# Birkhoff Averages, Invariant Sets, and Adaptive Filtering


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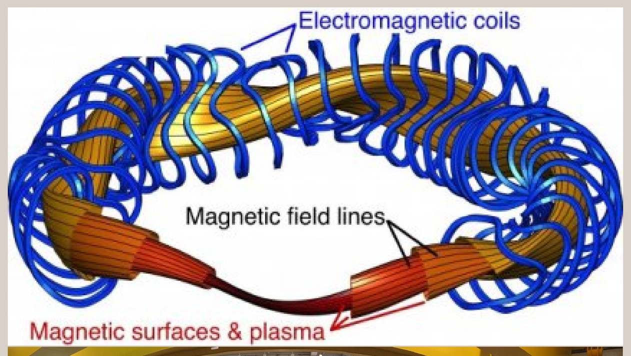
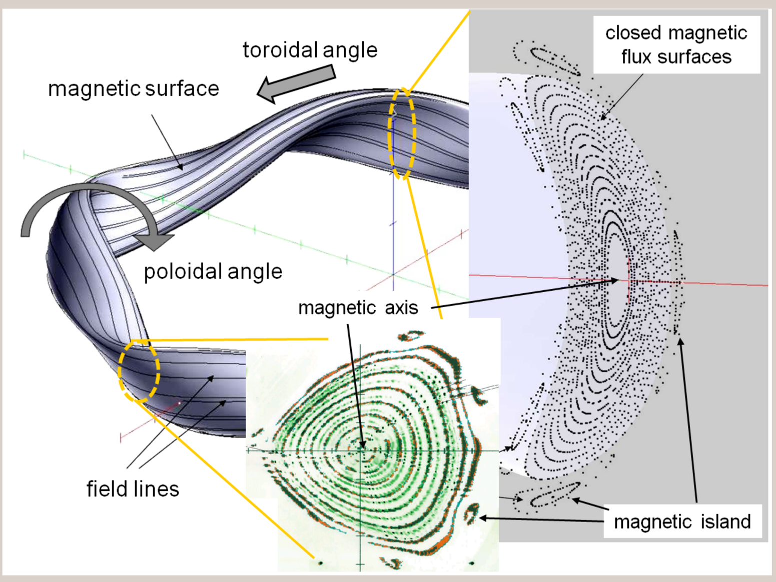
## Motivation: Stellarators and Fusion Plasmas



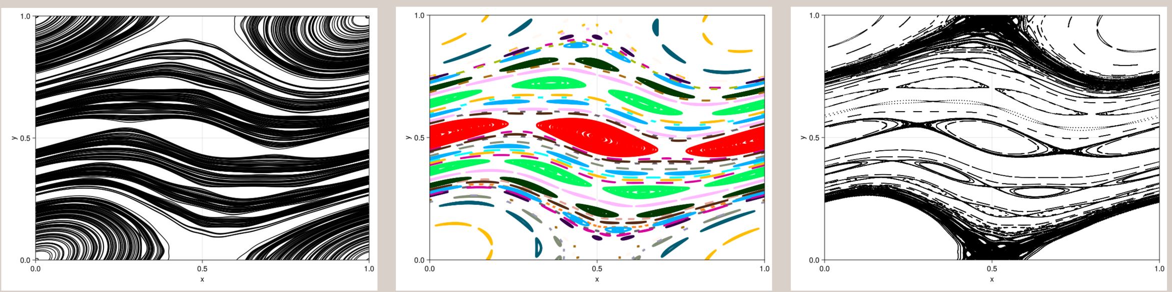

- Charged particles in plasma (roughly) follow magnetic field.
- Field flow defines a symplectic map on a Poincaré section:

$$x_{t+1} = F(x_t), \quad F : \mathbb{R}^2 \to \mathbb{R}^2.$$

- Goal: Auto-identify structures (quasi-periodic orbits, chaos)

## Processing Poincaré Plots

Circles Islands Chaos



1. Make a Poincaré plot and eyeball it
2. Parameterization method
3. Form a function with invariant level sets
4. Directly model dynamics for a field line

## Parameterization Method

Goal: Find invariant circles $z : \mathbb{T} \to \mathbb{R}^2$ s.t.

$$F(z(\theta)) = z(\theta + \omega).$$

Determined up to choice of $z(0)$ (phase + which circle). Expand:

$$z(\theta) = \sum_{n=-\infty}^{\infty} \hat{z}_n \exp(2\pi i n\theta).$$

Parameterization approach: truncate expansion and solve for coefficients via a nonlinear least squares at collocation points.

*Problem:* Requires initial guess, does not handle chaos.

