Separability in Domain Semirings

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Abstract. First, we show with two examples that in test semirings with an incomplete test algebra a domain operation may or may not exist. Second, we show that two notions of separability in test semirings coincide, respectively, with locality of composition and with extensionality of the diamond operators in domain semirings. We conclude with a brief comparison of dynamic algebras and modal Kleene algebras.

1 Basic Definitions

A *test semiring* [6] is a two-sorted structure $(K, \mathsf{test}(K))$, where K is an idempotent semiring and $\mathsf{test}(K) \subseteq K$ is a Boolean algebra embedded into K such that the operations of $\mathsf{test}(K)$ coincide with the restricted operations of K. In particular, $p \leq 1$ for all $p \in \mathsf{test}(K)$. In general, $\mathsf{test}(K)$ may be a proper subset of all elements below 1.

A *domain semiring* [1] is a structure $(K, \ulcorner)$, where K is an idempotent semiring such that the *domain operation* $\ulcorner \colon K \to \mathsf{test}(K)$ satisfies, for all $a, b \in K$ and $p \in \mathsf{test}(K)$,

$$a \leq \ulcorner a\, a, \tag{D1}$$

$$\ulcorner(pa) \leq p. \tag{D2}$$

The conjunction of (D1) and (D2) is equivalent to each of

$$\ulcorner a \leq p \Leftrightarrow a = pa, \tag{LLP}$$

$$\ulcorner a \leq p \Leftrightarrow \neg pa = 0, \tag{GLA}$$

which constitute elimination laws for domain. (LLP) says that $\ulcorner a$ is the least left preserver of a. (GLA) says that $\neg\ulcorner a$ is the greatest left annihilator of a. Both properties obviously characterize domain in set-theoretic relations. An important consequence of the axioms is strictness of the domain operation:

$$a = 0 \Leftrightarrow \ulcorner a = 0 \ . \tag{1}$$

Moreover, we have the following useful proof principle.

Lemma 1.1 (Indirect Domain Inequality)

$$\ulcorner a \leq \ulcorner b \;\Leftrightarrow\; (\forall\, p \,.\, pb = 0 \Rightarrow pa = 0) \ .$$

Proof. $\ulcorner a \leq \ulcorner b$

$\Leftrightarrow$ $\{\![$ contraposition $]\!\}$

$\neg\ulcorner b \leq \neg\ulcorner a$

$\Leftrightarrow$ $\{\![$ indirect inequality $]\!\}$

$\forall\, p \,.\, p \leq \neg\ulcorner b \Rightarrow p \leq \neg\ulcorner a$

$\Leftrightarrow$ $\{\![$ by (GLA) $]\!\}$

$\forall\, p \,.\, pb = 0 \Rightarrow pa = 0$.

□

A domain semiring is called *modal* if it additionally satisfies

$$\ulcorner(a\ulcorner b) \leq \ulcorner(ab). \tag{DLoc}$$

In a modal semiring, domain is *local*:

$$\ulcorner(ab) = \ulcorner(a\ulcorner b).$$

Without (DLoc), only the inequality $\ulcorner(ab) \leq \ulcorner(a\ulcorner b)$ holds. The additional axiom (DLoc) guarantees that the domain of ab is independent from the inner structure and the "far end" of b; information about the domain of b in interaction with a suffices. By this property the multimodal operators that can be defined in terms of domain become well-behaved under composition.

A *codomain* operation $\urcorner$ can easily be defined as a domain operation in the opposite semiring, where, as usual in algebra, opposition just swaps the order of multiplication. We call a semiring K with local domain and codomain a *modal semiring*.

A *Kleene algebra* [5] is a structure $(K, ^*)$ such that K is an idempotent semiring and the *star* * satisfies, for $a, b, c \in K$, the *unfold* and *induction laws*

$$\begin{array}{rl} 1 + aa^* \leq a^*, & 1 + a^*a \leq a^*, \\ b + ac \leq c \Rightarrow a^*b \leq c, & b + ca \leq c \Rightarrow ba^* \leq c. \end{array}$$

A Kleene algebra is $*$-*continuous* if it satisfies

$$ab^*c = \bigsqcup \{ab^nc \mid n \in \mathbb{N}\} \ .$$

A *Kleene algebra with tests (*KAT*)* [6] is a test semiring $(K, \mathsf{test}(K))$ such that K is a Kleene algebra. Finally, a *Kleene algebra with domain (*KAD*)* [1] is a Kleene algebra with tests that also provides a domain operation.

Below we examine in how far the existence of a domain operation depends on completeness of the test algebra and discuss the relation of the domain axioms to separability of elements by tests.

