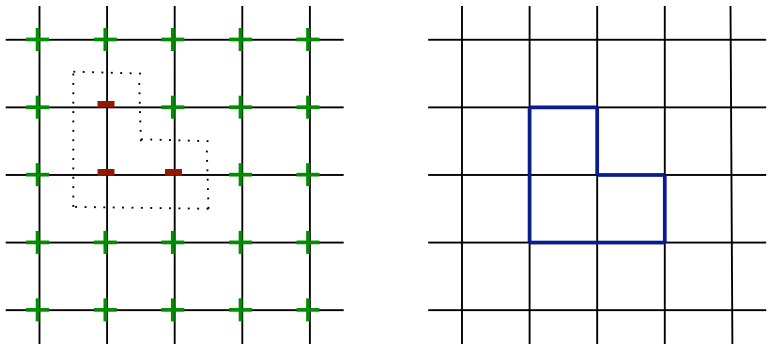
The 2d Ising model on a square lattice consists of spins $\sigma_{\vec{n}} = \pm 1$ at the sites of the lattice, an energy $E = -(J/k_B T)\sum_{\text{n.n.}} \sigma\sigma'$, where the sum is over nearest neighbor couplings ($\sum_{\text{n.n.}} \sigma\sigma' \equiv \sum_{\vec{n},\hat{k}=\hat{x},\hat{y}} \sigma_{\vec{n}}\sigma_{\vec{n}+\hat{k}}$), and the sign of the coupling is such that neighboring spins tend to align (ferromagnet). In terms of the dimensionless coupling $L \equiv J/k_B T$, the partition function is written

$$Z(L) = \sum_{\{\sigma_{\vec{n}}=\pm 1\}} \mathrm{e}^{L \sum_{\text{n.n.}} \sigma\sigma'}$$



(a) (b)

The zero temperature ground state is doubly degenerate, with either all spins + or all −. The total energy is $-N_\ell L$, where N_ℓ is the number of the links on the lattice (finite before taking the infinite lattice thermodynamic limit). At low temperatures, L is large so the typical configurations can be enumerated as small fluctuations from the ground state, with a few spins flipped to the opposite sign. A typical configuration is shown in part (a) of the figure. Note the energy for this state receives a contribution of $+2L$ from each link that connects a + to − spin. In the figure, these are the 8 links crossed by the dotted line enclosing the island of − spins in the sea of +'s. The "low temperature expansion" for the above partition function can thus be written

$$Z(L) = 2\mathrm{e}^{N_\ell L} \sum_{\text{paths } P'} \mathrm{e}^{-2L\mathcal{L}(P')} \,, \tag{1}$$

where the sum is over all possible (connected and disconnected) closed paths P' on the dual lattice, and $\mathcal{L}(P')$ is the length of the path. (The factor of 2 is due to the overall +/− symmetry.) The above summation can be evaluated, though that will not be necessary to locate the phase transition point of the model.

Now consider the "high temperature expansion", an expansion in small $K \equiv J/k_B T$. We use the identity $\mathrm{e}^{K\sigma} = \cosh K + \sigma \sinh K$ (true for $\sigma = \pm 1$), to write

$$\begin{aligned} Z(K) &= \sum_{\{\sigma_{\vec{n}}=\pm 1\}} \mathrm{e}^{K \sum_{\text{n.n.}} \sigma\sigma'} = \sum_{\{\sigma_{\vec{n}}=\pm 1\}} \prod_{\text{n.n.}} \mathrm{e}^{K\sigma\sigma'} \\ &= (\cosh K)^{N_\ell} \sum_{\{\sigma_{\vec{n}}=\pm 1\}} \prod_{\text{n.n.}} (1 + \sigma\sigma' \tanh K) \end{aligned}$$

The product in the last expression is over links in the lattice, and expanded out results in either a 1 or a factor of $\sigma\sigma' \tanh K$ for each link. But terms in the expansion with an odd number of any given $\sigma_{\vec{n}}$ will automatically cancel in the overall summation over $\sigma_{\vec{n}} = \pm 1$. This means that the only contributing terms in the expansion correspond to closed loops of links, for which any given σ appears an even number of times ($\sigma^{2m} = 1$ for $\sigma = \pm 1$).The "high temperature expansion" for the above partition function can thus be written

$$Z(K) = 2^{N_s} (\cosh K)^{N_\ell} \sum_{\text{paths } P} (\tanh K)^{\mathcal{L}(P)} \tag{2}$$

where the sum is over all possible (connected and disconnected) closed paths P on the lattice, $\mathcal{L}(P)$ is the length of the path (and the factor of 2^{N_s} comes from the summation over $\sigma_{\vec{n}} = \pm 1$ at each of the N_s sites).

Comparing equations (1) and (2), we see that the summations over paths are identical under the identification of couplings

$$\mathrm{e}^{-2L} = \tanh K \ .$$

This is a remarkable relation, since it relates large L to small K, and vice-versa. Although the low termperature (ordered) phase and high temperature (disordered) phase have completely different associated physics, their thermodynamic behaviors as a function of the coupling are directly related, and are thus said to be *dual*. (Note the prefactors in equations (1) and (2) are perfectly regular and do not affect the critical behavior in the thermodynamic limit.) By simple algebraic manipulation,

$$\mathrm{e}^{2L} = \frac{\cosh K}{\sinh K} = \frac{\mathrm{e}^{2K} + 1}{\mathrm{e}^{2K} - 1} = 1 + \frac{2}{\mathrm{e}^{2K} - 1} \ ,$$

the above duality relation can be written in the more symmetric form:

$$(\mathrm{e}^{2L} - 1)(\mathrm{e}^{2K} - 1) = 2 \ . \tag{3}$$

The identification of the phase transition point requires one additional piece of physical input: that there is only one such point. This is based on the physical picture of a non-zero spontaneous magnetization at low temperature (ordered phase), and a zero magnetization at high temperature (disordered phase), with only a single critical point separating the two. But eq. (3) relates every temperature but one to a different temperature. Only the temperature L^*, satisfying $(\mathrm{e}^{2L^*} - 1)^2 = 2$, is reflected back onto itself under the duality relation. If there is only a single transition, then it must thus occur at this "self-dual" point,

$$L^* = \frac{1}{2} \ln(\sqrt{2} + 1) \ .$$

Additional Comments:

The location and other properties of the phase transition in the 2d Ising model are verified by the exact solution by Onsager (1944) (for a more modern treatment, see Schultz, Mattis, and Lieb (1964)).

The expectation value of the average magnetization per site, $\langle m \rangle = \langle \sum_{\vec{n}} \sigma_{\vec{n}} \rangle / N_s$, can be calculated by coupling to an external magnetic field h:

$$\langle m \rangle = -\frac{1}{N_s} \frac{\partial}{\partial h} F \Big|_{h=0}$$

where

$$\mathrm{e}^{-F/k_B T} = Z = \sum_{\{\sigma_{\vec{n}} = \pm 1\}} \mathrm{e}^{\left(J \sum_{\text{n.n.}} \sigma \sigma' + h \sum_{\vec{n}} \sigma_{\vec{n}} \right) / k_B T} \ .$$

It is found to be non-zero for temperatures below the critical point, and vanishes for temperatures at or above that point.

Note also that the duality relation of the previous page carries over to all multi-spin correlation functions of the model, not just the partition function.

For a historical overview of the exact solution to the 2d Ising model and its implications for the development of statistical mechanics, see http://arxiv.org/cond-mat/9511003 (Bhattacharjee and Khare, Fifty Years of the Exact Solution of the Two-Dimensional Ising Model by Onsager), written on the 50th anniversary of Onsager's solution.