## Birkhoff Average

Consider $F : \Omega \to \Omega$ symplectic, $h \in \mathscr{C}^\infty(\Omega)$. Define *Birkhoff average*:

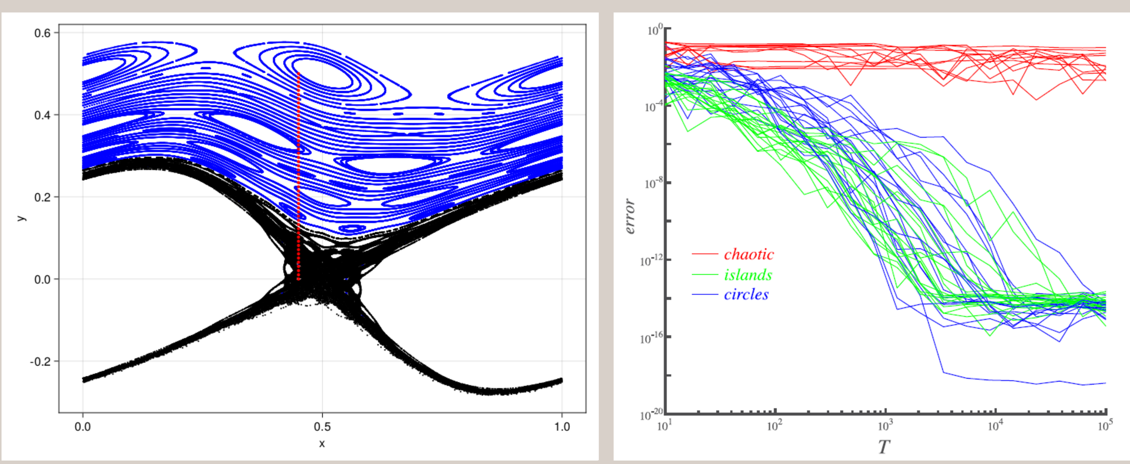
$$\mathscr{B}_K[h](x) = \frac{1}{K+1} \sum_{k=0}^{K} (h \circ F^k)(x).$$

*Birkhoff-Khinchin*: for $h \in \mathscr{L}^1$, converges a.e. to conditional expectation of an invariant measure on an invariant set.

Convergence of $\mathscr{B}_K[h](x) - \bar{h}(x)$ signals regular vs chaotic:
Invariant circle/island? $O(K^{-1})$ Chaos? $O(K^{-1/2})$

*Problem:* $K^{-1}$ is slow convergence! And $\mathscr{B}_K[h]$ may not be smooth.

