# Music of the Microspheres

## Eigenvalue problems from micro-gyro design

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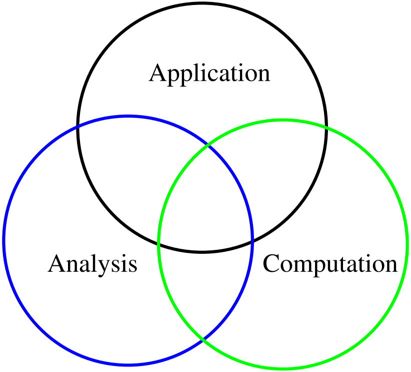
UTRC, 19 Nov 2013

# The Computational Science & Engineering Picture

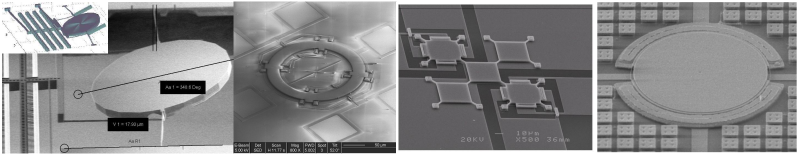



- MEMS (today)
- Smart grids
- Networks

- Linear algebra
- Approximation theory
- Symmetry + structure

- HPC / cloud
- Simulators
- Little languages

# A Favorite Application: MEMS

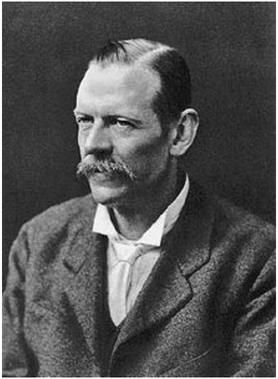



I've worked on this for a while:

- SUGAR (early 2000s) – SPICE for MEMS
- HiQLab (2006) – high-Q mechanical resonator device modeling
- AxFEM (2012) – solid-wave gyro device modeling

Goal today: an illustrative snapshot.

# G. H. Bryan (1864–1928)



- Fellow of the Royal Society (1895)
- *Stability in Aviation* (1911)
- Thermodynamics, hydrodynamics

*Bryan was a friendly, kindly, very eccentric individual...*

*(*Obituary Notices of the FRS*)*

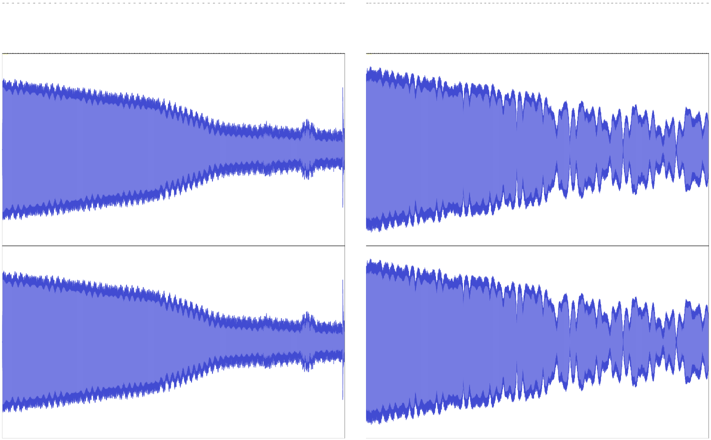
*... if he sometimes seemed a colossal buffoon, he himself did not help matters by proclaiming that he did his best work under the influence of alcohol.*

*(Williams, J.G.,* The University College of North Wales, 1884–1927*)*

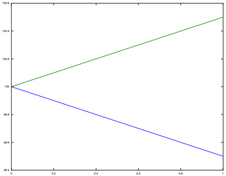
# Bryan's Experiment

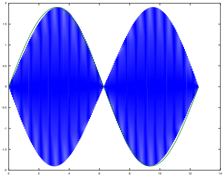
"On the beats in the vibrations of a revolving cylinder or bell"
by G. H. Bryan, 1890

