean combinations is standardly defined. M is said to

satisfy a closed formula of the form $\forall x \psi$ $(\exists x \psi)$ if it satisfies $\psi \left\{ \frac{\overline{a}}{x} \right\}$ for every (some) $a \in D$. M satisfies an open formula $\psi(x_1, \ldots, x_n)$ if it satisfies $\psi \left\{ \frac{\overline{a_1}}{x_1}, \ldots, \frac{\overline{a_n}}{x_n} \right\}$ for every $a_1, \ldots, a_n \in D$.

In what follows, we fix a first-order language $\mathcal{L}$ which has at least one constant symbol.[1] By a structure we mean a first-order structure for $\mathcal{L}$. We denote the classical satisfaction relation between structures and formulas by $\models$, and its induced consequence relation by $\vdash$.

2.2   Herbrand and Semi-Herbrand Structures

We start the investigation of Herbrand structures by disassembling their standard definition into its two components: the existence of a name for every domain element, and its uniqueness.[2]

Definition 1. *A structure $M = \langle D, I \rangle$ for $\mathcal{L}$ is called semi-Herbrand if for every $a \in D$ there is some $t \in cl(\mathcal{L})$ such that $I(t) = a$. If for each $a \in D$, there is a unique $t \in cl(\mathcal{L})$ such that $I(t) = a$, M is called a Herbrand structure.*

Since in Herbrand structures every element has a unique name, it is possible (and beneficial from a computational point of view) to avoid isomorphic structures, by fixing the domain to consist solely of these names. Thus, in the reminder of this paper, we utilize the more common definition of a Herbrand structure, requiring that $D = cl(\mathcal{L})$ and $I(t) = t$ for every $t \in D$.

Herbrand structures allow for an even more computationally-oriented definition, as they admit a strong notion of definability, which we call "structure-definability". That is, every Herbrand structure is uniquely determined by a set of closed atoms, by setting $I(P) = \{ \langle t_1, \ldots, t_n \rangle \in cl(\mathcal{L})^n : P(t_1, \ldots, t_n) \in A \}$ for every predicate symbol P.[3]

In (semi-)Herbrand structures there is no need to explicitly add the fresh constants $\overline{a}$ for every $a \in D$, as done in the substitutional approach, since each element of the domain already has a corresponding closed term. This property allows for an equivalent, more convenient definition of quantifier semantics and of satisfaction of open formulas.

Lemma 1. *Let M be a semi-Herbrand structure.*

- $M \models \exists x \psi$ *iff* $M \models \psi \left\{ \frac{t}{x} \right\}$ *for some* $t \in cl(\mathcal{L})$.

- $M \models \forall x \psi$ *iff* $M \models \psi \left\{ \frac{t}{x} \right\}$ *for every* $t \in cl(\mathcal{L})$.

Definition 2. *Let $\varphi(x_1, \ldots, x_n)$ be an open formula whose free variables are a subset of $\{x_1, \ldots, x_n\}$. An $\mathcal{L}$-instance of φ is a formula of the form $\varphi \left\{ \frac{t_1}{x_1}, \ldots, \frac{t_n}{x_n} \right\}$, where $t_1, \ldots, t_n \in cl(\mathcal{L})$.[4]*

Lemma 2. *Let M be a semi-Herbrand structure. Then, $M \models \varphi$ iff $M \models \varphi'$ for every $\mathcal{L}$-instance φ' of φ.*

The consequence relations induced by (semi-)Herbrand structures are defined as follows:

Definition 3. *Let $T \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. $T \vdash^H \varphi$ $(T \vdash^{SH} \varphi)$ if for every (semi-) Herbrand structure M, $M \models T$ (i.e. $M \models \psi$ for every $\psi \in T$) implies $M \models \varphi$.*

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[1]Dismissing this requirement yields a Herbrand counterpart of free-logic [23].

[2]These two components correspond to well-known concepts from inductive reasoning: a set of operators generating a set, and the question whether each element is generated in a unique way [16].

[3]This is how Herbrand structures were defined in [19].

[4]Note that in [34], instances of formulas depend not only on the language $\mathcal{L}$, but also on the structure M under consideration. One of the advantages of semi-Herbrand structures is that the definition of instances is simplified, and depends solely on $\mathcal{L}$.

Notice that the relations $\vdash^{SH}$ and $\vdash^{H}$ are parametrized by the underlying language $\mathcal{L}$, and the answer to derivability questions highly depends on the identity of $\mathcal{L}$. For readability, we refrain from including $\mathcal{L}$ in the notation itself, as it is fixed throughout the paper.

The next proposition summarizes the relations between the logics that are induced by Herbrand, semi-Herbrand and arbitrary structures. Note that the proposition assumes languages without a distinguished equality symbol (such languages are the subject of the next subsection).

Proposition 1. $\vdash \subsetneq \vdash^{SH} = \vdash^{H}$.

