

Goldilocks' discovery of Newton's method for approximation
required surprisingly few changes.

<https://www.xkcd.com/872/>

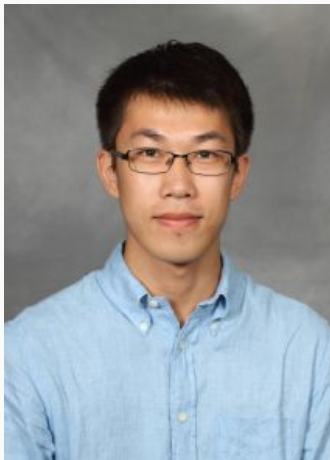
The Structure and Interpretation of Graph Spectral Densities

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8 April 2022

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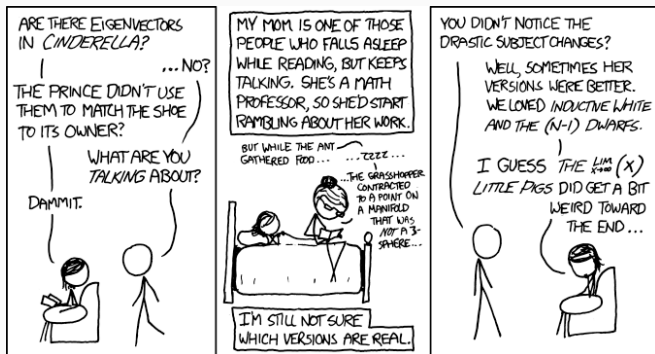
Acknowledgements



Thanks to **Kun Dong** and Austin Benson, along with Anna Yesypenko, Moteleolu Onabajo, Jianqiu Wang.

Also: NSF DMS-1620038.

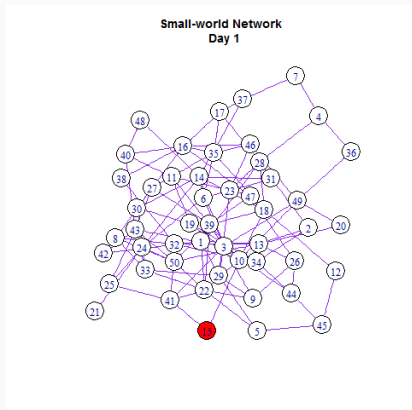
Prelude: Tales from the Spectrum's Edge



Geometry and graph theory share many spectral stories:

- Dynamics (operator on functions over manifold or graph)
- Counting and measure (quadratic form)
- Geometric embedding (pos semidef form / kernel)

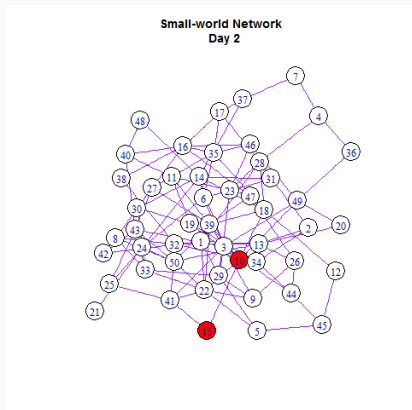
Tales from the Spectrum's Edge



Dynamics (operator on functions over manifold or graph)

- Diffuse according to heat kernel $\exp(-tLD^{-1})$
- Mixing rates and bottlenecks appear through spectrum

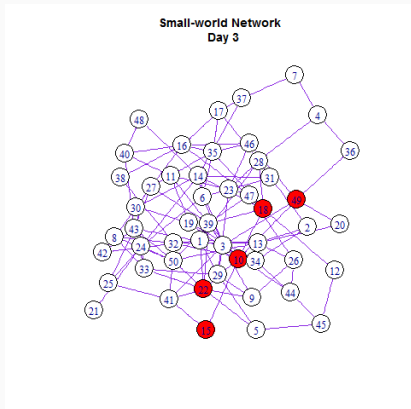
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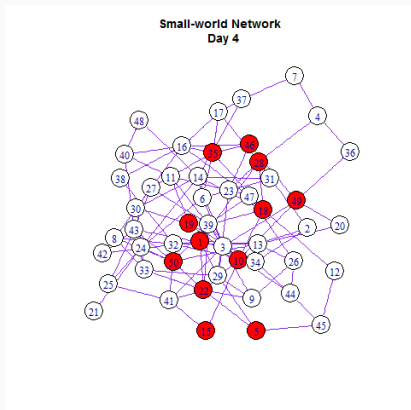
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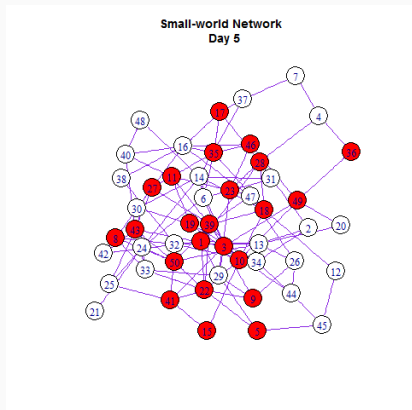
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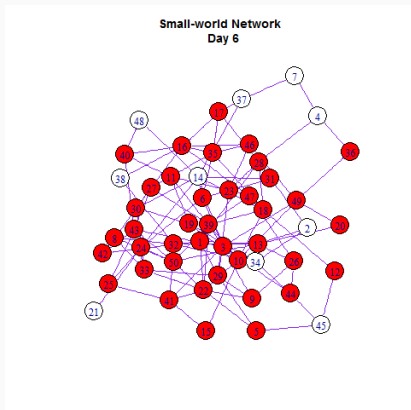
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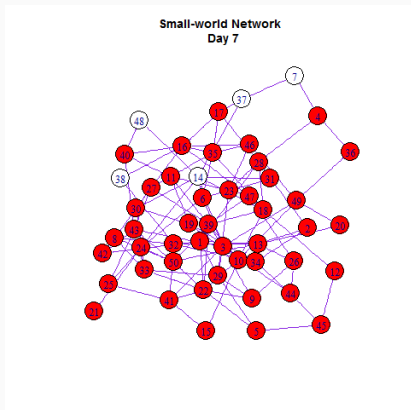
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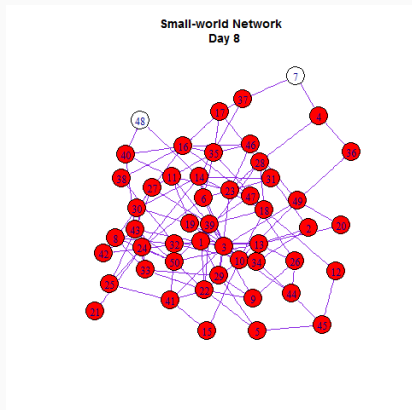
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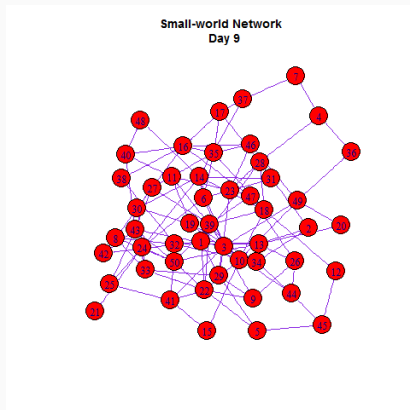
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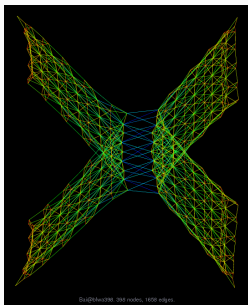
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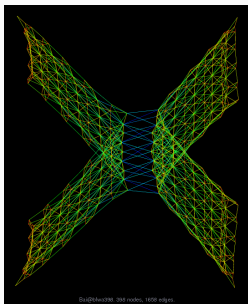
Tales from the Spectrum's Edge: Counting and Measure



