Linear Stability: Some Thoughts

David Bindel

08 Dec 2023

Department of Computer Science
Cornell University

Ideal MHD linear stability

Standard approach:

- Analyze Hessian of W (stable if positive definite)
- Or analyze $-\rho\omega^2\xi = F\xi$.

Numerical steps either way:

- Discretize
- Possibly spectral transform (shift-invert)
- Compute a few eigenpairs via Lanczos or Arnoldi
- Maybe compute more with filtering (e.g. EVSL, FEAST)

There are good textbooks on this stuff:

Spectral Approximation of Linear Operators (Chatelin);

Templates for the Solution of Algebraic Eigenvalue Problems (Bai, Demmel, Dongarra, Ruhe, van der Vorst, eds)

Plan today

Some salient points (if you're a numerical analyst)

- Discrete + essential spectrum (slow and Alfvén)
- Highly symmetric geometry
- Maybe we just care about stability (vs spectrum)

Two mini-talks:

- Symmetry and why it matters
- Stability constraints in optimization

And a question: what do we want to compute?

Group reminder

A group $\mathcal{G}$ is a set with

- an identity element $e \in \mathcal{G}$
- an associative operation (multiplication)
- inverses.

Can be continuous $(\mathrm{GL}(\mathcal{V}), O(\mathcal{V}), SO(\mathcal{V}))$ or discrete.

Symmetry group

Common use of group theory to describe *symmetries*:

I *think* stellarator symmetry corresponds to a dihedral group.

Group representations

A *representation* of a group $\mathcal{G}$ is a homomorphism

$$\rho : \mathcal{G} \to \mathsf{GL}(\mathcal{V}).$$

For a finite group and a Hilbert space, the map goes to $O(\mathcal{V})$.

Decomposition ideas:

- *Subrepresentation* is $\mathcal{U} \subset \mathcal{V}$ s.t. $\forall g \in \mathcal{G}$, $\rho(g)\mathcal{U} \subset \mathcal{U}$.
- Requires $\rho(g)\mathcal{U} \subset \mathcal{U}$ (invariant subspace)
- *Irreducible* if no nontrivial subrepresentations.
- *Character table* gives basic types of irreps.
- *Canonical decomposition* of $\mathcal{V}$ by type of irrep.

Ex: $\mathcal{V} = \mathcal{L}^2(\mathbb{R}), \quad \mathcal{G} = \mathbb{Z}/2\mathbb{Z}, \quad [\rho(g)f](x) = f(-x)$
canonical decomposition into even and odd functions.

Why we care

Suppose

- Operators A and G commute
- $\mathcal{U}$ a *maximal invariant subspace* of G for eigenvalue μ

Then for any $u \in \mathcal{U}$,

$$Gu = \mu u \implies AGu = \mu Au \implies GAu = \mu Au \implies Au \in \mathcal{U}$$

$\mathcal{U}$ is invariant for A (i.e. $A\mathcal{U} \subset \mathcal{U}$).

Note: $AG = GA$ and $AH = HA$ with $GH \neq HG$
$\implies \exists u : Au = \mu u, Gu \neq Hu$
$\implies A$ has multiple eigenvalues

Why does this matter?

Symmetry-adapted bases block-diagonalize A:

- Can construct by hand or use canonical projectors.
- Sets up for less expensive computations.
- Sometimes split degenerate modes across blocks $\implies$ convergence easier for eigensolvers.

NB: Works on continuous spectrum as well.

Caira Anderson working on a solver that uses this.

Stability without eigenvalues

- Sometimes maybe we want part of spectrum.
- But for checking stability, eigenvalues are overkill.
- What to do instead?

Factorization

Simplest check for positive definiteness: *Cholesky factorization*

$$A = R^T R, \quad R \text{ upper triangular}$$

- Succeeds (with nonzero diagonal) iff R nonsingular.
- Can take advantage of sparsity.
- For block diagonal A, just factor the blocks.

But this gives a binary determination – how to use in optimization?

Log determinant

Given Cholesky factorization $A = R^T R$

$$\det(A) = \det(R)^2 = \prod_j r_{jj}^2$$

Better scaled:

$$\log\det(A) = 2\sum_j \log r_{jj}$$

Log barrier idea: given vector θ of design parameters,

$$\text{minimize } \phi(\theta) - \lambda \log\det(A(\theta))$$

Differentiation of log barrier:

$$\delta[\log\det(A)] = \operatorname{tr}(A^{-1}\delta A) = \langle A^{-1}, \delta A\rangle_F.$$

Note: Can compute relatively quickly when δA low rank.

Log determinant

Log-det pros and cons

- Does need to be interior point!
- Natural from an interior point perspective (though have to be careful with nonconvexity of objective – see, e.g. Kocvara 2002 in context of stability-constrained truss design).
- May worry about scalability for large discretizations
    - Tricks like low-rank δA help
    - Can also use stochastic trace estimators

Could probably use this with DCON3D approach...

Bordered system

Considered bordered system

$$\begin{bmatrix} A(\theta) & b \\ b^T & 0 \end{bmatrix} \begin{bmatrix} f(\theta) \\ g(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- KKT for $\mathcal{K} = \frac{1}{2} f^T A f + g^T (b^T f - 1)$
- Well-posed if $\dim \mathcal{N}(A(\theta)) \leq 1$ (and $b \not\perp \mathcal{N}(A(\theta))$)
- $g(\theta) = 0$ iff $A(\theta) f(\theta) = 0$
- Sign $g(\theta)$ is $(-1)^{(1+\text{nneg})}$ (nneg = # negative eigs)
- Have tricks for fast solves (Govaerts and Pryce)

Bordered system

Considered bordered system

$$\begin{bmatrix} A(\theta) & b \\ b^T & 0 \end{bmatrix} \begin{bmatrix} f(\theta) \\ g(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Differentiate:

$$\begin{bmatrix} A(\theta) & b \\ b^T & 0 \end{bmatrix} \begin{bmatrix} \delta f(\theta) \\ \delta g(\theta) \end{bmatrix} = \begin{bmatrix} -\delta A(\theta) f(\theta) \\ 0 \end{bmatrix}.$$

Cost per derivative: one bordered solve (re-use factorizations).

Bordered system

Bordered system pros and cons

- Might worry about big steps with even change in nneg
- Issue with higher-dimensional kernels
  - Can fix with wider borders (Govaerts BEMW)
  - Can also ameliorate issue of even nneg
- Can't use same b everywhere
  - But random will work most of the time

What do we want to compute?

Things we can compute *without* getting all eigenvalues

- Partial decomposition via symmetry groups
- Stability test functions (in this talk)
- Inertia (counts of positive, negative, zero eigs)
- Bounds on distance to instability
- Extremal eigenvalues and derivatives
- Densities of states
- And more!

The question you ask matters a lot! So what do we want?