## Weighted Birkhoff average

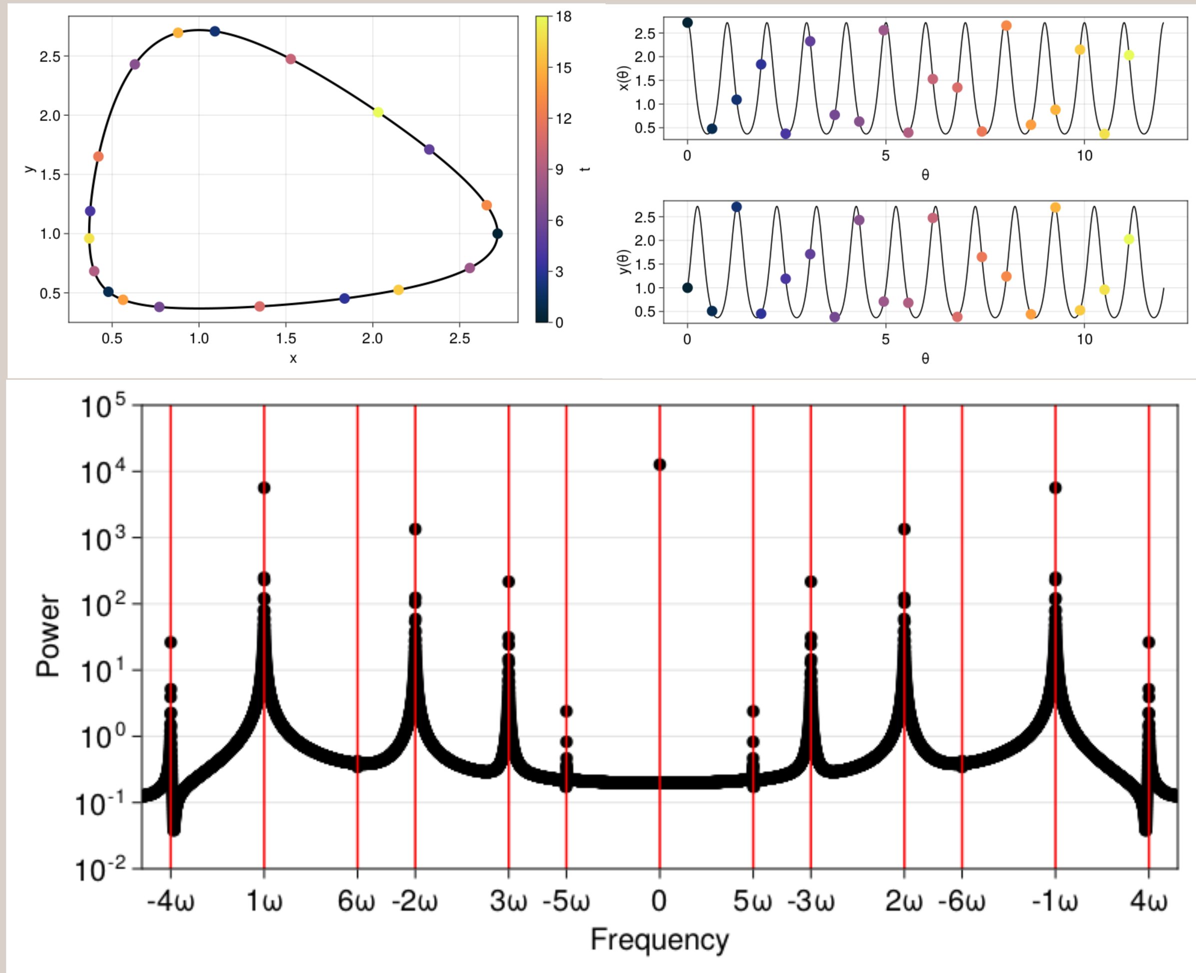



Sander and Meiss, Physica D, 411 (2020) p. 132569;
Das, Sander, and Yorke, Nonlinearity, 30 (2017), pp. 4111-4140

Weighting accelerates regular convergence to super-algebraic:

$$\mathscr{W}\mathscr{B}_K[h](x) = \sum_{k=0}^{K} w_{k,K} (h \circ F^k)(x).$$

## Signal Processing Perspective

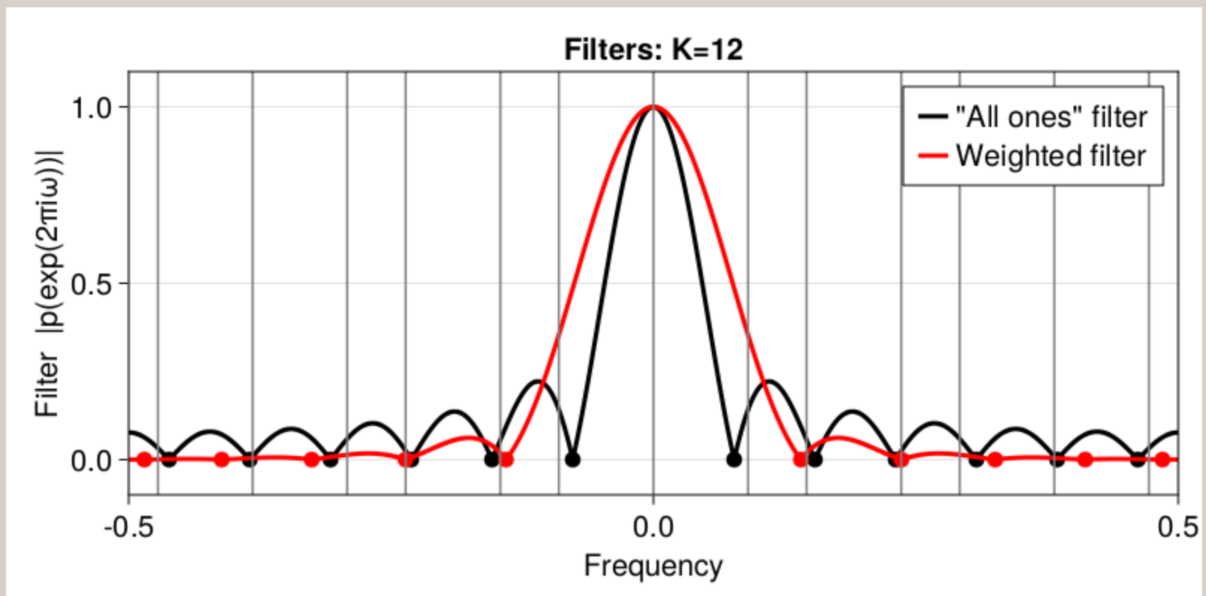



Expand signals and plot coefficient magnitude vs angle:

$$x_t = z(\omega t) = \sum_{n\in\mathbb{Z}} \hat{z}_n \xi^{nt}, \qquad \xi = \exp(2\pi i \omega)$$