Counting and measure (quadratic form)

- Example: Spectral partitioning
- Measure cut sizes via $x^T L x / 4$, $x \in \{\pm 1\}^n$, relax to $\mathbb{R}^n$
- $\lambda_2(L)$ bounds cuts (Cheeger), partition with Fiedler vector

Tales from the Spectrum's Edge

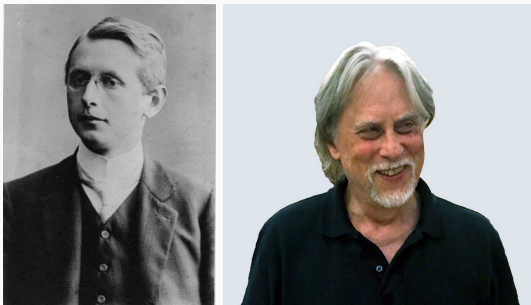


Geometric embedding (pos semidef form / kernel)

- Example: Spectral coordinates via $R = L^\dagger$
- Diffusion distances: $d_{ij}^2 = r_{ii}^2 - 2r_{ij} + r_{jj}^2$
- First few eigenvectors of R give coordinates that approximate distance

Eigenvalues Two Ways

What can we tell from *partial* spectral information (eigenvalues and/or vectors)?

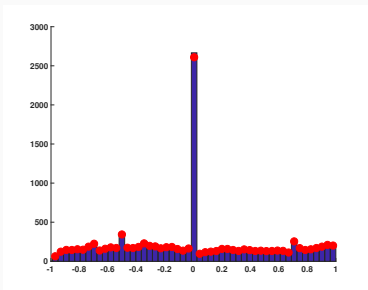


Claim: Most spectral analyses involve one of two perspectives:

- Approximate something via a *few* (extreme) eigenvalues.
- Look at *all* the eigenvalues (or all in a range).

Eigenvalues Two Ways

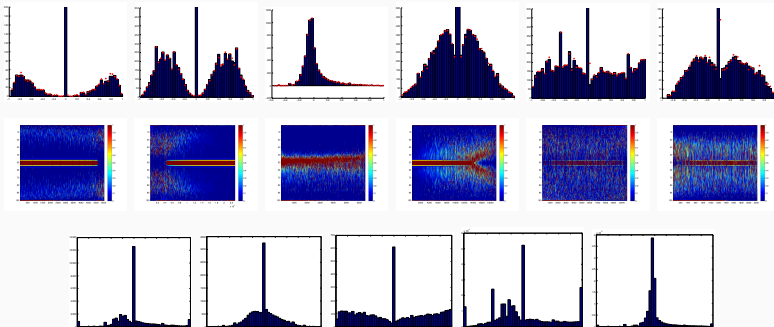
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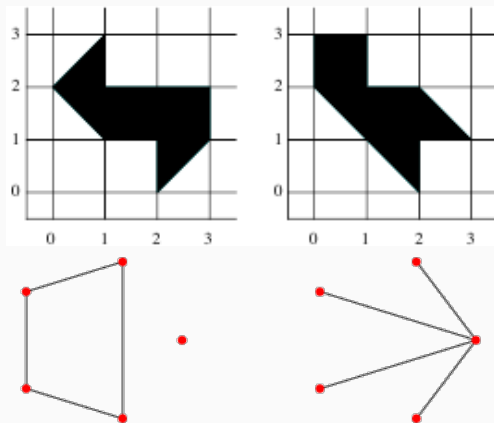
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Today in Three Acts



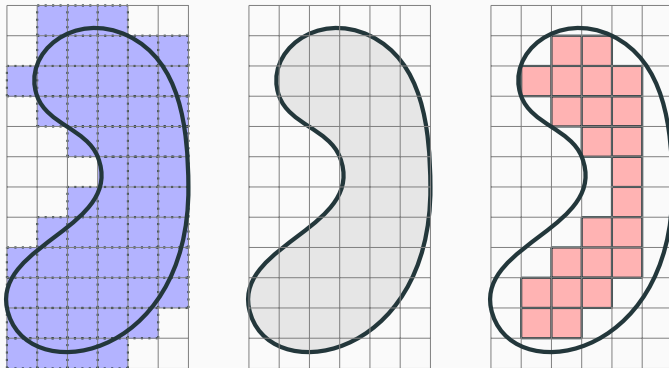
- Act 1: Spectral densities
- Act 2: Algorithms and approximations
- Act 3: Spectral densities of “real world” graphs

Can One Hear the Shape of a Drum?



