Ballot Length in Instant Runoff Voting

AAAI '23

Kiran Tomlinson  Johan Ugander  Jon Kleinberg
Given the preferences of voters, how do we pick a winner?
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Plurality voting

choose the candidate with the most first-place votes
**Plurality voting**

choose the candidate with the most first-place votes

![Plurality Voting Diagram]

more preferred

Winner!
Instant runoff voting (IRV)
repeatedly eliminate the candidate with fewest first-place votes

Elimination order
Instant runoff voting (IRV)
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Elimination order

more preferred
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**Instant runoff voting (IRV)**
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Elimination order

![Diagram illustrating the process of IRV with different fruits representing candidates and voters.](Image)
**Instant runoff voting (IRV)**

repeatedly eliminate the candidate with fewest first-place votes

![Elimination order](image)
Instant runoff voting (IRV)
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Instant runoff voting (IRV)
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Elimination order

more preferred
**Instant runoff voting (IRV)**

Repeatedly eliminate the candidate with fewest first-place votes.

Elimination order:

1. Grape
2. Apple
3. Strawberry
**Instant runoff voting (IRV)**

repeatedly eliminate the candidate with fewest first-place votes
*Instant runoff voting (IRV)*

Repeatedly eliminate the candidate with fewest first-place votes.
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repeatedly eliminate the candidate with fewest first-place votes
Instant runoff voting (IRV)
repeatedly eliminate the candidate with fewest first-place votes

Elimination order
Winner!

more preferred

a.k.a. STV, AV, RCV, Hare method, preferential voting
Who uses IRV?

Cities and counties: ● In use ○ Upcoming use
States: ▼ Used statewide ▀ Local elections in some jurisdictions
▌ Military and overseas voters ▄ 2020 Democratic presidential primary
▌ Special elections ▀ Party primary elections
Who uses IRV?
Who uses IRV?
Ballot length: how many candidates can you rank?
**Ballot length:** how many candidates can you rank?

Cities and counties: ● In use ○ Upcoming use
States: □ Used statewide ○ Local elections in some jurisdictions
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San Fransisco
**Ballot length**: how many candidates can you rank?

New York City

Cities and counties: ● In use ○ Upcoming use
States: □ Used statewide □ Local elections in some jurisdictions
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**Ballot length**: how many candidates can you rank?

Cities and counties: ● In use  ○ Upcoming use

States: USED statewide  ○ Local elections in some jurisdictions
Military and overseas voters  ○ 2020 Democratic presidential primary
Special elections  ○ Party primary elections

Maine

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**State of Maine Sample Ballot**
Democratic Primary Election, June 12, 2018

**Instructions to Voters**

To vote, fill in the oval like this ●

To rank your candidate choices, fill in the oval:
- In the first column for your first choice candidate.
- In the second column for your second choice candidate, and so on.

Continue until you have ranked as many or as few candidates as you like.

Fill in no more than one oval for each candidate or column.

To rank a write-in candidate, write the person's name in the write-in space and fill in the oval for the ranking of your choice.

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**Governor**

<table>
<thead>
<tr>
<th>1st Choice</th>
<th>2nd Choice</th>
<th>3rd Choice</th>
<th>4th Choice</th>
<th>5th Choice</th>
<th>6th Choice</th>
<th>7th Choice</th>
<th>8th Choice</th>
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<tr>
<td>Sweet, Elizabeth A.</td>
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</tr>
<tr>
<td>Write-in</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

SOURCE: Maine Secretary of State Office
**Ballot length**: how many candidates can you rank?

How much does ballot length matter?
Ballot length can change the winner

fix the profile, truncate all rankings

$h = \text{ballot length} \quad k = \text{# candidates}$

$h$       winner

1       

Ballot length can change the winner

fix the profile, truncate all rankings

$h = \text{ballot length}$

$k = \# \text{ candidates}$

$h$

winner

1

2
Ballot length can change the winner

fix the profile, truncate all rankings

$h = \text{ballot length}$

$k = \# \text{candidates}$
Ballot length can change the winner

fix the profile, truncate all rankings

\[ h = \text{ballot length} \]
\[ k = \# \text{candidates} \]
Ballot length can change the winner

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$h = \text{ballot length}$

$k = \# \text{candidates}$
Prior work
Prior work

voluntary truncation

[Saari & Newenhizen, Public Choice 1988]
[Baumeister et al, AAMAS ’12]
[Narodytska & Walsh, ECAI ’14]
[Menon & Larson, JAAMAS 2017]
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forced truncation (i.e., ballot length)