# Bryan's Experiment Today



# The Beat Goes On



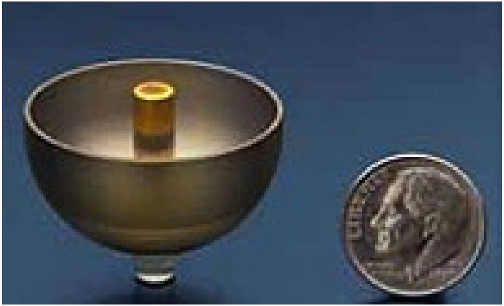

Free vibrations in a rotating frame (simplified):

$$\ddot{\mathbf{q}} + 2\beta\Omega\mathbf{J}\dot{\mathbf{q}} + \omega_0^2\mathbf{q} = 0, \qquad \mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalue problem: $\left(-\omega^2\mathbf{I} + 2i\omega\beta\Omega\mathbf{J} + \omega_0^2\right) q = 0.$

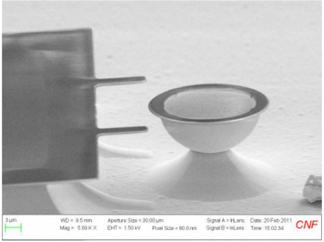
Solutions: $\omega \approx \Omega_0 \pm \beta\Omega$. $\implies$ beating $\propto \Omega$!

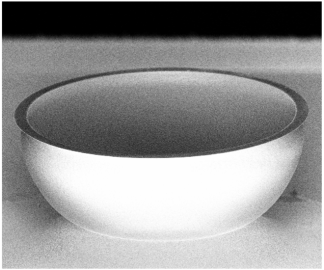
# A Small Application

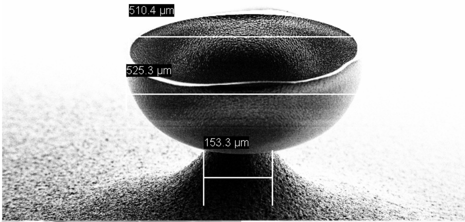


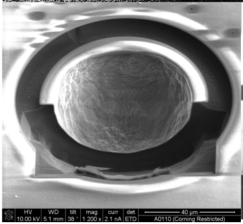
Northrup-Grummond HRG
(developed c. 1965–early 1990s)

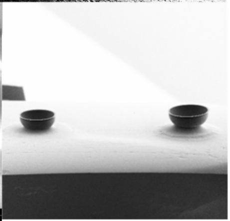
# A Smaller Application (Cornell)