"You mean, if you had perfect pitch could you find the shape of a drum." — Mark Kac (quoting Lipmann Bers)
American Math Monthly, 1966

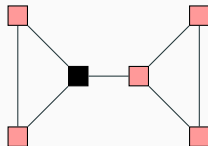
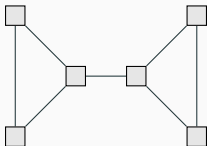
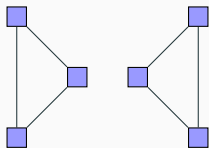
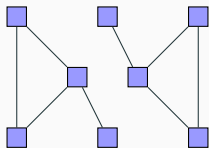
What Do You Hear?



Use $\mathcal{H}_{\text{lo}} \supset \mathcal{H} \supset \mathcal{H}_{\text{hi}}$ to get $\lambda_{k,\text{lo}} \leq \lambda_k \leq \lambda_{k,\text{hi}}$

$$\lambda_k = \min_{\dim(\mathcal{V})=k, \mathcal{V} \subset \mathcal{H}} \left(\max_{v \in \mathcal{V}} \rho_L(v) \right)$$

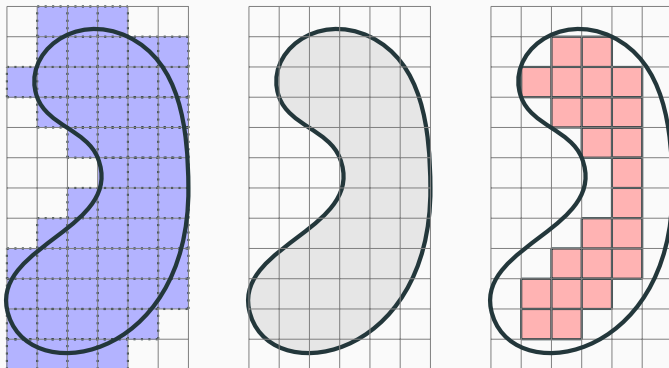
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What Do You Hear?



Weyl law (asymptotics for $N(x) = \{\# \text{ eigenvalues } \leq x\}$):

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega).$$

What Do You Hear?

What information hides in the eigenvalue distribution?

1. Discretizations of Laplacian: something like Weyl's law
2. Sparse E-R random graphs: Wigner semicircular law
3. Some other random graphs: Wigner semicircle + a bit
(Farkas *et al*, Phys Rev E (64), 2001)
4. "Real" networks: less well understood

But computing all eigenvalues seems *expensive*!

A Bestiary of Matrices

- Adjacency matrix: A
- Laplacian matrix: $L = D - A$
- Unsigned Laplacian: $L = D + A$
- **Random walk matrix:** $P = AD^{-1}$ (or $D^{-1}A$)
- **Normalized adjacency:** $\bar{A} = D^{-1/2}AD^{-1/2}$
- **Normalized Laplacian:** $\bar{L} = I - \bar{A} = D^{-1/2}LD^{-1/2}$
- Modularity matrix: $B = A - \frac{dd^T}{2n}$
- Motif adjacency: $W = A^2 \odot A$

All have examples of co-spectral graphs

... through spectrum uniquely identifies *quantum graphs*

Reminder: Spectral Mapping

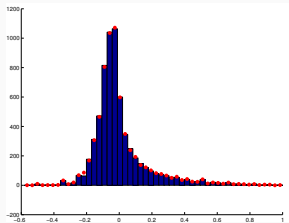
Consider a matrix H , and let f be analytic on the spectrum.

Then if $H = V\Lambda V^{-1}$,

$$f(H) = Vf(\Lambda)V^{-1}.$$

(generalizes to non-diagonalizable case)

Another Perspective: Density of States

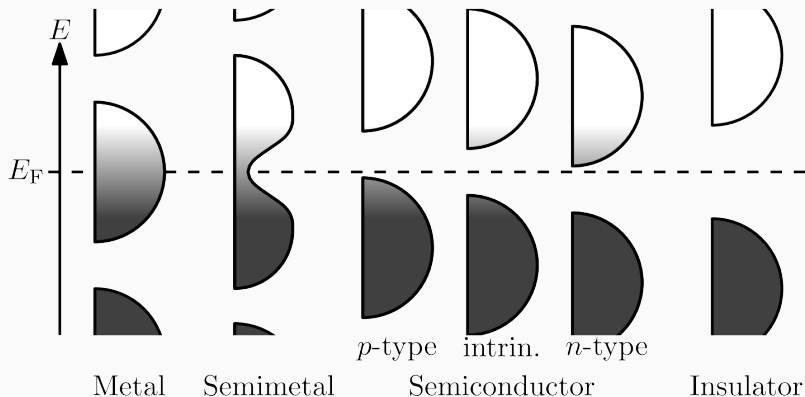


Spectra define a *generalized function* (a *density*):

$$\mathrm{tr}(f(H)) = \int f(\lambda) \mu(\lambda) dx = \sum_{k=1}^N f(\lambda_k)$$

where f is an analytic test function. Smooth to get a picture: a *spectral histogram* or *kernel density estimate*.

Density of States in Physics



How is this useful in the graph case?

Example: Estrada Index

Consider

$$\text{tr}(\exp(\alpha A)) = \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \cdot (\# \text{ closed random walks of length } k).$$

- Global measure of connectivity in a graph.
- Can clearly be computed via DoS.
- Generalizes to other weights.