[Ayadi et al., *AAMAS* ’19]

The prevalence and consequences of ballot truncation in ranked-choice elections

D. Marc Kilgour¹ · Jean-Charles Grégoire² · Angèle M. Foley¹
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The prevalence and consequences of ballot truncation in ranked-choice elections

D. Marc Kilgour¹ · Jean-Charles Grégoire² · Angèle M. Foley¹

“A natural question […] is whether the outcome of the election stays the same as the extent of truncation increases from 0 (complete ballots) to k – 1. If not, how many different winners are possible?”
A natural question [...] is whether the outcome of the election stays the same as the extent of truncation increases from 0 (complete ballots) to \( k - 1 \). If not, how many different winners are possible?

In thousands of simulations involving \( k = 4, 5, \) and 6 candidates, we found instances of up to \( k - 2 \) different winners.
A $k-1$ winner construction for $k = 4$
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voter count 2 5 6 6 3 2

A D C
A B D A
C D B C
A $k$-1 winner construction for $k = 4$

voter count 2 5 6 6 3 2

ballot length $h$: 1 2 3

winner: A B C
A $k$-1 winner construction for $k = 4$

Voter count: 2 5 6 6 3 2

A B C D
D A B C

Ballot length $h$: 1 2 3

Winner: A B C

No smaller 3-winner $k = 4$ profile exists.
A $k$-1 winner construction for $k = 4$

voter count: 2 5 6 6 3 2

<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ballot length $h$: 1 2 3

winner: A B C

no smaller 3-winner $k = 4$ profile exists

we generalize this construction to any $k$
$k - 1$ different winners are possible as ballot length varies
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*K key assumption:* voters report as long a prefix of their ideal ranking as allowed
$k - 1$ different winners are possible as ballot length varies

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possibly incomplete
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require unique winners at each $h$: consequential-tie-free

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*key assumption:* voters report as long a prefix of their ideal ranking as allowed require unique winners at each $h$: *consequential-tie-free* possibly incomplete

*truncation winners:* candidates who win at some $h$
\( k - 1 \) different winners are possible as ballot length varies

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\( k - 1 \) distinct values of \( h \)
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$k - 1$ distinct values of $h$

**Theorem 2**
For every $k > 3$, there are consequential-tie-free profiles with $2k^2 - 2k$ voters and $k - 1$ truncation winners.
**$k - 1$ different winners are possible as ballot length varies**

*key assumption:* voters report as long a prefix of their ideal ranking as allowed

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- possibly incomplete

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**Theorem 2**
For every $k > 3$, there are consequential-tie-free profiles with $2k^2 - 2k$ voters and $k - 1$ truncation winners.

---

**Theorem 1**
For every $k > 3$, a consequential-tie-free profile needs at least $2k^2 - 2k$ voters to have $k - 1$ truncation winners.
Actually, it’s even worse....
Actually, it’s even worse….

label candidates in IRV elimination order: elimination order

1 2 3 4

🫐🍎🍓🍋
Actually, it’s even worse….

label candidates in IRV elimination order:

elimination order

1 2 3 4

\[\text{grape} \quad \text{apple} \quad \text{strawberry} \quad \text{lemon}\]

\[\text{truncation winner sequence}\]

ballot length \( h \) 1 2 3 4

\[\begin{array}{c}
\text{strawberry} \\
\text{lemon} \\
\text{lemon}
\end{array}\]

winner 3 4 4
Actually, it’s even worse….

Label candidates in IRV elimination order:

<table>
<thead>
<tr>
<th>elimination order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>🍇</td>
<td>🍎</td>
<td>🍓</td>
<td>🍊</td>
</tr>
</tbody>
</table>

Truncation winner sequence

- Ballot length: $h$, 1, 2, 3, 4
- Winner: 3, 4, 4

A truncation winner sequence is feasible if it’s element-wise $> 1, 2, \ldots, k - 1$. 
Actually, it’s even worse…

label candidates in IRV elimination order:

\[
\begin{array}{cccc}
\text{truncation winner sequence} & \text{elimination order} \\
\text{ballot length} & h & 1 & 2 & 3 & 4 \\
\text{winner} & 3 & 4 & 4 \\
\end{array}
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**truncation winner sequence**

- **ballot length** $h$: 1, 2, 3, 4
- **winner**: 3, 4, 4

**elimination order**

1, 2, 3, 4

**label candidates in IRV elimination order:**

- feasible: 1
- infeasible: 2
- feasible: 3
- infeasible: 4
Actually, it’s even worse….