2 Completeness of the Test Algebra

Consider a test semiring $(K, \mathsf{test}(K), +, 0, \cdot, 1)$. In the sequel, $a, b, ...$ range over K and $p, q, ...$ range over $\mathsf{test}(K)$.

Define, for $a \in K$, the sets $\mathrm{LP}(a), \mathrm{LA}(a) \subseteq \mathsf{test}(K)$ of *left-preservers* and *left-annihilators* of a by

$$\mathrm{LP}(a) \stackrel{\text{def}}{=} \{p \mid a = pa\} \qquad \qquad \mathrm{LA}(a) \stackrel{\text{def}}{=} \{p \mid pa = 0\}$$

Lemma 2.1 *1.* $\mathrm{LP}(a)$ *is closed under upper bounds of arbitrary non-empty subsets and under infima of finite subsets. Hence,* $1 \in \mathrm{LP}(a)$*. In particular,* $\mathrm{LP}(a)$ *is a filter. Moreover,*

$$\mathrm{LP}(a) \cap \mathrm{LP}(b) \subseteq \mathrm{LP}(a + b).$$

2. $\mathrm{LA}(a)$ *is closed under lower bounds of arbitrary non-empty subsets and under suprema of finite subsets. Hence,* $0 \in \mathrm{LA}(a)$*. In particular,* $\mathrm{LA}(a)$ *is an ideal. Moreover,*

$$\mathrm{LA}(a) \cap \mathrm{LA}(b) \;=\; \mathrm{LA}(a+b).$$

3. *If* K *is a domain semiring then* $\ulcorner a$ *is the least element of* $\mathrm{LP}(a)$ *and hence the infimum of* $\mathrm{LP}(a)$*, whereas* $\neg\ulcorner a$ *is the greatest element of* $\mathrm{LA}(a)$ *and hence the supremum of* $\mathrm{LA}(a)$*.*

This implies that a test semiring can be a domain semiring only if its test algebra has sufficiently many infima and suprema. Since infima are unique, we have

Corollary 2.2 *Every test semiring with a finite test algebra can be extended uniquely to a domain semiring.*

We now give an example of a test semiring that lacks infima and suprema to such an extent that it cannot be extended to a domain semiring.

Example 2.3 Consider an infinite set M and the test semiring $K = (\mathcal{P}(M), \mathcal{Q}, \cup, \emptyset, \cap, M)$ where

$$\mathcal{Q} \stackrel{\text{def}}{=} \{N \subseteq M \mid N \text{ finite or cofinite}\}.$$

Then $\mathcal{Q} = \mathsf{test}(K)$ is a Boolean algebra that is incomplete.

For $A \in K$ and $Q \in \mathcal{Q}$ we have

$$Q \in \mathrm{LP}(A) \;\Leftrightarrow\; A \subseteq Q\;, \qquad\qquad Q \in \mathrm{LA}(A) \;\Leftrightarrow\; Q \subseteq \overline{A}\;.$$

Let A now be infinite and coinfinite. Suppose $\mathrm{LP}(A)$ has an infimum $Q \in \mathcal{Q}$. Then $A \subseteq Q$ but $A \neq Q$, since $A \notin \mathcal{Q}$. Let $x \in Q - A$. Now, Q must be cofinite, since $A \subseteq Q$. But then also $Q - \{x\}$ is cofinite with $Q - \{x\} \in \mathrm{LP}(A)$ and $Q - \{x\} \subsetneq Q$, a contradiction. Hence $\mathrm{LP}(A)$ does not have an infimum in $\mathcal{Q}$. Symmetrically, $\mathrm{LA}(\overline{A})$ does not have a supremum in $\mathcal{Q}$.

Thus, $\ulcorner A$ is undefined in K for infinite and coinfinite A. □

Hence the question arises whether the test algebra needs to be complete to admit a domain operation. This is not the case, as the following example shows.

Example 2.4 Consider an infinite set M. Choose a finite subset $N \subseteq M$ and set

$$K \stackrel{\text{def}}{=} \{R \subseteq M \times M \mid R \cap I \text{ finite or cofinite} \wedge R - I \subseteq N \times N\}$$

where $I \stackrel{\text{def}}{=} \{(x,x) \mid x \in M\}$. Moerover, set

$$\mathsf{test}(K) \stackrel{\text{def}}{=} \{R \mid R \subseteq I \,.\, R \text{ finite or cofinite}\}.$$

Then K is closed under $\cup$ and $;$, so that $(K, \mathsf{test}(K), \cup, \emptyset, ;, I)$ is a test semiring which under the standard definition

$$\ulcorner R \stackrel{\text{def}}{=} \{(x,x) \mid \exists\, y \,.\, (x,y) \in R\}$$

becomes a domain semiring. But $\mathsf{test}(K)$ is incomplete.