DoS information equivalent to looking at the *heat kernel trace*:

$$h(s) = \text{tr}(\exp(-sH)) = \mathcal{L}[\mu](s)$$

Use $H = LD^{-1}$ (continuous time random walk generator) $\implies$
 $h(s)/N = P(\text{self-return after time } s \text{ from uniform start}).$

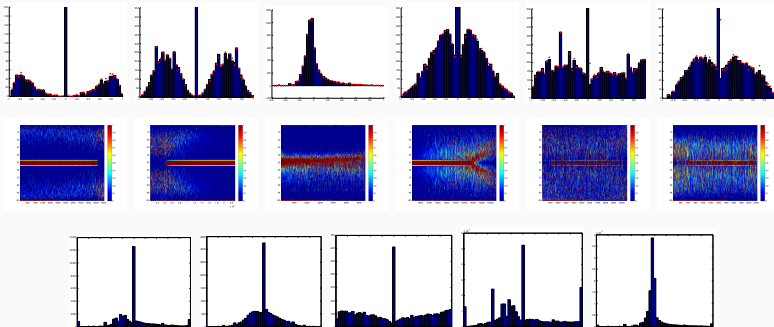
DoS information equivalent to looking at the *power moments*:

$$\text{tr}(H^j).$$

Natural interpretation for matrices associated with graphs:

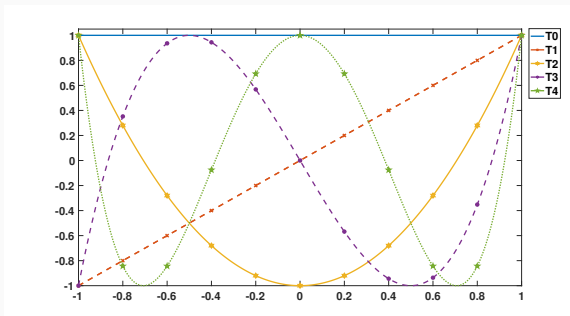
- A : number of length k cycles.
- $\bar{A}$ or P : return probability for k -step random walk (times N).
- L : ??

Today in Three Acts



- Act 1: Spectral densities
- **Act 2: Algorithms and approximations**
- Act 3: Spectral densities of “real world” graphs

Chebyshev Moments



For numerics, prefer Chebyshev moments to power moments:

$$d_j = T_j(A)$$

where $T_j(z) = \cos(j \cos^{-1}(z))$ is the j th Chebyshev polynomial:

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_{k+1}(z) = 2zT_k(z) - T_{k-1}(z).$$

Exploring Spectral Densities

Kernel polynomial method (see Weisse, Rev. Modern Phys.)

- Spectral distribution on $[-1, 1]$ is a generalized function:

$$\int_{-1}^1 \mu(x) f(x) dx = \frac{1}{N} \sum_{k=1}^N f(\lambda_k)$$

- Write $f(x) = \sum_{j=1}^{\infty} c_j T_j(x)$ and $\mu(x) = \sum_{j=1}^{\infty} d_j \phi_j(x)$, where $\int_{-1}^1 \phi_j(x) T_k(x) dx = \delta_{jk}$
- Estimate $d_j = \text{tr}(T_j(H))$ by stochastic methods
- Truncate series for $\mu(x)$ and filter (avoid Gibbs)

Much cheaper than computing all eigenvalues!

Alternatives: Lanczos (Golub-Meurant), maxent (Röder-Silver)

Stochastic Trace and Diagonal Estimation

$Z \in \mathbb{R}^n$ with independent entries, mean 0 and variance 1.

$$E[(Z \odot HZ)_i] = \sum_j h_{ij} E[Z_i Z_j] = h_{ii}$$

$$\text{Var}[(Z \odot HZ)_i] = \sum_j h_{ij}^2.$$

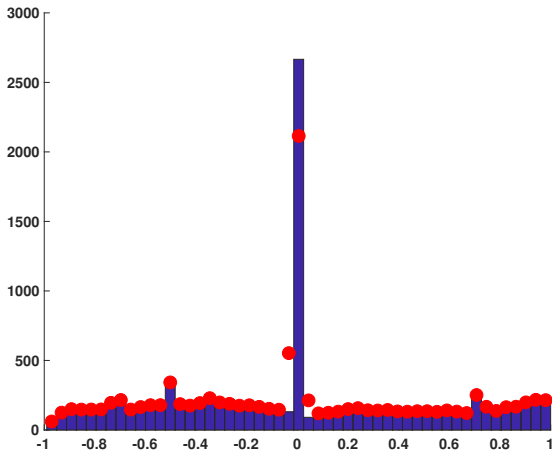
Serves as the basis for stochastic estimation of

- Trace (Hutchinson, others; review by Toledo and Avron)
- Diagonal (Bekas, Kokiopoulou, and Saad)

Independent probes $\implies 1/\sqrt{N}$ convergence (usual MC).

Can do better with some cleverness.

Example: PGP Network



Spike (non-smoothness) at eigenvalues of 0 leads to inaccurate approximation.

Motifs and Symmetry

Suppose $PH = HP$. Then

$\mathcal{V}$ a max invariant subspace for $P \implies$

$\mathcal{V}$ a max invariant subspace for H

So *local symmetry* $\implies$ *localized eigenvectors*.

Simplest example: P swaps (i, j)

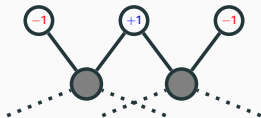
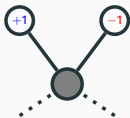
- $e_i - e_j$ an eigenvector of P with eigenvalue -1
- $e_i - e_j$ an eigenvector of $\bar{A}$ with eigenvalue

$$\lambda = \rho_{\bar{A}}(e_i - e_j) = \begin{cases} d^{-1}, & (i, j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

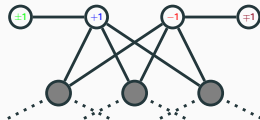
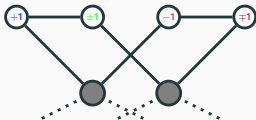
- All other eigenvectors (eigenvalue -1) satisfy $v_i = v_j$

Motifs in Spectrum

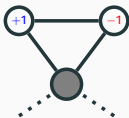
- $\lambda = 0$



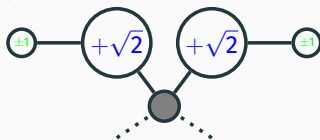
- $\lambda = \pm 1/2$



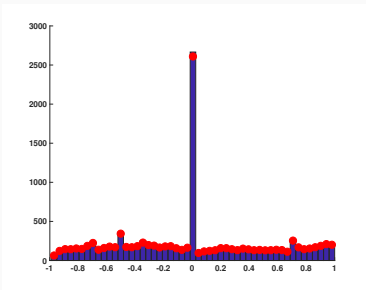
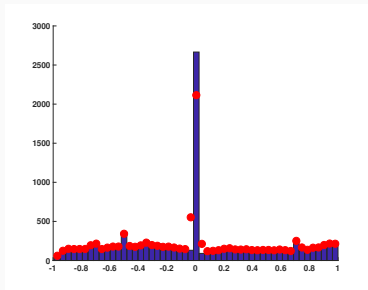
- $\lambda = -1/2$



$$\lambda = \pm 1/\sqrt{2}$$



Motif Filtering



Motif “spikes” slow convergence – deflate motif eigenvectors!