$$h_t = \sum_{n\in\mathbb{Z}} \hat{h}_n \xi^{nt}, \qquad \bar{h} = \hat{h}_0$$

## Birkhoff and weighted Birkhoff

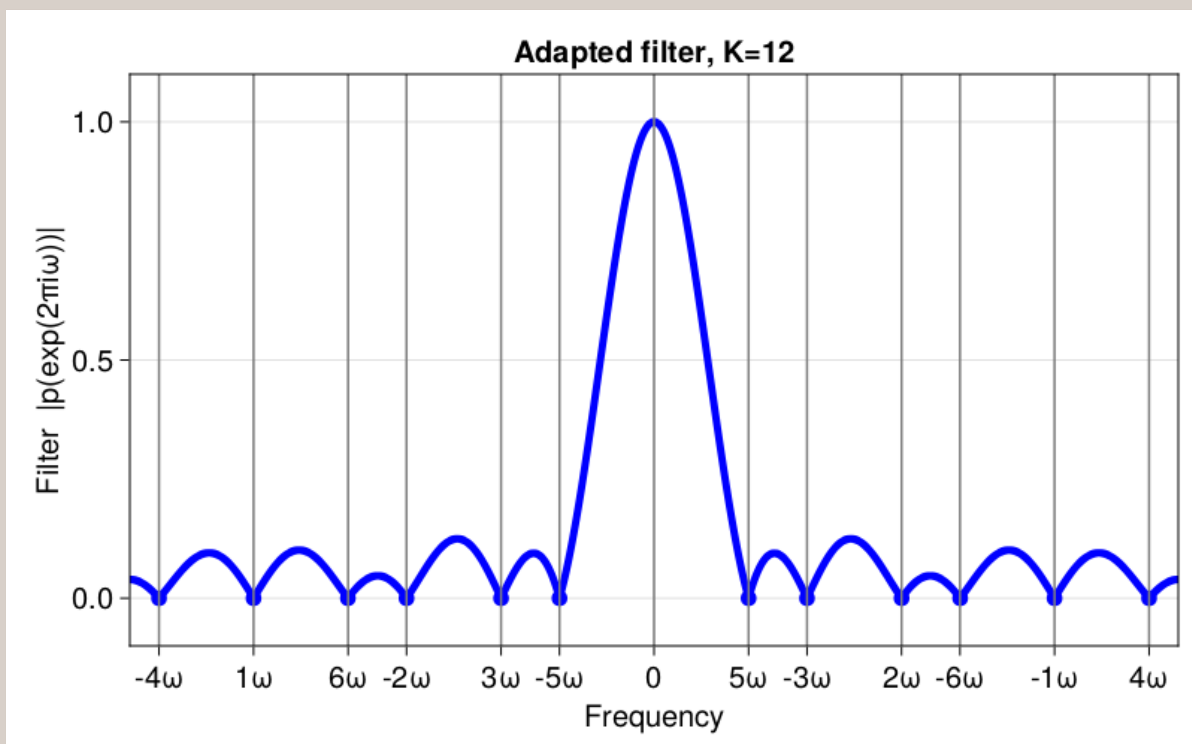



Weighted Birkhoff starting from $x_0$

$$\mathscr{W}\mathscr{B}_K[h](x_0) = \sum_{n\in\mathbb{Z}} \hat{h}_n p_K(\xi^n), \quad p_K(\zeta) = \sum_{k=0}^{K} w_{k,K} \zeta^k$$

Ordinary Birkhoff (uniform weights) gives cyclotomic polynomial.

## Adaptive Filtering




Difference to remove mean: $u_t = h_{t+1} - h_t = \sum_{n\neq 0} (\xi - 1)\xi^{nt} \hat{h}_n$.
Seek *adaptive* weighted Birkhoff coeffs $c_k$ to minimize

$$\sum_{t=0}^{T-1} \left( \sum_{k=0}^{K} c_k u_{k+t} \right)^2 \text{ s.t. } \sum_{k=0}^{K} c_k = 1 \text{ and } c \text{ palyndromic.}$$

Beyond filtering: Extract $\xi$ by inspecting roots for the polynomial $\sum_{k=0}^{K} \zeta^k$, project signal onto Fourier modes to recover shape for invariant sets — no more need to analyze $h$!

## For Much More

- On this method: Ruth and Bindel, Chaos, 34(12), 2024
- A related approach: Ruth and Bindel, SIADS, 24(1), 2025
- Imbert-Gérard, Paul, Wright, *An Introduction to Stellarators* (SIAM)
- Simons Collaboration: `hiddensymmetries.princeton.edu`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