Proof. Since every Herbrand structure is also semi-Herbrand, and every semi-Herbrand structure is a classical structure, we have that $\vdash \subseteq \vdash^{SH} \subseteq \vdash^{H}$. To see that $\vdash \neq \vdash^{SH}$, consider e.g. a language with only one constant symbol c, and a unary predicate P. Clearly, $P(c) \vdash^{SH} \forall x P(x)$, whereas $P(c) \not\vdash \forall x P(x)$.[5] As for $\vdash^{SH}$ and $\vdash^{H}$, for every semi-Herbrand structure $M = \langle D, I \rangle$, an equivalent Herbrand structure $H_M = \langle cl(\mathcal{L}), I' \rangle$ can be constructed by taking $I'(t) = t$ for every $t \in cl(\mathcal{L})$, and $I'(P) = \{\langle t_1, \ldots, t_n \rangle : \langle I(t_1), \ldots, I(t_n) \rangle \in I(P)\}$ for each predicate symbol P. It can be shown using induction that M and H_M satisfy the same formulas. $\square$

2.3   Handling Equality

In this section we use the classical axiomatization of equality to study Herbrand logics with equality. Throughout, we assume $\mathcal{L}$ includes a binary predicate symbol $=$, and abbreviate $\neg(s = t)$ by $(s \neq t)$. A structure $M = \langle D, I \rangle$ is called normal if $I(=)$ is $\{\langle a, a \rangle \mid a \in D\}$.

Definition 4. $\vdash^{H=}$ ($\vdash^{SH=}$) *is defined similarly to* $\vdash^{H}$ ($\vdash^{SH}$), *but is restricted to normal Herbrand (semi-Herbrand) structures.*

The addition of equality separates the consequence relations that are induced by Herbrand and semi-Herbrand structures.

Proposition 2. *For languages with at least two closed terms,* $\vdash^{SH=} \subsetneq \vdash^{H=}$.[6]

Proof. Let t_1 and t_2 be two distinct closed terms. While we have $\vdash^{H=} t_1 \neq t_2$, this does not hold for $\not\vdash^{SH=}$. $\square$

Semi-Herbrand structures allow for the same axiomatization of equality that is used in classical logic.

Definition 5. *Let* $Equiv = \{x = x, x = y \supset y = x, x = y \wedge y = z \supset x = z\}$, *and let* $E(\mathcal{L})$ *be the set consisting of the following formulas:*

- $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ *for every n-ary (n > 0) function symbol f.*

- $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n))$ *for every n-ary predicate symbol P.*

Finally, let $Eq(\mathcal{L})$ be the set $Equiv \cup E(\mathcal{L})$.

Proposition 3. $T \vdash^{SH=} \varphi$ *iff* $T \cup Eq(\mathcal{L}) \vdash^{SH} \varphi$.

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[5]A more general argument was given in [19] for the proper inclusion of $\vdash$ in $\vdash^{H}$, which amounts to the failure of the compactness theorem for $\vdash^{H}$.

[6]If the language has only one closed term the consequence relations are identical.

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$Eq(\mathcal{L})$, however, is insufficient for characterizing normal Herbrand structures. For example, $\vdash^{H=} t_1 \neq t_2$, but $Eq(\mathcal{L}) \not\vdash^H t_1 \neq t_2$, whenever t_1 and t_2 are distinct closed terms. Interestingly, in Herbrand structures, capturing equality requires the *inequalities* of the language, which requires the following inequality axioms. Together with the equality axioms above, we obtain a variant of Clark equality theory [12].

Definition 6. *Let $inE(\mathcal{L})$ be the set consisting of the following formulas:*

- $f(x_1, \ldots, x_n) \neq g(y_1, \ldots, y_m)$ *for every distinct function symbols f and g with arities n and m, respectively.*[7]

- $x_i \neq y_i \supset f(x_1, \ldots x_i, \ldots, x_n) \neq f(x_1, \ldots y_i, \ldots, x_n)$ *for every n-ary ($n > 0$) function symbol f and $1 \leq i \leq n$.*

Notice that $inE(\mathcal{L})$ is determined solely by $func(\mathcal{L})$. Therefore, whenever this set is finite, so is $inE(\mathcal{L})$. This is in contrast to $Eq(\mathcal{L})$, which also depends on $pred(\mathcal{L})$.

Example 1. *[19] defines a theory (which is here denoted by NAT) in a language (which is here denoted by $\mathcal{L}_{NAT}$) for arithmetics, that consists of a constant 0, a unary function s, two ternary predicate symbols plus and times, and a binary predicate symbol equal. For this particular language, we have $inE(\mathcal{L}_{NAT}) = \{0 \neq s(x), s(x) \neq 0, x \neq y \supset s(x) \neq s(y)\}$. These axioms, together with $x = x$, are essentially the axioms that were used for the predicate equal in NAT.*

Lemma 3. *The followings hold:*

- $inE(\mathcal{L}) \vdash^H s \neq t$ *for every distinct closed terms s and t.*

- *Let M be a Herbrand structure. $M \models inE(\mathcal{L}) \cup \{x = x\}$ iff M is normal.*

- *Let M be a semi-Herbrand structure. $M \models inE(\mathcal{L})$ iff M is Herbrand.*

With $inE(\mathcal{L})$, we obtain the following counterpart of Proposition 3.