Theorem 2
For every $k > 3$ and every feasible truncation winner sequence, there is a consequential-tie-free profile with $2k^2 - 2k$ voters achieving that sequence.
What if we assume structured preferences?
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e.g., single-peaked preferences:
What if we assume structured preferences?

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E.g., *single-peaked preferences*:

![Graph showing single-peaked preferences](image-url)
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Truncation winners with single-peaked preferences
Theorem 3
For every $k \geq 5$, no single-peaked profile has $k - 1$ truncation winners.
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**Theorem 5**
For every $k = c(c + 1)/2$, where $c \geq 3$, there is a single-peaked profile with $3k$ voters and $c$ truncation winners.
Truncation winners with single-peaked preferences

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For every $k = c(c + 1)/2$, where $c \geq 3$, there is a single-peaked profile with $3k$ voters and $c$ truncation winners.

Open question: more than $\Theta(\sqrt{k})$ truncation winners with single-peaked profiles?
Truncation winners with single-peaked preferences
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Theorem 3
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Proof. For $k - 1$ truncation winners: winner sequence is $2, 3, \ldots, k$
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$2$ is eliminated second, but has the most 1st place votes at the start
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2 is eliminated second, but has the most 1st place votes at the start
\( \Rightarrow \) 1’s elimination must cause 3, \ldots, k to overtake 2
Truncation winners with single-peaked preferences

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For every \( k \geq 5 \), no single-peaked profile has \( k - 1 \) truncation winners.

**Proof.** For \( k - 1 \) truncation winners: winner sequence is \( 2, 3, \ldots, k \)

\[ \text{elimination order is } 1, 2, \ldots, k \]

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but with single-peaked preferences, ballots listing 1 first can only list two different candidates second (to the left and right of 1)
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⇒ ! contradiction if $k \geq 5$
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but with single-peaked preferences, ballots listing 1 first can only list two different candidates second (to the left and right of 1)

⇒ ! contradiction if $k \geq 5$  □
What about real elections?
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168 elections (PrefLib.org)

[Mattei & Walsh, ADT ’13]

2011 San Francisco Mayor
2009 Burlington Mayor
1999-2008 APA President
What about real elections?

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[Mattei & Walsh, ADT ’13]

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2009 Burlington Mayor
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Truncating real-world elections
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• \(\frac{42}{168} = 25\%\) of PrefLib elections have multiple truncation winners (2 or 3)
Truncating real-world elections

• $42 / 168 = 25\%$ of PrefLib elections have multiple truncation winners (2 or 3)

• Under resampling, up to 6 truncation winners
Truncating real-world elections

- \( \frac{42}{168} = 25\% \) of PrefLib elections have multiple truncation winners (2 or 3)
- Under resampling, up to 6 truncation winners

Ballot length under resampling:

![Ballot length under resampling graph](image)
Truncating real-world elections

- \( \frac{42}{168} = 25\% \) of PrefLib elections have multiple truncation winners (2 or 3)

- Under resampling, up to 6 truncation winners

Ballot length under resampling:
More things in the paper
More things in the paper

- Single-crossing preferences: $k - 1$ truncation winners impossible
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- Other restrictions on ties with voter lower bounds and matching constructions
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• Construction with $k - 1$ truncation winners and only $\Theta(k)$ voter types
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- Full-ballot construction with $k/2$ truncation winners
More things in the paper

- Single-crossing preferences: \( k - 1 \) truncation winners impossible
- Other restrictions on ties with voter lower bounds and matching constructions
- Construction with \( k - 1 \) truncation winners and only \( \Theta(k) \) voter types
- Full-ballot construction with \( k/2 \) truncation winners
- Linear program for finding full-ballot \( k - 1 \) truncation winner profiles
More things in the paper

- Single-crossing preferences: $k - 1$ truncation winners impossible
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- Construction with $k - 1$ truncation winners and only $\Theta(k)$ voter types
- Full-ballot construction with $k/2$ truncation winners
- Linear program for finding full-ballot $k - 1$ truncation winner profiles
- Simulations
Thank you!

Code and data: 
github.com/tomlinsonk/irv-ballot-length

Extended version: 
arxiv.org/abs/2207.08958

Funding from:

![Funding Logos](image)