If $P \in \mathbb{R}^{n \times m}$ an orthonormal basis for the quotient space,

- Apply estimator to $P^T \overline{A} P$ to reduce size for $m \ll n$.
- or use $Proj_P(Z)$ to probe the desired subspace.

Diagonal Estimation and LDoS

Diagonal estimation also useful for *local* DoS $\nu_k(x)$;
in the symmetric case with $H = Q\Lambda Q^T$, have

$$\int f(x)\nu_k(x) dx = f(H)_{kk} = e_k^T Q f(\Lambda) Q^T e_k$$

$$\nu_k(x) = \sum_{j=1}^n q_{kj}^2 \delta(x - \lambda_j)$$

DoS is sum of local densities of states:

$$\mu(x) = \sum_{k=1}^n \nu_k(x)$$

Same game, different moments:

- Estimate $d_j = [T_j(H)]_{kk}$ by diag estimation
- Truncate series for $\mu(x)$ and filter (avoid Gibbs)

Diagonal estimator gives moments *for all k simultaneously!*

Alternatives: Lanczos (Golub-Meurant), maxent (Röder-Silver)

Can compute common *centrality measures* with LDoS

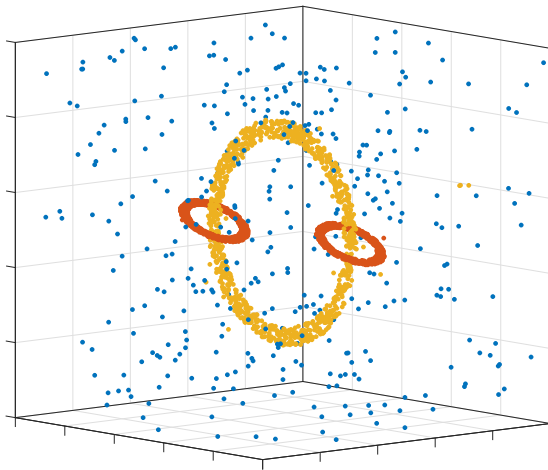
- Estrada centrality: $\exp(\gamma A)_{kk}$
- Resolvent centrality: $[(I - \gamma \bar{A})^{-1}]_{kk}$

Some motifs associated with localized eigenvectors:

- Chief example: Null vectors of $\bar{A}$ supported on leaves.
- Use LDoS + topology to find motifs?

What else?

LDoS and Clustering



Phase Retrieval in Graph Reconstruction

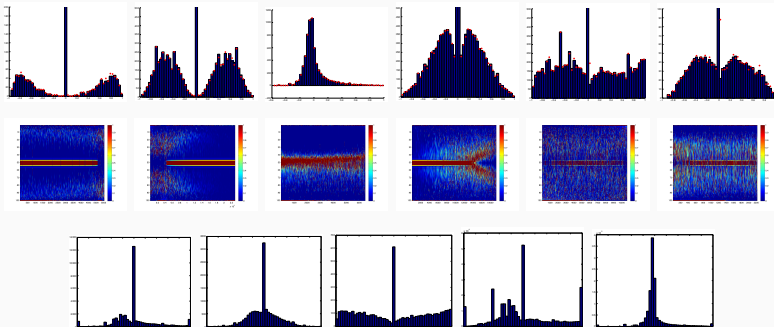
Reconstruct graph from *fully resolved* LDoS at all nodes?

- Assume $H = Q\Lambda Q^T$
- No multiple eigenvalues $\implies$ know $|Q|$ and Λ
- Can we recover signs in Q ?

Feels a little like phase retrieval...

Of course, we usually have noisy LDoS estimates!

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Exploring Spectral Densities (with David Gleich)

- Compute spectrum of normalized Laplacian / RW matrix
- Compare KPM to full eigencomputation

Things we know

- Eigenvalues in $[-1, 1]$; nonsymmetric in general
- Stability: change d edges, have

$$\lambda_{j-d} \leq \hat{\lambda}_j \leq \lambda_{j+d}$$

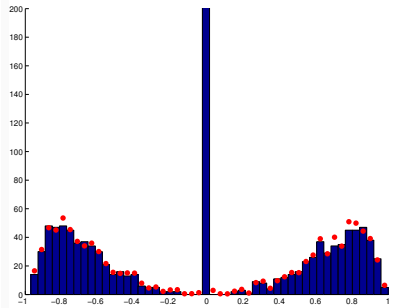
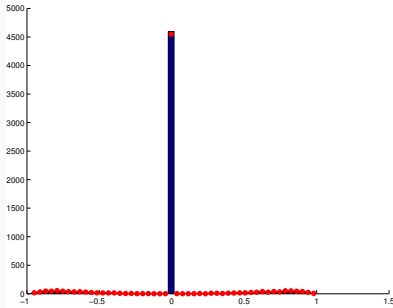
- k th moment = $P(\text{return after } k\text{-step random walk})$
- Eigenvalue cluster near 1 $\sim$ well-separated clusters
- Eigenvalue cluster near -1 $\sim$ bipartite structure
- Eigenvalue cluster near 0 $\sim$ leaf clusters

What else can we “hear”?