Proposition 4. $T \vdash^{H=} \varphi$ iff $T \cup inE(\mathcal{L}) \cup \{x = x\} \vdash^H \varphi$.

Corollary 1. $T \vdash^{H=} \varphi$ iff $T \cup inE(\mathcal{L}) \vdash^{SH=} \varphi$.

To conclude this section, we make a brief remark about the axiomatization of arithmetics within the framework of Herbrand structures. The theory NAT, mentioned in Example 1, provided an axiomatization that categorically characterizes the natural numbers under Herbrand structures. This axiomatization is actually (the relational version of) Peano Arithmetics (PA) without the induction scheme, called Π_2 (see, e.g., [26]). Now, Robinson [30] added to this system the axiom $\forall x. x \neq 0 \supset \exists y. x = s(y)$, thus forming his famous system Q. When only considering Herbrand structures for $\mathcal{L}_{NAT}$, this additional axiom is valid, and thus the induction-free part of PA suffices. Moreover, the induction scheme itself is also valid in such structures. Therefore, Π_2, Q and PA are equivalent in Herbrand structures.

3   Infinitary Proof Systems for Herbrand Logics

A plausible proof-theoretical counterpart of *thinking* inside the box amounts to prohibiting the use of free variables, as they are the syntactical representatives of nameless elements. Accordingly, in this section we provide derivation systems for Herbrand logics that do not make use of free variables, thus providing a mechanism for *reasoning* inside the box.

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[7]Note that f and g may be constant symbols.

Figure 1: Derivation Rules for Herbrand and semi-Herbrand Logics

$$(\Rightarrow \forall)_H \ \frac{\left\{ \Gamma \Rightarrow \varphi \left\{ \frac{t}{x} \right\}, \Delta : t \in cl\,(\mathcal{L}) \right\}}{\Gamma \Rightarrow \forall x \varphi, \Delta} \qquad (\exists \Rightarrow)_H \ \frac{\left\{ \Gamma, \varphi \left\{ \frac{t}{x} \right\} \Rightarrow \Delta : t \in cl\,(\mathcal{L}) \right\}}{\Gamma, \exists x \varphi \Rightarrow \Delta}$$

$$(paramodulation) \ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, s = t \Rightarrow \varphi', \Delta} \quad s, t \in cl\,(\mathcal{L})$$

where φ' is obtained from φ by replacing occurrences of s by t

$$(\Rightarrow =) \ \frac{}{\Gamma \Rightarrow t = t, \Delta} \ \ t \in cl\,(\mathcal{L}) \qquad (= \Rightarrow) \ \frac{}{\Gamma, s = t \Rightarrow \Delta} \ \ s \neq t \in cl\,(\mathcal{L})$$

It was already shown in [19] that $\vdash^H$ is not recursively enumerable, and the same holds for the other variants of Herbrand logics (due to the above translations between them). This entails that Herbrand logics are inherently incomplete. Nevertheless, as shown below, there are natural formal systems which allow for infinitary proofs, that are sound and complete for Herbrand logics.

The proof-theoretic mechanism we use is that of sequent calculi [20], which is widely applied in automated reasoning, especially when one is interested in the computational aspects of a logic (e.g., [29]). It has been employed in a variety of logical frameworks, e.g., many-valued and fuzzy logics [10,27], modal logics [28,36], paraconsistent logics [5], and also logical argumentation [3]. To obtain completeness for our systems, however, the standard definition of a sequent must be relaxed to allow infinite sequents.

Definition 7. *A sequent is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are (possibly infinite) sets of formulas. A sequent containing only closed formulas is called closed; otherwise, it is called open.*

Let $\mathcal{LK}$ denote a variant of Gentzen's calculus for classical logic [20], in which sequents are taken to be pairs of finite sets, rather than pairs of lists (in particular, contraction, expansion and permutation are not needed). Denote by G the calculus obtained from $\mathcal{LK}$ by allowing only closed (possibly infinite) sequents, dismissing $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ (which are the $\forall$-introduction rule and $\exists$-elimination rule, respectively), replacing the axiom $\varphi \Rightarrow \varphi$ by $\Gamma, \varphi \Rightarrow \varphi, \Delta$ for every closed sequent $\Gamma, \varphi \Rightarrow \varphi, \Delta$, and also replacing the two original weakening rules by the following single weakening rule: $\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}$.

The key idea behind capturing the essence of Herbrand logics in a formal proof system is to formalize the syntactic restriction on the domain. This is obtained by replacing the standard $(\Rightarrow \forall)$ rule by a language-based introduction rule for $\forall$, which is a generalization of the ω-rule employed in some proof systems for PA (see, e.g., [32]). Similar modification is required in the $(\exists \Rightarrow)$ rule. The adjustments in the axiom and weakening rules allow the introduction of infinite sequents.