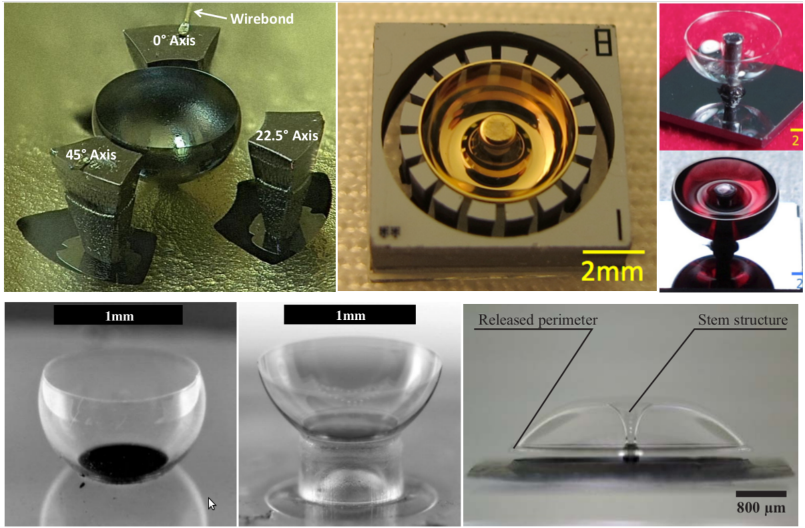





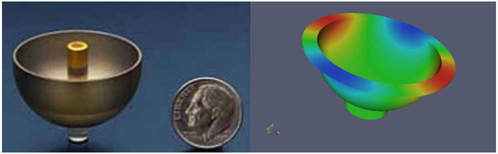



# A Smaller Application (UMich, GA Tech, Irvine)



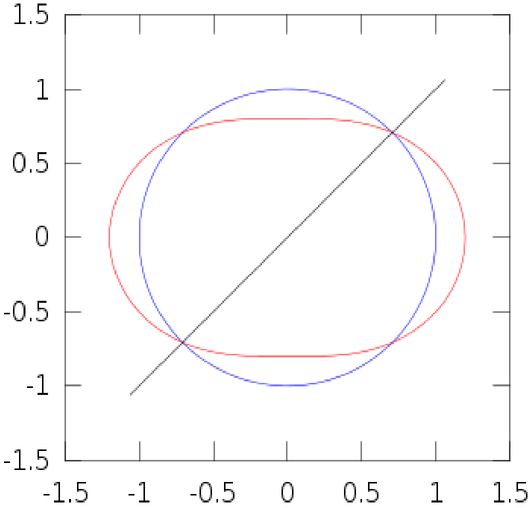
# Micro-HRG / GOBLiT / OMG



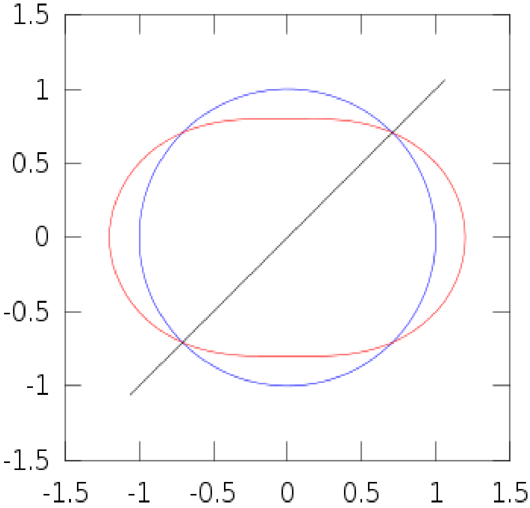
- Goal: Cheap, small (1mm) HRG
- Collaborator roles:
  - Basic design
  - Fabrication
  - Measurement
- Our part:
  - Detailed physics
  - Fast software
  - Sensitivity
  - Design optimization