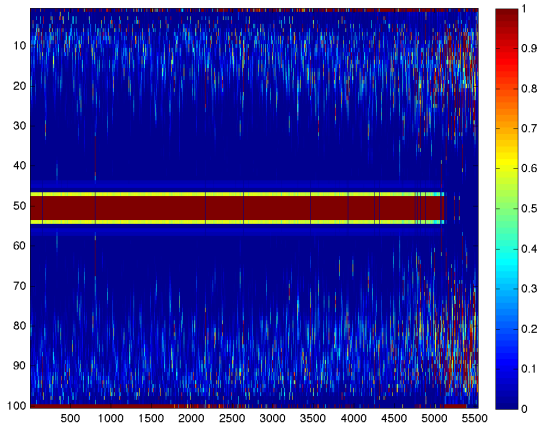
Experimental setup

- Global DoS
 - 1000 Chebyshev moments
 - 10 probe vectors (componentwise standard normal)
 - Histogram with 50 bins
- Local DoS
 - 100 Chebyshev moments
 - 10 probe vectors (componentwise standard normal)
 - Plot smoothed density on $[-1, 1]$
 - Spectrally order nodes by density plot

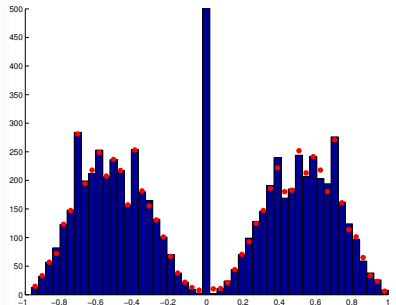
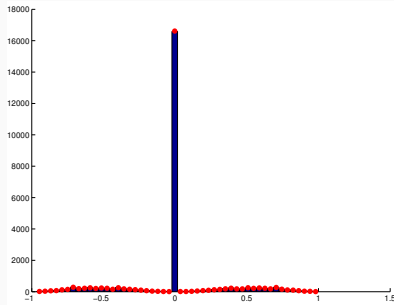
Suggestions for better pics are welcome!



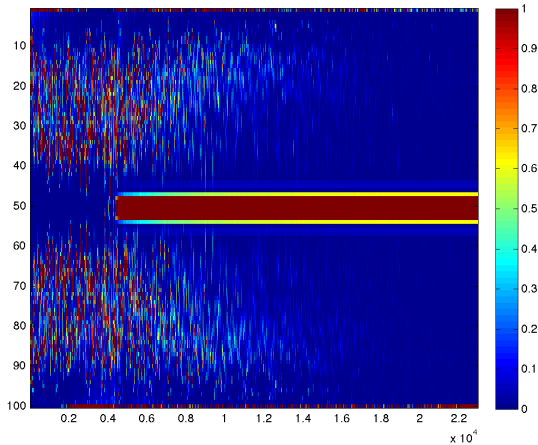
Erdos (local)



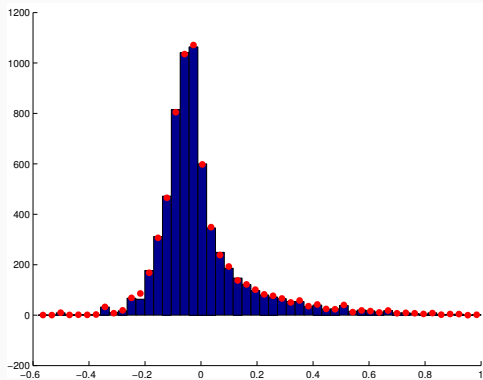
Internet topology



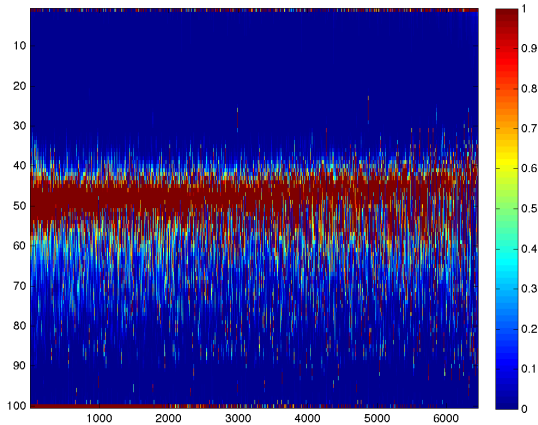
Internet topology (local)



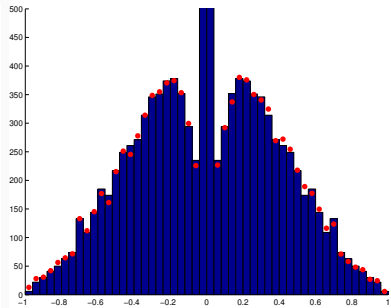
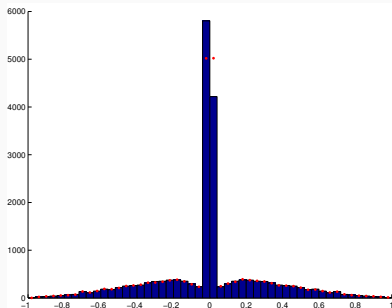
Marvel characters



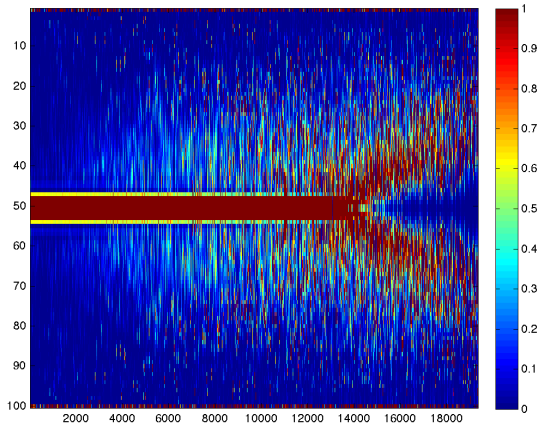
Marvel characters (local)