Definition 8. *Figure 1 includes the various derivation rules employed.*

1. $G_H = G + (\Rightarrow \forall)_H + (\exists \Rightarrow)_H$.

2. $G_{SH_=} = G_H + (\Rightarrow =) + (paramodulation)$.

  3. $G_{H_=} = G_H + (\Rightarrow =) + (= \Rightarrow)$.

Note that (*paramodulation*) is not included in $G_{H_=}$ since it is derivable using $(\Rightarrow =)$ and $(= \Rightarrow)$.

  Let X be one of the three calculi in the above definition. A derivation of a sequent s in X is a (possibly infinite) tree with a finite height, rooted with s, in which every node in the tree is the result of an application of some rule of X on the set of its predecessors. We write $\vdash_X s$ if there is a derivation of s in X.

  It is important to note that every derivation in the above calculi does not contain occurrences of free variables. Also note that, in general, the above calculi are not effective, as they allow for infinite derivations and infinite sequents. They are effective, however, in case $cl\,(\mathcal{L})$ is finite.

  First we show that all three calculi extend $\mathcal{LK}$. For this we consider $\mathcal{L}$-instances of sequents.

Definition 9. *A substitution is a function σ assigning an element of $cl\,(\mathcal{L})$ to every variable. Given a formula φ, φ^σ is obtained from φ by substituting every free occurrence of a variable x with $\sigma\,(x)$. An $\mathcal{L}$-instance of a sequent $\Gamma \Rightarrow \Delta$ is any sequent of the form $\Gamma^\sigma \Rightarrow \Delta^\sigma$ for some σ, where $X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$.*

Proposition 5. *Let s be a finite closed sequent derivable in $\mathcal{LK}$. Then s is derivable in G_H.*

Proof outline. We prove a stronger claim: if s is a finite (possibly open) sequent derivable in $\mathcal{LK}$, then every $\mathcal{L}$-instance of it is derivable in G_H. The proof is carried out by induction on the derivation of s in $\mathcal{LK}$. Applications of the right introduction rule of $\forall$ and left introduction rule of $\exists$ in $\mathcal{LK}$ are replaced by applications of $(\Rightarrow \forall)_H$ and $(\exists \Rightarrow)_H$, respectively. The proposition then easily follows from this claim, as the only closed instance of a closed sequent is itself. □

  We now prove that all three calculi are sound and complete with respect to their corresponding Herbrand logics. The main challenge in the completeness proofs is the unavailability of Lindenbaum's lemma, due to the infinitary nature of the systems. This, in turn, renders the standard construction of a maximal unprovable sequent inapplicable. To solve this problem, we employ a similar method to the one used in [15, 35], and generate a sequent that admits all the necessary properties for inducing a countermodel. This is achieved by the addition of Henkin witnesses from $\mathcal{L}$ itself (unlike in the classical case, where the language is extended).

Definition 10. *Let M be a structure for $\mathcal{L}$ and $s = \Gamma \Rightarrow \Delta$ a closed sequent. $M \models s$ if $M \not\models \varphi$ for some $\varphi \in \Gamma$ or $M \models \psi$ for some $\psi \in \Delta$. We write $\vdash_{G_H} s$ if $M \models s$ for every Herbrand structure M for $\mathcal{L}$, and use this notation similarly for $\vdash_{G_{SH_=}}$ and $\vdash_{G_{H_=}}$.*

Theorem 1 (Completeness)**.** *Let s be a closed sequent. The followings hold:*

  1. $\vdash_{G_H} s$ *iff* $\vdash^H s$.

  2. $\vdash_{G_{SH_=}} s$ *iff* $\vdash^{SH_=} s$.

  3. $\vdash_{G_{H_=}} s$ *iff* $\vdash^{H_=} s$.

Proof outline. Soundness is proven by a usual induction on derivations. Completeness for G_H is proven as follows. First, we prove the following key lemmas:

  - If $\nvdash_{G_H} \Gamma \Rightarrow \Delta$ and $\forall x \varphi \in \Delta$, then there exists $t \in cl\,(\mathcal{L})$ such that $\nvdash_{G_H} \Gamma \Rightarrow \varphi \left\{\frac{t}{x}\right\}, \Delta$.

  - If $\nvdash_{G_H} \Gamma \Rightarrow \Delta$ and $\exists x \varphi \in \Gamma$, then there exists $t \in cl\,(\mathcal{L})$ such that $\nvdash_{G_H} \Gamma, \varphi \left\{\frac{t}{x}\right\} \Rightarrow \Delta$.

