COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 7 (4) (1987) 141—142

THE GAP BETWEEN MONOTONE AND NON-MONOTONE CIRCUIT COMPLEXITY IS EXPONENTIAL

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Received October 6, 1986

A. A. Razborov has shown that there exists a polynomial time computable monotone Boolean function whose monotone circuit complexity is at least $n^{c\log n}$. We observe that this lower bound can be improved to $\exp(cn^{1/6-o(1)})$. The proof is immediate by combining the Alon—Boppana version of another argument of Razborov with results of Grötschel—Lovász—Schrijver on the Lovász — capacity, ϑ of a graph.

A. A. Razborov [6, 7] recently proved surprising superpolynomial ($n^{c\log n}$) lower bounds for the monotone circuit complexity of the following two properties of an input graph X on v vertices ($n=v^2$ is the number of input bits):

(a) X has a perfect matching,
(b) X has a clique of size $f(v)$ for some simple function $f(v)$.

The lower bound (b) has been improved to a properly exponential function ($\exp(cn^{1/6-o(1)})$) by N. Alon and R. Boppana [1].

It is a conceptual advantage of (a) that the problem considered there is polynomial time solvable and therefore can be computed by a polynomial size non-monotone Boolean circuit, thus establishing a superpolynomial gap between the monotone and non-monotone circuit complexities of monotone Boolean functions.

The aim of this is note to show that the gap is properly exponential. This follows fairly easily from the Alon—Boppana improvement of Razborov's argument for (b), combined with results of Lovász [4] and Grötschel—Lovász—Schrijver [2] on the Shannon — capacity of a graph.

It is easy to see that the argument of Razborov actually applies not only to the clique number $\omega(X)$ but to any graph function $\varphi(X)$ satisfying $\omega(X)\leqq\varphi(X)\leqq\chi(X)$ where $\chi(X)$ denotes the chromatic number. This observation carries over to the Alon—Boppana improvement and yields the following corollary:

Corollary (A. A. Razborov; N. Alon and R. Boppana). Let $\varphi(X)$ be any monotone graph function such that

$$(*)\qquad \omega(X) \leqq \varphi(X) \leqq \chi(X).$$

Then for any function $3\leqq f(v)\leqq(v/\log v)^{2/3}/4$ the monotone circuit complexity of deciding whether or not $\varphi(X)\leqq f(v)$ is at least $\exp\left(c\cdot f(v)^{1/2}\right)$.

AMS subject classification: 68 C 25.

Now, in order to justify the claim that the gap is properly exponential, we just have to point out that there exists a polynomial time computable monotone function $\varphi(X)$ satisfying $(*)$.

In his seminal paper on the Shannon-capacity of graphs [4] Lovász introduced the capacity $\vartheta(X)$. The function $\varphi(X)=\vartheta(\overline{X})$, where $\overline{X}$ denotes the complement of X, is a monotone function satisfying $(*)$. Grötschel, Lovász and Schrijver [GLS] gave a polynomial time approximation algorithm for ϑ. That is, given a graph X and a rational number $\varepsilon>0$ the algorithm computes, in polynomial time, a function $g(X, \varepsilon)$ such that

$$\vartheta(X) \leqq g(X, \varepsilon) \leqq \vartheta(X)+\varepsilon.$$

Now, for any $0<\varepsilon<1/2$ the function $\lfloor g(\overline{X}, \varepsilon)\rceil$, where $\lfloor\alpha\rceil$ denotes the integer nearest to the number α, is a polynomial time computable function satisfying $(*)$. But this function might not be monotone. Let us introduce instead the function

$$\varphi(X) = \lfloor g(\overline{X}, v^{-2})+e(X)\cdot v^{-2}\rceil,$$

where $e(X)$ denotes the number of edges in X. $\varphi(X)$ is a polynomial time computable monotone function satisfying $(*)$.

Acknowledgements. I would like to thank László Babai for many helpful discussions, and László Lovász for pointing out an error in an earlier version of this note.

References

[1] N. Alon and R. Boppana, The monotone circuit complexity of Boolean functions, *Combinatorica* **7** (1987), 1—23.

[2] M. Grötschler, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169—197.

[3] G. L. Khachiyan, A polynomial algorithm in linear programming, *Doklady Akademii Nauk SSSR* **244** (1979), 1093—1096 (English translation: *Soviet Math. Dokl.* **20**, 191—194).

[4] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. on Information Theory* **25** (1979), 1—7.

[5] L. Lovász, An Algorithmic Theory of Numbers, Graphs and Convexity, SIAM Philadelphia 1986.

[6] A. A. Razborov, Lower bounds on the monotone complexity of some Boolean functions, *Doklady Akademii Nauk SSSR* **281** (1985), 798—801.

[7] A. A. Razborov, A lower bound on the monotone network complexity of the logical permanent, *Matematischi Zametki* **37** (1985), 887—900.

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