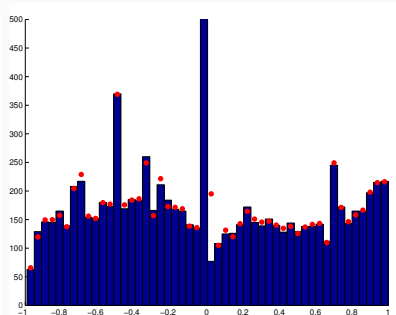
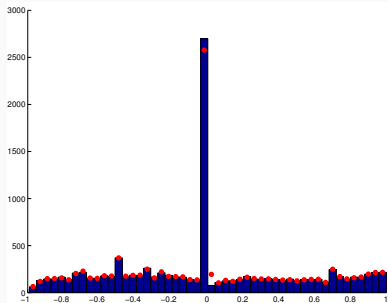


Marvel comics

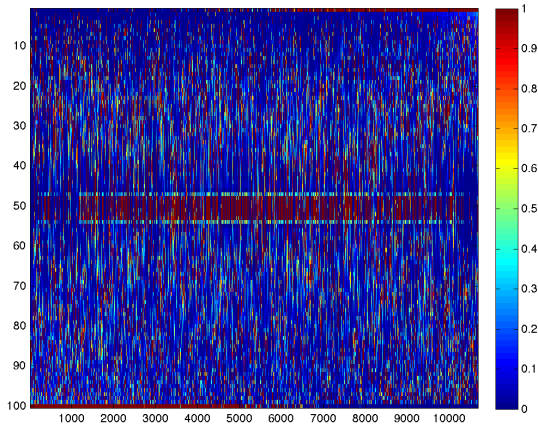


Marvel comics (local)

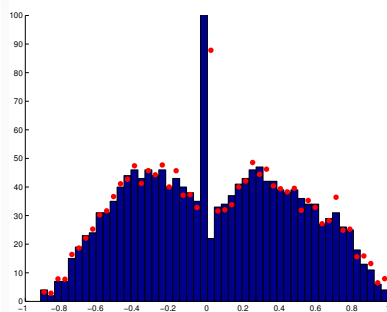
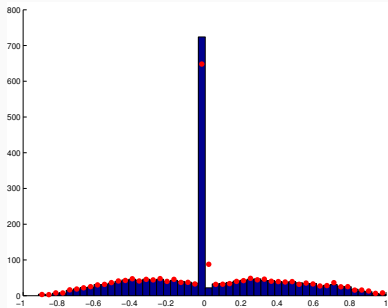




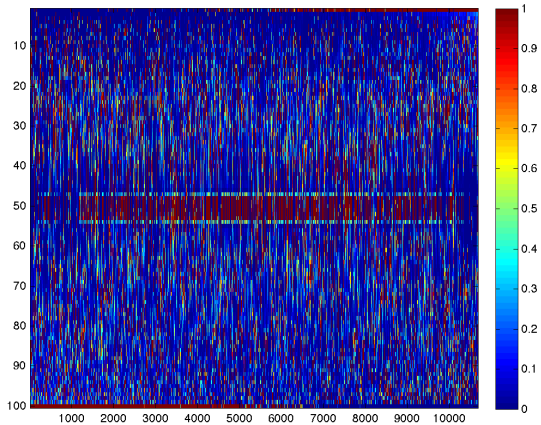
PGP (local)



Yeast

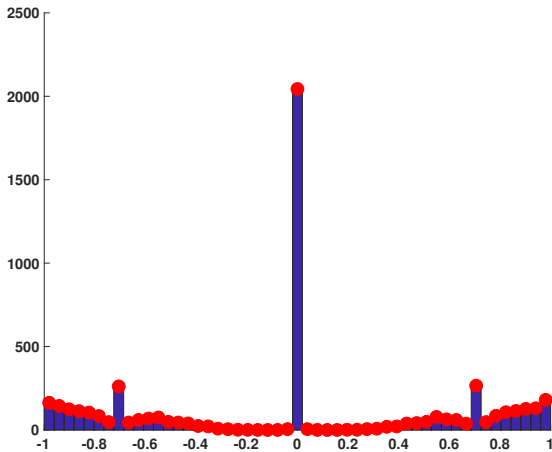


Yeast (local)



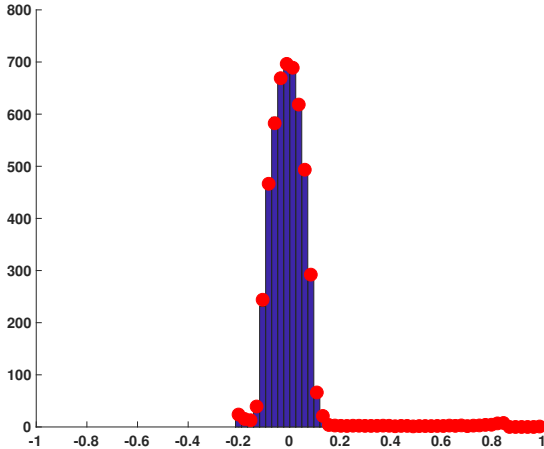
What about random graph models?

Barabási–Albert model



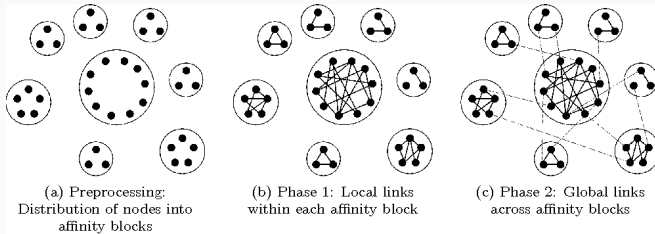
Scale-free network (5000 nodes, 4999 edges)

Watts-Strogatz



Small world network (5000 nodes, 260000 edges)

Model Verification: BTER



Kolda et al, SISC (36), 2014

Block Two-Level Erdős-Rényi model (BTER)

- First Phase: Erdős-Rényi Blocks
- Second Phase: Using Chung-Lu Model to connect blocks with $p_{ij} = p(d_i, d_j)$

Model Verification: BTER

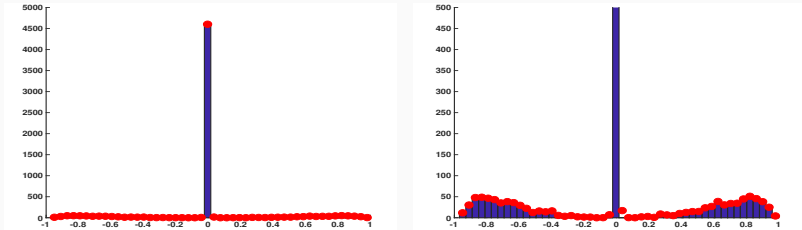


Figure 1: Erdos collaboration network.