Then, assuming $\nvdash_{G_H} s$, a sequence of sequents $s_0, s_1, \ldots$ is inductively defined, based on an enumeration $\psi_1, \psi_2, \ldots$ of the closed formulas of the language: $s_0 = s$, and for every $i \geq 0$:

$$s_{2i+1} = \begin{cases} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i} & \nvdash_{G_H} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i} \\ \Gamma_{2i} \Rightarrow \psi_i, \Delta_{2i} & otherwise \end{cases}$$

$$s_{2i+2} = \begin{cases} \Gamma_{2i+1} \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta_{2i+1} & \psi_i = \forall x \varphi \in \Delta_{2i+1} \\ \Gamma_{2i+1}, \varphi\left\{\frac{t}{x}\right\} \Rightarrow \Delta_{2i+1} & \psi_i = \exists x \varphi \in \Gamma_{2i+1} \\ s_{i+1} & otherwise \end{cases}$$

where t is chosen with accordance to the lemmas. We then show that $\nvdash_{G_H} s_i$ for every $i \in \mathbb{N}$ by induction on i. The case where i is odd is shown using cut, while the case where i is even makes use of the lemmas. A sequent $L \Rightarrow R$ is then defined by $L = \bigcup_{i \in \mathbb{N}} \Gamma_i$ and $R = \bigcup_{i \in \mathbb{N}} \Delta_i$.

Finally, a Herbrand structure $M = \langle D, I \rangle$ is defined by $D = cl(\mathcal{L})$, $I(t) = t$ for every $t \in cl(\mathcal{L})$, and $I(P) = \{\langle t_1, \ldots, t_n \rangle \in D^n \mid P(t_1, \ldots, t_n) \in L\}$ for every predicate symbol P. It is proven that $M \models \varphi$ iff $\varphi \in L$ for every closed formula φ, which in turn entails that $M \not\models s$. The proof is carried out by standard induction, however, since we do not have that $L \Rightarrow R$ is maximal unprovable, we consider the elements of the sequence $s_0, s_1, \ldots$ themselves in the induction steps. The Henkin witnesses introduced in steps s_{2i+2} are used for the cases of universal and existential formulas.

Completeness of $G_{SH_=}$ is proven similarly, however, the domain D of the constructed countermodel $M = \langle D, I \rangle$ is taken to be the quotient set of $cl(\mathcal{L})$ induced by the equivalence relation $\sim = \left\{\langle s, t \rangle \in cl(\mathcal{L})^2 \mid s = t \in L\right\}$. In turn, $I(t)$ is defined as the equivalence class of t under $\sim$, and $I(P)$ is defined by $\langle [t_1]_\sim, \ldots, [t_n]_\sim \rangle \in I(P)$ iff $P(t_1, \ldots, t_n) \in L$. This changes the proof of the base case of the induction, while the induction steps remain intact.

Completeness of $G_{H_=}$ is also shown similarly to that of G_H, by interpreting $=$ as $\{\langle t, t \rangle \mid t \in cl(\mathcal{L})\}$. The required modifications in the induction base are obtained using $(\Rightarrow =)$ and $(= \Rightarrow)$. $\qquad \square$

Note that Theorem 1 provides an alternative, less constructive proof of Proposition 5.

Next, we restate the soundness and completeness theorems in terms of formulas, rather than sequents.

Corollary 2. *Let $T \cup \{\varphi\}$ be a set of closed formulas. Then:*

1. $\vdash_{G_H} T \Rightarrow \varphi$ *iff* $T \vdash^H \varphi$.

2. $\vdash_{G_{SH_=}} T \Rightarrow \varphi$ *iff* $T \vdash^{SH_=} \varphi$.

3. $\vdash_{G_{H_=}} T \Rightarrow \varphi$ *iff* $T \vdash^{H_=} \varphi$.

Due to Proposition 1, G_H is also sound and complete for $\vdash^{SH}$.

4 Effective Approximations

While the proof systems presented in the previous section are sound and complete with respect to Herbrand logics, they are not effective, in the sense that proofs cannot be verified. Thus, for systems that are more suitable for automated reasoning, one needs to compromise completeness for the sake of effectiveness. Accordingly, in this section, we provide finitary counterparts of

Figure 2: Effective Derivation Rules for Herbrand and semi-Herbrand Logics

$$(\Rightarrow \forall)_{IND} \ \frac{\left\{\Gamma, \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}, \Delta : f \in func\,(\mathcal{L})\right\}}{\Gamma \Rightarrow \forall x \varphi, \Delta}$$

$$(\exists \Rightarrow)_{IND} \ \frac{\left\{\Gamma, \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\} \Rightarrow \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\}, \Delta : f \in func\,(\mathcal{L})\right\}}{\Gamma, \exists x \varphi \Rightarrow \Delta}$$

In $(\Rightarrow \forall)_{IND}$ and $(\exists \Rightarrow)_{IND}$, $x_1, \ldots, x_n$ do not occur free in $\Gamma \cup \Delta$.

$$(\Rightarrow =)_{IND} \ \frac{}{\Gamma \Rightarrow x = x, \Delta}$$

$$(= \Rightarrow)_1 \ \frac{}{\Gamma, f\,(x_1, \ldots, x_n) = g\,(y_1, \ldots, y_m) \Rightarrow \Delta} \quad f \neq g \in func\,(\mathcal{L})$$

$$(= \Rightarrow)_2 \ \frac{\Gamma, x_i = y_i \Rightarrow \Delta}{\Gamma, f\,(x_1 \ldots, x_i, \ldots, x_n) = f\,(x_1, \ldots, y_i, \ldots, x_n) \Rightarrow \Delta} \quad f \in func^+\,(\mathcal{L})$$

the infinitary systems presented in the previous section. To recover some of the strength of the original systems, we allow ourselves some "peeking outside the box", in the sense that the systems do incorporate free variables.