# Foucault in Solid State

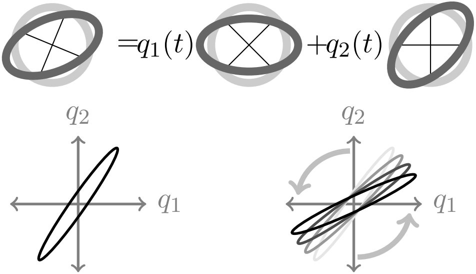
# Rate Integrating Mode



# Rate Integrating Mode



# A General Picture




$$\ddot{\mathbf{q}} + 2\beta\Omega\mathbf{J}\dot{\mathbf{q}} + \omega_0^2\mathbf{q} = 0, \qquad \mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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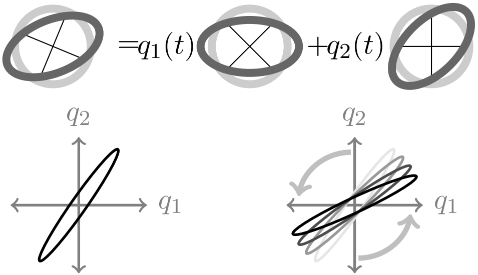
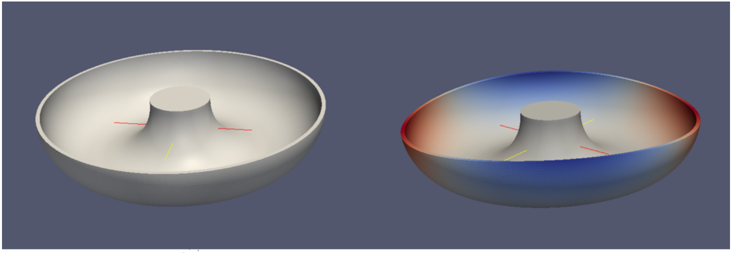
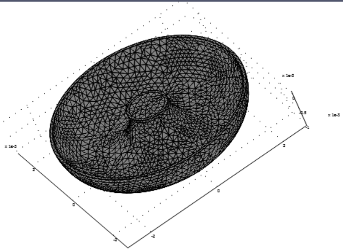
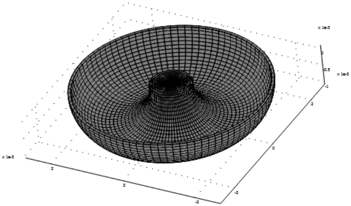



$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \approx \begin{bmatrix} \cos(-\beta\Omega t) & -\sin(-\beta\Omega t) \\ \sin(-\beta\Omega t) & \cos(-\beta\Omega t) \end{bmatrix} \begin{bmatrix} q_1^0(t) \\ q_2^0(t) \end{bmatrix}.$$

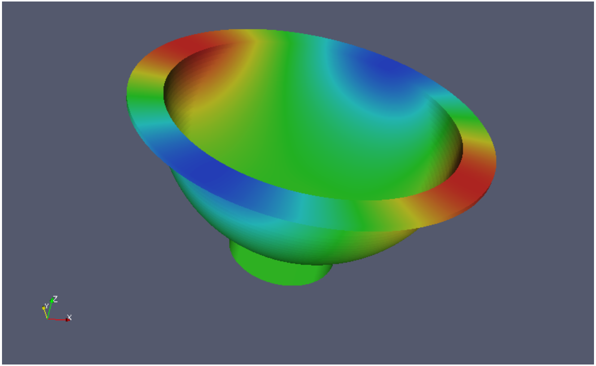
# An Uncritical FEA Approach







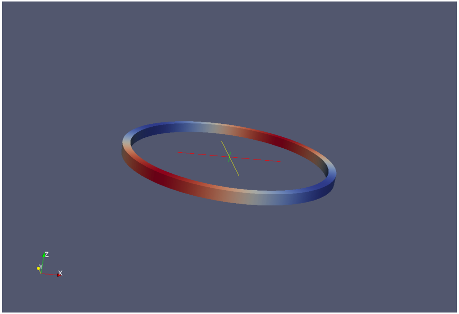
# The Perturbation Picture




Perturbations split degenerate modes:

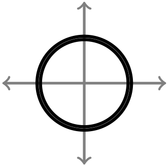
- Coriolis forces (good)
- Imperfect fab (bad, but physical)
- Discretization error (non-physical)

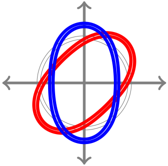
# Three Step Program



1. Perfect geometry, no rotation
2. Perfect geometry, rotation
3. Imperfect geometry

# Step I: Perfect Geometry, No Rotation





# Step I: Perfect Geometry, No Rotation

Free vibration problem in weak form

$$\forall \mathbf{w}, \quad b(\mathbf{w}, \ddot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

Symmetry: $Q$ any rotation or reflection

$$b(Q\mathbf{w}, Q\mathbf{u}) = b(\mathbf{w}, \mathbf{u})$$
$$a(Q\mathbf{w}, Q\mathbf{u}) = a(\mathbf{w}, \mathbf{u})$$

Decompose by invariant subspaces of $Q \implies$ Fourier analysis!