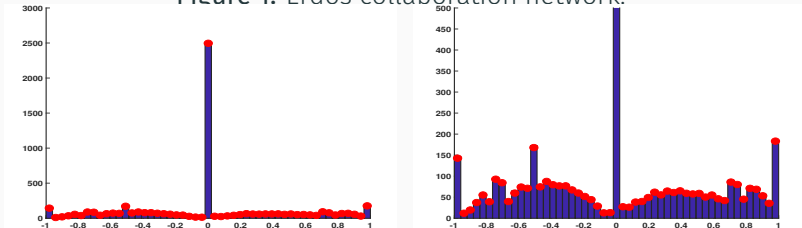
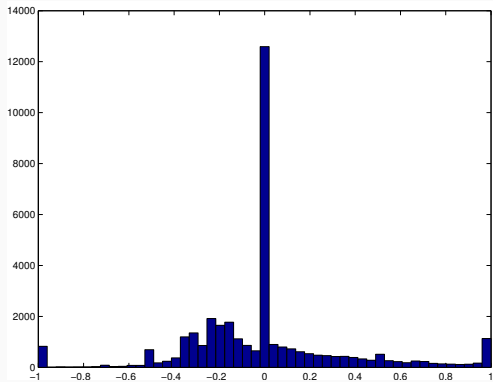


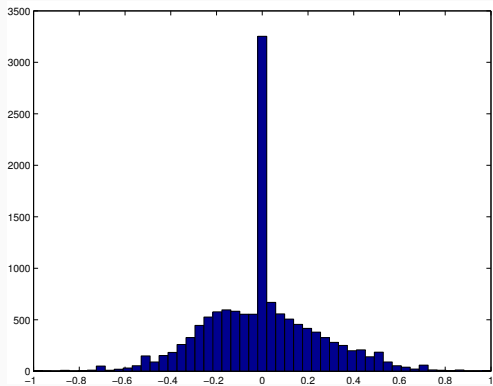
Figure 2: BTER model for Erdos collaboration network.

And a few more...

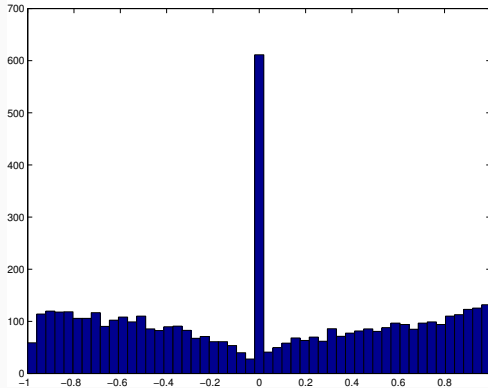
Enron emails (SNAP)

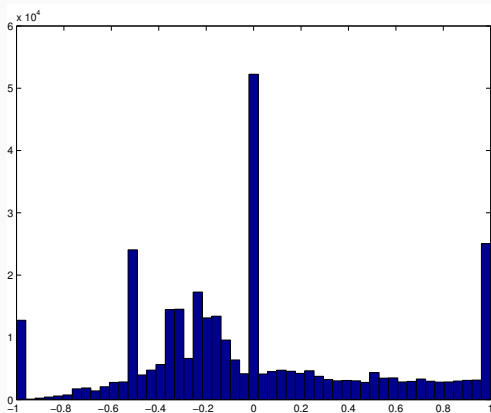


Reuters911 (Pajek)



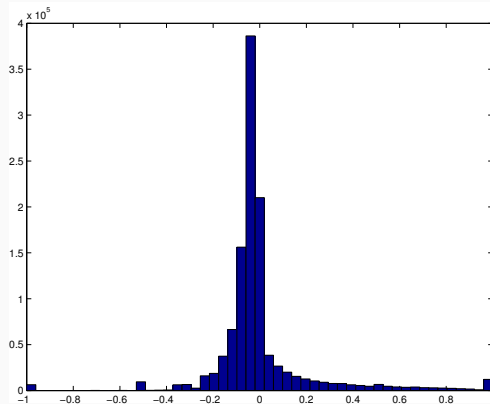
US power grid (Pajek)





$N = 326186$, $nnz = 1615400$, 80 s (1000 moments, 10 probes)

Hollywood 2009 (LAW)

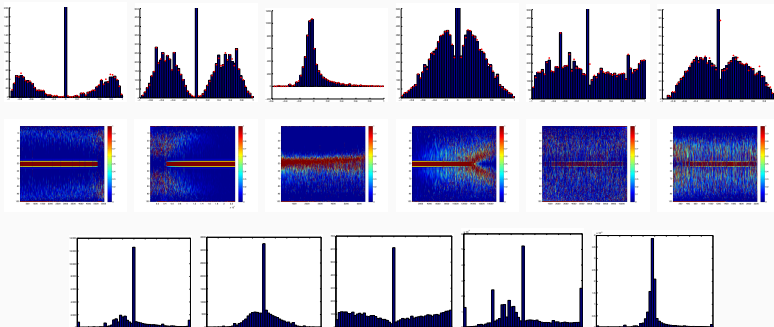


$N = 1139905$, $nnz = 113891327$, 2093 s (1000 moments, 10 probes)

Questions for You?

- Any isospectral graphs for multiple matrices?
- Can we recover topology from (exact) LDoS?
- Variance reduction in diagonal estimators?
- Random graphs with spectra that look “real”?
- Compression of moment information for diag estimators?
- More applications?

What Do You Hear?



Dong, Benson, Bindel. “Network Density of States.”

Proceedings of KDD 2019.