Let G^{IND} be the sequent calculus obtained from G by allowing only finite sequents in applications of rules, and permitting open sequents. To obtain finitary systems, in this section we assume $func\,(\mathcal{L})$ is finite.

Definition 11. *Figure 2 includes the additional derivation rules employed.*

1. $G_H^{IND} = G^{IND} + (\Rightarrow \forall)_{IND} + (\exists \Rightarrow)_{IND}$.

2. $G_{SH_=}^{IND} = G_H^{IND} + (\Rightarrow =)_{IND} + (paramodulation)$.

3. $G_{H_=}^{IND} = G_{SH_=}^{IND} + (= \Rightarrow)_1 + (= \Rightarrow)_2$

The key idea in constructing the above systems is to replace the infinitary rules $(\Rightarrow \forall)_H$ and $(\exists \Rightarrow)_H$ by finitary approximations. This is achieved by taking PA's induction scheme as a finite approximation of the ω-rule. Similarly, we replace $(\Rightarrow \forall)_H$, which is a language-based version of the ω-rule, with a new rule, $(\Rightarrow \forall)_{IND}$, which is a language-based version of Gentzen's induction rule for PA. $(\exists \Rightarrow)_{IND}$ is treated symmetrically.

Following Example 1, we note that for $\mathcal{L}_{NAT}$, applications of $(\Rightarrow \forall)_{IND}$ have the form

$$\frac{\Gamma \Rightarrow \varphi\left\{\frac{0}{x}\right\}, \Delta \ \ \Gamma, \varphi\left\{\frac{x_1}{x}\right\} \Rightarrow \varphi\left\{\frac{s(x_1)}{x}\right\}, \Delta}{\Gamma \Rightarrow \forall x \varphi, \Delta} \tag{1}$$

Actually, Gentzen's original induction scheme for PA is easily derivable in the above systems, using (1) and $(\forall \Rightarrow)$. What enables the effective formulation of PA's induction rule is the fact that the language of PA is finite. The above systems can manipulate languages which are more expressive than that of PA, since they allow multiple constant symbols, as well as function

symbols with arbitrary arities. However, we still assume that there are finitely many basic constructs for terms. This explains the requirement employed in this section that $func(\mathcal{L})$ is finite. The fact that $func(\mathcal{L})$ is finite entails that G_H^{IND}, $G_{SH_=}^{IND}$ and $G_{H_=}^{IND}$ are finitary, in the sense that their proof trees are always finite.

In search of effectiveness, we replace $(=\Rightarrow)$ in the system for Herbrand logics with equality by two rules $(=\Rightarrow)_1$ and $(=\Rightarrow)_2$, that correspond to sequent rules for inequalities in [31]. Those achieve precisely the same power, while being actual finite sets of rules, rather than schemes. In order to be able to derive in $G_{H_=}^{IND}$ all instances of the rule $(=\Rightarrow)$ of $G_{H_=}$ we include (*paramodulation*) in $G_{H_=}^{IND}$ (unlike in the original system $G_{H_=}$).

Lemma 4. *If t_1 and t_2 are two distinct closed terms of $\mathcal{L}$, then $\vdash_{G_{H_=}^{IND}} t_1 = t_2 \Rightarrow$.*

Proof. This is shown using induction on the sums of complexities of t_1 and t_2. The only interesting case is when $t_1 = f(s_1, \ldots, s_n)$ and $t_2 = f(r_1, \ldots, r_n)$ for some n-ary function symbol f. Since t_1 and t_2 are distinct, we must have some $1 \le i \le n$ such that $s_i \ne r_i$. By the induction hypothesis, $\vdash_{G_{H_=}^{IND}} s_i = r_i \Rightarrow$.