# Fourier Expansion and Axisymmetric Shapes

Decompose into symmetric $\mathbf{u}^c$ and antisymmetric $\mathbf{u}^s$ in $y$:

$$\mathbf{u}^c = \sum_{m=0}^{\infty} \mathbf{\Phi}_m^c(\theta)\mathbf{u}_m^c(r,z), \qquad \mathbf{u}^s = \sum_{m=0}^{\infty} \mathbf{\Phi}_m^s(\theta)\mathbf{u}_m^s(r,z)$$

where

$$\begin{aligned}\mathbf{\Phi}_m^c(\theta) &= \mathrm{diag}\left(\phantom{-}\cos(m\theta),\ \sin(m\theta),\ \phantom{-}\cos(m\theta)\right)\\ \mathbf{\Phi}_m^s(\theta) &= \mathrm{diag}\left(-\sin(m\theta),\ \cos(m\theta), -\sin(m\theta)\right).\end{aligned}$$

Modes involve only one azimuthal number $m$; degenerate for $m > 1$.

Preserve structure in FE: shape functions $N_j(r,z)\mathbf{\Phi}_m^{c,s}(\theta)$

# Block Diagonal Structure

Finite element system: $\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{K}\mathbf{u}^h = 0$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0^{cc} & & & & & & & \\ & \mathbf{K}_1^{ss} & & & & & & \\ & & \mathbf{K}_1^{cc} & & & & & \\ & & & \mathbf{K}_2^{ss} & & & & \\ & & & & \mathbf{K}_2^{cc} & & & \\ & & & & & \ddots & & \\ & & & & & & \mathbf{K}_M^{ss} & \\ & & & & & & & \mathbf{K}_M^{cc} \end{bmatrix}$$

Mass has same structure.

# Step II: Perfect Geometry, Rotation

Free vibration problem in weak form

$$\forall \mathbf{w}, \quad b(\mathbf{w}, \mathbf{a}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

where

$$\mathbf{a} = \ddot{\mathbf{u}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{u}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) + \dot{\boldsymbol{\Omega}} \times \mathbf{x}$$

Discretize by finite elements as before:

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{C}\dot{\mathbf{u}}^h + \mathbf{K}\ddot{\mathbf{u}}^h = 0$$

where $\mathbf{C}$ comes from Coriolis term ($2b(\mathbf{w}, \boldsymbol{\Omega} \times \dot{\mathbf{u}})$).

# Block Structure of Finite Element Matrix

Discretize $2b(\mathbf{w}, \boldsymbol{\Omega} \times \dot{\mathbf{u}})$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} & & & \\ \mathbf{C}_{10} & \mathbf{C}_{11} & \mathbf{C}_{12} & & \\ & \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Off-diagonal blocks come from cross-axis sensitivity:

$$\boldsymbol{\Omega} = \Omega_z \mathbf{e}_z + \boldsymbol{\Omega}_{r\theta} = \begin{bmatrix} 0 \\ 0 \\ \Omega_z \end{bmatrix} + \begin{bmatrix} \cos(\theta)\Omega_x - \sin(\theta)\Omega_y \\ \sin(\theta)\Omega_x + \cos(\theta)\Omega_y \\ 0 \end{bmatrix}.$$

Neglect cross-axis effects ($O(\Omega^2/\omega_0^2)$, like centrifugal effect).

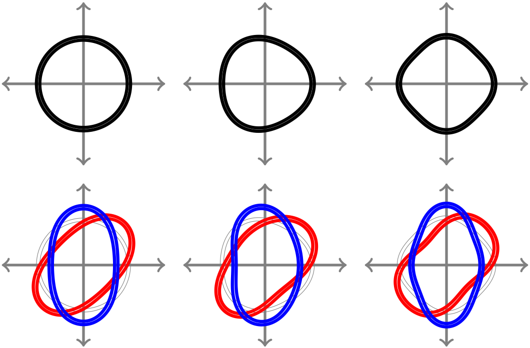
# Analysis in Ideal Case

Only need to mesh a 2D cross-section!