K can even be extended into a KAD, since, for $R \in K$,

$$\begin{aligned} R^* &= ((R \cap I) \cup (R - I))^* = (R \cap I)^* \,;\, ((R - I) \,;\, (R \cap I)^*)^* \\ &= (R - I)^* = I \cup (R - I)^+ \in K \,. \end{aligned}$$

The third step uses that $R \cap I \subseteq I$ implies $(R \cap I)^* = I$. □

3 Separability, Locality and Extensionality

In [7] the following two separability properties are studied:

$$ab = 0 \Rightarrow \exists\, q \,.\, a = aq \wedge b = \neg q b \,, \qquad \text{(Sep1)}$$

$$a \not\leq b \Rightarrow \exists\, p, q \,.\, paq \neq 0 \wedge pbq = 0 \,. \qquad \text{(Sep2)}$$

Actually, both properties can be strengthened to equivalences. For (Sep1), from the assumption $a = aq \wedge b = \neg q b$ we get

$$ab = aq\neg q b = a0b = 0 \,.$$

So we will use

$$ab = 0 \Leftrightarrow \exists\, q \,.\, a = aq \wedge b = \neg q b\ . \qquad \text{(Sep1')}$$

For (Sep2) we first note that, by contraposition, it is equivalent to

$$(\forall\, p, q \,.\, pbq = 0 \Rightarrow paq = 0) \Rightarrow a \leq b\ . \qquad \text{(Sep2')}$$

Now from the assumptions $a \leq b$ and $pbq = 0$ we obtain $paq = 0$ by isotonicity of multiplication.

It turns out that, in a domain semiring, (Sep1) is equivalent to (DLoc) and (Sep2) is equivalent to

$$(\forall\, q \,.\, |a\rangle q \leq |b\rangle q) \Rightarrow a \leq b\ , \qquad \text{(IDext)}$$

where the forward modal diamond operator $|a\rangle$ is defined as

$$|a\rangle q \stackrel{\text{def}}{=} \ulcorner(aq)\ .$$

This operator is the same as $\langle a\rangle$ in Dynamic Logic (see Section 4); the notation $|a\rangle$ has been chosen, since in a test semiring with codomain one can analogously define a backward diamond operator $\langle a|$.

Property (IDext) means *extensional isotonicity*. It can be expressed more succinctly using the pointwise ordering on test transformers $f, g : \mathsf{test}(K) \rightarrow \mathsf{test}(K)$:

$$f \leq g \stackrel{\text{def}}{\Leftrightarrow} \forall\, q \,.\, f(q) \leq g(q)\ ;$$

(IDext) then becomes

$$|a\rangle \leq |b\rangle \Rightarrow a \leq b\ .$$

By the definition of the natural order on idempotent semirings it is equivalent to the property of extensionality:

$$(\forall\, q \,.\, |a\rangle q = |b\rangle q) \Rightarrow a = b\ , \qquad \text{(Dext)}$$

or, point-free,

$$|a\rangle = |b\rangle \Rightarrow a = b\ .$$

Property (Sep2) induces the following extensionality property.

Corollary 3.1 *In a domain semiring*

$$(\forall\ p, q \,.\, pbq = 0 \Leftrightarrow paq = 0) \quad \Leftrightarrow \quad |a\rangle = |b\rangle\ .$$

If the test semiring K also has a codomain operation, we refer to the analogues of the above locality and extensionality properties as *co-locality/co-extensionality* properties.

Now we make the connections between the various properties precise.

Lemma 3.2 *Consider a domain semiring K.*

1. *$(Sep1) \Leftrightarrow (DLoc)$.*
2. *$(Sep2) \Leftrightarrow (IDext)$.*

Proof. 1. We prove the equivalent statement $(Sep1') \Leftrightarrow (DLoc)$.
$(\Rightarrow)$