It is easy to verify that the sequent $x = s_i, y = r_i, x_i = y_i \Rightarrow s_i = r_i$ is derivable in $G_{H_=}$. It is also straightforward to derive the rule: $(sub) \dfrac{\Gamma \Rightarrow \Delta}{\Gamma\left\{\frac{s}{x}\right\} \Rightarrow \Delta\left\{\frac{s}{x}\right\}}$ $s \in cl(\mathcal{L})$. This, in turn, enables the proof of $t_1 = t_2 \Rightarrow$ in $G_{H_=}^{IND}$, using cuts on the conclusion of the following derivation:

$$\dfrac{\dfrac{\overline{x_i = s_i, y_i = r_i, x_i = y_i \Rightarrow s_i = r_i}}{x_i = s_i, y_i = r_i, f(x_1 \ldots x_i \ldots x_n) = f(x_1 \ldots y_i \ldots x_n) \Rightarrow s_i = r_i} (=\Rightarrow)_2}{s_i = s_i, r_i = r_i, f(x_1 \ldots s_i \ldots x_n) = f(x_1 \ldots r_i \ldots x_n) \Rightarrow s_i = r_i} (sub) \qquad \square$$

All the effective systems are complete for classical logic (as the rules $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ of $\mathcal{LK}$ are derivable in them). They are, however, not sound for it (since they include consecutions that are only valid in Herbrand logics). As a consequence of the finitary nature of the systems, one cannot expect that they are complete with respect to Herbrand logics. In the case of the language of the natural numbers, we obtain exactly the difference between the provable theorems of PA and the true statements of arithmetics. They are, however, sound, and expressive enough to capture meaningful Herbrand-valid statements (such as PA's induction scheme).

Theorem 2 (Soundness). *Let s be a closed sequent. The followings hold:*

1. *If $\vdash_{G_H^{IND}} s$ then $\vdash^H s$.*

2. *If $\vdash_{G_{SH_=}^{IND}} s$ then $\vdash^{SH_=} s$.*

3. *If $\vdash_{G_{H_=}^{IND}} s$ then $\vdash^{H_=} s$.*

Proof. The finitary systems allow free variables to appear in derivations. Therefore, in order to be able to use induction on derivations, we prove the theorem for open sequents as well. For this purpose, Definition 10 is extended to open sequents, by setting $M \models s$ iff M satisfies every $\mathcal{L}$-instance of s. For G_H^{IND}, the only non-trivial cases are $(\Rightarrow \forall)_{IND}$ and $(\exists \Rightarrow)_{IND}$. We here provide the proof for the first, as the second is symmetrical.

Let M be a Herbrand model of the premises of $(\Rightarrow \forall)_{IND}$. We prove $M \models \Gamma \Rightarrow \forall x \varphi, \Delta$. If $M \models \Gamma \Rightarrow \Delta$, this clearly holds. Otherwise, $M \models \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}$ for every $f \in func(\mathcal{L})$. Thus, M satisfies every $\mathcal{L}$-instance of such sequents. To show that

$M \models \forall x\varphi$, we consider an arbitrary $\mathcal{L}$-instance $\forall x\varphi'$ of $\forall x\varphi$, and prove that $M \models \varphi' \left\{ \frac{t}{x} \right\}$ for every $t \in cl(\mathcal{L})$, by inner induction on t. If $t \in consts(\mathcal{L})$, then this follows from the induction hypothesis (in this case, $n = 0$ and hence the premise has the form $\Gamma \Rightarrow \varphi \left\{ \frac{t}{x} \right\}, \Delta$). Otherwise, $t = f(t_1, \ldots, t_n)$ for some $f \in func^+(\mathcal{L})$ and $t_1, \ldots, t_n \in cl(\mathcal{L})$ such that $t_1, \ldots, t_n$ all have lower complexities than t. By the induction hypothesis, $M \models \varphi' \left\{ \frac{t_i}{x} \right\}$ for every $1 \leq i \leq n$. Since $M \models \varphi' \left\{ \frac{t_1}{x} \right\}, \ldots, \varphi' \left\{ \frac{t_n}{x} \right\} \Rightarrow \varphi' \left\{ \frac{f(t_1,\ldots,t_n)}{x} \right\}$, we also have that $M \models \varphi' \left\{ \frac{t}{x} \right\}$.

Soundness for $G_{SH_=}^{IND}$ is obtained similarly, with the addition of the same arguments for equality as in the classical case. For $G_{H_=}^{IND}$, the fact that only Herbrand structures are considered is used to show the validity of the rules $(=\Rightarrow)_1$ and $(=\Rightarrow)_2$. □

5   Further Research: Other Herbrand Logics

Herbrand structures are a robust concept, not limited to classical first-order logic. The structure-definability property embedded in them can be studied in every logic that employs structures, including modal logics, intuitionistic logic, second order logic, many-valued logics and more. Due to the definitional aspect of Herbrand logics, it seems that the study of Herbrand structures in computationally-useful logics is of special interest for automated reasoning and AI. Below we briefly outline research directions regarding two such prominent logics.

Perhaps the immediate candidate is intuitionistic first-order logic. Injecting the notion of structure-definability into the intuitionistic approach, which already carries constructive computational content, seems to hold great potential. A natural way to combine these two frameworks would be to consider Kripke's semantics for intuitionistic logic [22], restricting the structures at each possible world to Herbrand structures. There is, however, a fundamental difference between Herbrand logics and Kripke's semantics for intuitionistic logic. While the domain of Herbrand structures is fixed, in Kripke's semantics the domains of the possible worlds are allowed to expand. There are several possible ways to reconcile this conflict. The first is to relax the intuitionistic requirement by replacing it with its constant-domain counterpart, namely CD (see, e.g., [21]). The second is to relax the strict notion of structure definability, by considering semi-Herbrand structures, thus allowing expanding domains that are all partial to the set of closed terms. A third possibility is to associate with each possible world a different language, such that not only the domains are expanding, but also the languages.