- Compute an operating mode $\mathbf{u}_c$ for the non-rotating geometry.
- Compute associated modal mass and stiffness $m$ and $k$.
- Compute $g = b(\mathbf{u}_c, e_z \times \mathbf{u}_s)$.
- Model: motion is approximately $q_1 \mathbf{u}_c + q_2 \mathbf{u}_s$, and

$$m\ddot{\mathbf{q}} + 2g\Omega\mathbf{J}\dot{\mathbf{q}} + k\mathbf{q} = 0,$$

# Step III: Imperfect Geometry



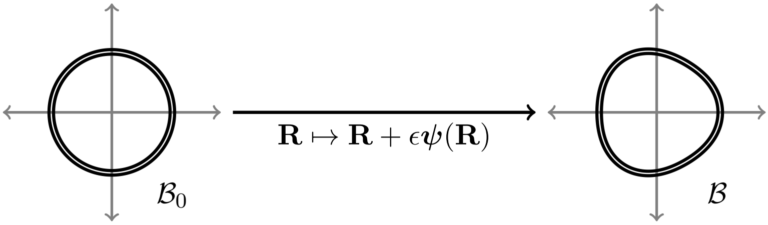
# What Imperfections?

Let me count the ways...

- Over/under etch
- Mask misalignment
- Thickness variations
- Anisotropy of etching single-crystal Si

These are *not* arbitrary!

# Representing the Perturbation




Write weak form in $\mathcal{B}_0$ geometry:

$$b(\mathbf{w}, \mathbf{a}) = \int_{\mathcal{B}_0} \rho \mathbf{w} \cdot \mathbf{a} \, J \, d\mathcal{B}_0,$$

$$a(\mathbf{w}, \mathbf{u}) = \int_{\mathcal{B}_0} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{C} : \boldsymbol{\varepsilon}(\mathbf{u}) \, J \, d\mathcal{B}_0,$$

where $J = \det(\mathbf{I} + \epsilon \mathbf{F})$ with $\mathbf{F} = \partial\psi / \partial\mathbf{R}$.

# Decomposing $\psi$

Do Fourier decomposition of $\psi$, too! Consider case where

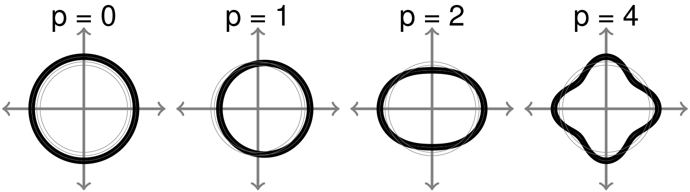
$$\begin{aligned} m &= \text{only azimuthal number of } \mathbf{w} \\ n &= \text{only azimuthal number of } \mathbf{u} \\ p &= \text{only azimuthal number of } \boldsymbol{\psi} \end{aligned}$$

Then we have *selection rules*

$$a(\mathbf{w}, \mathbf{u}) = \begin{cases} O(\epsilon^k), & |m - n| = kp \\ 0, & \text{otherwise} \end{cases}$$

Similar picture for $b$.

# Typical Fabrication Imperfections



# Block matrix structure

Ex: $p = 2$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 & & \epsilon & & \epsilon^2 & & \epsilon^3 & \\ & \mathbf{K}_1 & & \epsilon & & \epsilon^2 & & \\ \epsilon & & \mathbf{K}_2 & & \epsilon & & \epsilon^2 & \\ & \epsilon & & \mathbf{K}_3 & & \epsilon & & \\ \epsilon^2 & & \epsilon & & \mathbf{K}_4 & & \epsilon & \\ & \epsilon^2 & & \epsilon & & \mathbf{K}_5 & & \\ \epsilon^3 & & \epsilon^2 & & \epsilon & & \mathbf{K}_6 & \\ & & & & & & & \ddots \end{bmatrix}$$

# Impact of Selection Rules

- Fast FEA: Can neglect most wave numbers / blocks
- Also *qualitative* information