$$\begin{aligned}
& \ulcorner(a\,\urcorner b) \leq \ulcorner(ab) \\
\Leftrightarrow\quad & \{\!\![\text{ by (GLA) }]\!\!\} \\
& \neg\ulcorner(ab)a\,\urcorner b = 0 \\
\Leftrightarrow\quad & \{\!\![\text{ by (Sep1') }]\!\!\} \\
& \exists\ q \,.\, \neg\ulcorner(ab)a = \neg\ulcorner(ab)aq \wedge \urcorner b = \neg q\,\urcorner b \\
\Leftrightarrow\quad & \{\!\![\text{ Boolean algebra }]\!\!\} \\
& \exists\ q \,.\, \neg\ulcorner(ab)a = \neg\ulcorner(ab)aq \wedge \urcorner b \leq \neg q \\
\Leftrightarrow\quad & \{\!\![\text{ by (LLP) }]\!\!\} \\
& \exists\ q \,.\, \neg\ulcorner(ab)a = \neg\ulcorner(ab)aq \wedge b = \neg q\, b \\
\Leftrightarrow\quad & \{\!\![\text{ by (Sep1') }]\!\!\} \\
& \neg\ulcorner(ab)ab = 0 \\
\Leftrightarrow\quad & \{\!\![\text{ by (GLA) }]\!\!\} \\
& \text{TRUE}\ .
\end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned}
& ab = 0 \\
\Leftrightarrow\quad & \{\!\![\text{ strictness (1) }]\!\!\}
\end{aligned}$$

$$\begin{array}{ll}
& \ulcorner(ab) = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{by (DLoc)}\ ]\!\!\} \\
& \ulcorner(a\,\ulcorner b) = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{strictness (1)}\ ]\!\!\} \\
& a\,\ulcorner b = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{by (GLA)}\ ]\!\!\} \\
& a = a\,\neg\ulcorner b \\
\Leftrightarrow & \quad \{\!\![\ \text{by(LLP)}\ ]\!\!\} \\
& a = a\,\neg\ulcorner b \ \wedge\ b = \ulcorner b\, b \\
\Rightarrow & \quad \{\!\![\ \text{setting } q = \neg\ulcorner b\ ]\!\!\} \\
& (Sep1')\ .
\end{array}$$

2. We prove the equivalent statement $(Sep2') \Leftrightarrow (IDext)$. It suffices to transform the premise of (Sep2') into that of (IDext):

$$\begin{array}{ll}
& \forall\ p, q\ .\ pbq = 0\ \Rightarrow\ paq = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{indirect domain inequality (Lemma 1.1)}\ ]\!\!\} \\
& \forall\ q\ .\ \ulcorner(aq) \leq \ulcorner(bq) \\
\Leftrightarrow & \quad \{\!\![\ \text{definition of } |_\rangle\ ]\!\!\} \\
& \forall\ q\ .\ |a\rangle q \leq |b\rangle q\ .
\end{array}$$

□

The main application of this lemma is a representation theorem that follows immediately from Theorem 3.11 in [7].

Theorem 3.3 *Every $*$-continuous extensional* KAD *is isomorphic to a* KAT *of relations which, however, may be non-standard in that the star operation need not coincide with reflexive-transitive closure.*

The paper [7] also provides a sufficient condition for that algebra of relations to be standard.

Another application of the Lemma is the following. Since (Sep1) and (Sep2) are symmetric in the tests involved, we have

Corollary 3.4 *A semiring with domain and codomain is extensional/local iff it is co-extensional/co-local.*

The part on locality has already been shown in [1] using that in a semiring with domain and codomain one has

$$(DLoc) \Leftrightarrow (ab = 0 \Leftrightarrow a^{\urcorner}\, {}^{\ulcorner}b = 0).$$

4 Modal Kleene Algebra and Dynamic Algebra

Obviously the setting of modal Kleene algebra with its forward diamond operators is very similar to the one of dynamic algebra. Let us therefore briefly relate them and point out some of their differences. Roughly, a *dynamic algebra* is a structure $(K, B, \langle\_\rangle\_)$ where K has operations $+$, $\cdot$ and *, B is a Boolean algebra and $\langle\_\rangle\_ : K \times B \to B$ is a scalar multiplication satisfying Segerberg's induction axiom. In the original definition by Kozen [4] K was required to be a $*$-continuous Kleene algebra. Later, Pratt [9] introduced a more liberal definition in which K was not required to satisfy any axioms at all.

The class KAD now blends Kleene algebras and dynamic algebras in a single-sorted structure. The forward diamond as defined above satisfies all axioms of dynamic algebra including the induction axiom (see [1] for details). Since the class KAD is based on KAT, it does not require $*$-continuity; rather it uses Kozen's later axiomatization [5] of $*$ as the least fixed point of an appropriate isotone function. Hence the class of dynamic algebras obtained via the diamond operator of KAD is larger than that of Kozen's dynamic algebras.

Another related structure is that of Kleene modules [3, 8]; these are like dynamic algebras in the sense of Kozen, only with $*$-continuity replaced by the least-fixpoint characterization of star. Further details on the precise relation between the various structures mentioned can be found in Section 5 of [2].

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