Another promising candidate is ancestral logic (AL), which is the extension of classical first-order logic by a reflexive transitive closure operator, RTC (see, e.g., [33]).[8]In AL, the intended meaning of the formula $(RTC_{x,y}\varphi)(s,t)$ is "s and t stand in the reflexive transitive closure of the relation that is defined by $\varphi(x,y)$". This logic has rich expressive power, and its computational strength is well established (see, e.g., [4,13]). Now, for languages with finitely many constant symbols and unary function symbols (as the case of $\mathcal{L}_{NAT}$, for example), it is possible to axiomatize the structure-definability property of semi-Herbrand structures using the transitive closure operator (and equality). This is achieved using the following axiom:

$$\forall w \left(\bigvee_{c \in const(\mathcal{L})} \left(RTC_{x,y} \bigvee_{f \in func^+(\mathcal{L})} y = f(x) \right) (c, w) \right)$$

Just as in Herbrand logics, the expressiveness of AL renders it inherently incomplete. However,

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[8] [19] axiomatizes the transitive closure of a binary relation in Herbrand structures by introducing a new predicate symbol. AL handles the more uniform notion of transitive closure, namely a transitive closure *operator*.

[13] introduced finitary and sound sequent calculi for it. By augmenting the systems $G_{SH_=}^{IND}$ and $G_{H_=}^{IND}$ with the appropriate sequent rules for the RTC operator, finitary sound sequent calculi for Herbrand logics with transitive closure are obtained, in which the above axiom is derivable.

6   Conclusion and Related Work

In this paper we provided a modular study of Herbrand logics by considering Herbrand and semi-Herbrand structures, with and without equality. Sound and complete infinitary sequent-based proof systems were introduced for the various logics, and sound finitary approximations of those systems were given as well. The adequacy of Herbrand logics as a convenient logical framework is supported by the naturality of the rules employed in these systems. The correspondence between Herbrand structures and the suggested systems is however non-trivial, as evident by the method employed in order to achieve completeness.

This work also supplements each of the applications of Herbrand logic in [19] with a proof theoretical interpretation. For instance, for educational purposes, it was there claimed that Herbrand structures are easier to grasp than arbitrary structures. Our systems too provide an approachable way for handling the quantifiers, which is the key component in the proof theory of Herbrand logic. In systems for classical logic, the introduction of a universal formula relies on arbitrary elements. In contrast, our systems allow for more concrete arguments: the infinitary ones introduce a universal formula on the grounds of its closed instances; and the finitary ones do so based on induction. Another example of the potential applications of our systems concerns logic programming (see, e.g., [25]). The expressiveness of Herbrand logics allows for a finite axiomatization [19] of the minimal model of any safe stratified logic program. Reasoning about such programs can thus gain from the proof systems proposed here, by replacing validity checking in the minimal model with efficient proof search, having the aforementioned axiomatization as the set of premises. To conclude, we believe that the systems proposed in this paper may set the ground for further developments and applications of Herbrand logics that are more naturally handled proof theoretically.

A great deal of effort has been made to design formal systems that support inductive reasoning, e.g. [2, 6, 8, 9, 17]. Herbrand structures, being inductively defined, can be handled in them via some syntactic manipulations and additional rules. However, the generality of inductive definitions in these systems renders their possible treatment of the consequence relations induced by considering *only* Herbrand structures somewhat cumbersome. Thus, the systems presented in this work offer a more congenial framework for Herbrand logics. Amongst these works, [8] seems most related to ours. It introduces a finitary sequent calculus LKID for inductively defined predicates, and an infinitary calculus LKID$^\omega$ for infinite descent. Proofs in our finitary systems can be embedded in LKID, by the addition of an auxiliary domain predicate, and in the case of $G_{G_=}^{IND}$, the inequality axioms of Sec. 2.3. The infinitary system LKID$^\omega$, however, employs a completely different approach to the one used here. In contrast to our infinitary systems, it allows proofs of infinite height. Moreover, using domain predicates in LKID$^\omega$ is only possible for languages with finitely many function symbols, while our infinitary systems do not have this limitation. Nonetheless, interesting research questions naturally arise from the connection between the current work and [8]. One task for further research is relating the infinite descent principle of LKID$^\omega$ to the ω-rule of our infinitary systems. Such an investigation would potentially build on the cut-admissibility of LKID$^\omega$ in order to prove a similar result for our infinitary systems. (Cut-admissibility for our finitary systems is, however, beyond reach, as they subsume Gentzen's calculus for PA.) Another research direction is employing the Henkin-style semantics for LKID in order to obtain completeness for our finitary systems.

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