# Qualitative Information

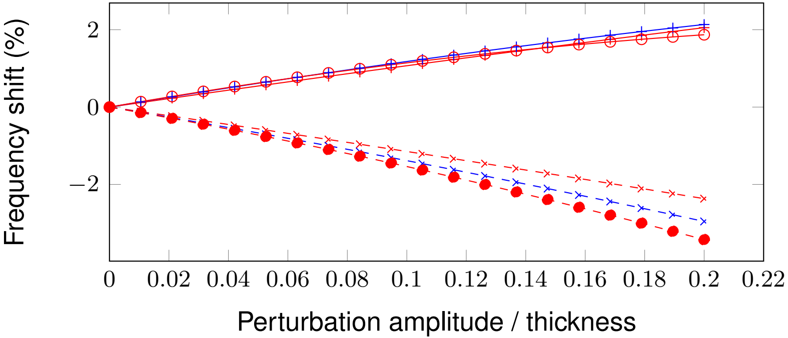
Operating wave number $m$, perturbation number $p$:

| | |
|---|---|
| $p = 2m$ | frequencies split by $O(\epsilon)$ |
| $kp = 2m$ | frequencies split at most $O(\epsilon^2)$ |
| $p \not| 2m$ | frequencies change at $O(\epsilon^2)$, *no* split |
| $p = 1$<br>$p = 2m \pm 1$ | $O(\epsilon)$ cross-axis coupling. |

Note:

- $m = 2$ affected at first order by $p = 0$ and $p = 4$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 2$).
- $m = 3$ affected at first order by $p = 0$ and $p = 6$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 3$).

# Mode Split for Rings: $\psi(r,\theta) = (\cos(2m\theta), 0)$.



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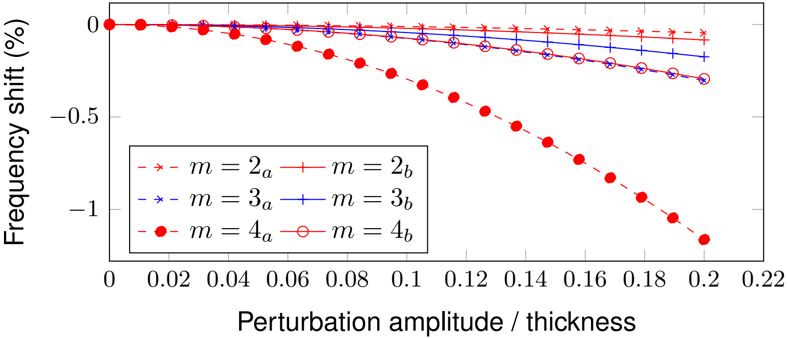
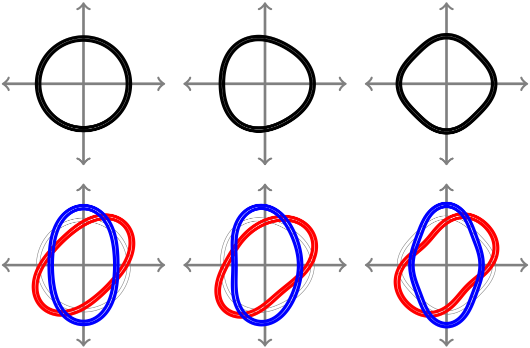


# Analyzing Imperfect Rings



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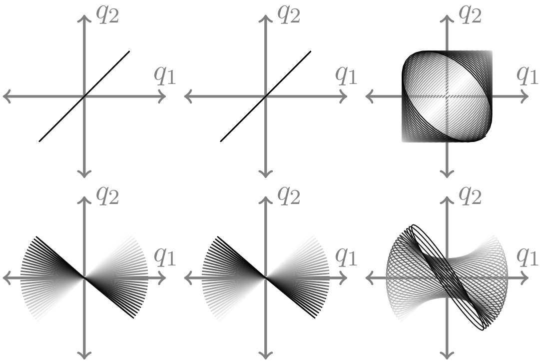
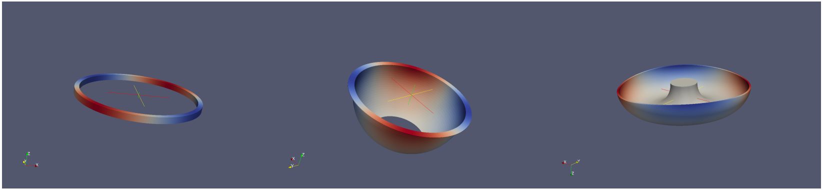
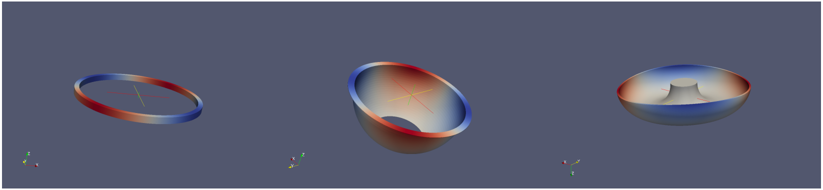


# Beyond Rings: AxFEM



- Mapped finite / spectral element formulation
- Low-order polynomials through thickness
- High-order polynomials along length
- Trig polynomials in $\theta$
- Agrees with results reported in literature
- Computes sensitivity to geometry, material parameters, etc.

# Read All About It!

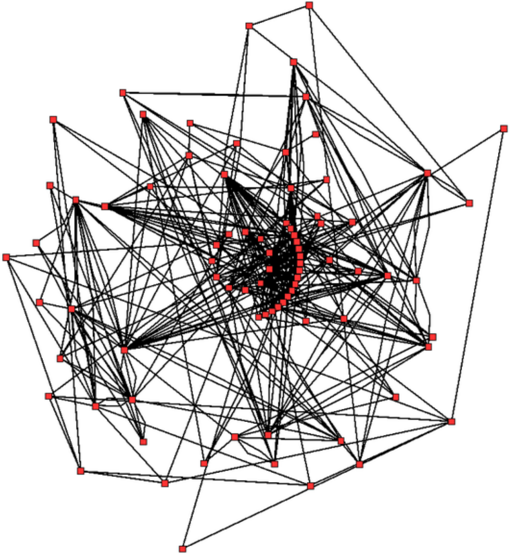


Yilmaz and Bindel
“Effects of Imperfections on Solid-Wave Gyroscope Dynamics”
Proceedings of IEEE Sensors 2013, Nov 3–6.

Thanks to DARPA MRIG + Sunil Bhave and Laura Fegely.

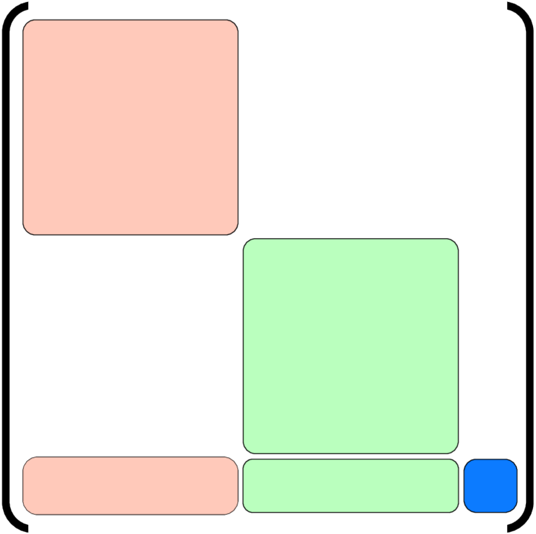
# Ad: Declarative HPC and Clouds

# Ad: Opinions, Game Theory, and Eigenvalues



# Ad: Finding Faults Fast in Smart Grids

# Ad: Super-Fast PDE Solvers



# Ad: Nonlinear Eigenvalues and Resonances