From Networks to Numerical Linear Algebra

David Bindel

Department of Computer Science
Cornell University

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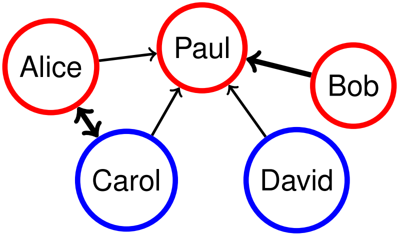
Example: Opinions in Networks



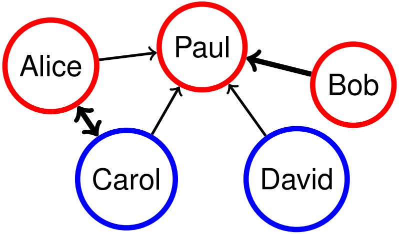
Basic model

Extension of DeGroot model due to Friedkin and Johnsen:

- Directed graph with nodes $1, \dots, n$, weights w_{ij}
- Each node has two quantities:
  - Fixed *internal opinion* $s_i \in \mathbb{R}$
  - Variable *expressed opinion* $z_i \in \mathbb{R}$
- Update equation:

$$z_i^{\text{new}} \leftarrow \frac{s_i + \sum_{j \in N(i)} w_{ij} z_j^{\text{old}}}{1 + \sum_{j \in N(i)} w_{ij}}$$

Example: Opinions in Networks



$$W = \begin{bmatrix} 0 & 0 & w_{AC} & 0 & 0 & w_{AP} \\ 0 & 0 & 0 & 0 & 0 & w_{BP} \\ w_{CA} & 0 & 0 & 0 & 0 & w_{CP} \\ 0 & 0 & 0 & 0 & 0 & w_{DP} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix reformulation

Scalar form:

$$z_i^{\text{new}} \leftarrow \frac{s_i + \sum_{j \in N(i)} w_{ij} z_j^{\text{old}}}{1 + \sum_{j \in N(i)} w_{ij}}$$

Matrix form:

$$(D + I) z^{\text{new}} = s + W z^{\text{old}}$$

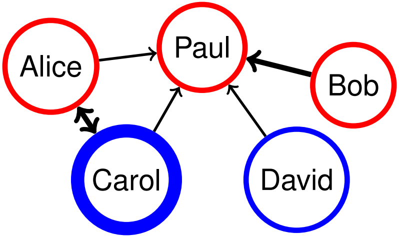
This is Jacobi iteration! Converges to solution of

$$(L + I) x = s$$

where $L = D - W$ is the (directed) graph Laplacian:

$$L_{ij} = \begin{cases} -w_{ij}, & i \neq j \\ \sum_{k \in N(i)} w_{ik}, & i = j \end{cases}$$

Another perspective



Carol is “pulled” to:

- To be true to her personal beliefs ($z_C - s_C$ small)
- To agree with Alice ($z_C - z_A$ small)
- To agree with Paul ($z_C - z_B$ small)

She is unhappy to the extent that she cannot reconcile these.

Equilibrium state and game theory

Define local cost function:

$$c_i = \frac{1}{2}\left((z_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}(z_i - z_j)^2\right)$$

Node i chooses opinion to optimize z_i:

$$\frac{\partial c_i}{\partial z_i} = (z_i - s_i) + \sum_{j \in N(i)} w_{ij}(z_i - z_j) = 0$$

Nash equilibrium satisfies $(L + I)x = s$.

Nash equilibrium vs social optimum

Define the *social cost*

$$c(z) = \sum_i c_i(z) = \frac{1}{2}\left(z^T(A+I)z - 2z^Ts + s^Ts\right).$$

where $A = L + L^T$

- Nash equilibrium: Node i chooses x_i to minimize c_i.
- Social optimimum: Choose y globally to minimize $c(y)$

Equations for social optimum: $(A+I)y = s$.

Price of anarchy

The *price of anarchy* is

$$\mathrm{PoA}(s) = \frac{c(y)}{c(x)} = \frac{s^T B s}{s^T C s}$$

where

$$B = (A+I)^{-1} - I + (A+I)^{-1} A (A+I)^{-1}$$
$$C = ((L+I)^{-1} - I)^T ((L+I)^{-1} - I) + (L+I)^{-T} A (L+I)^{-1}$$

Undirected case: L symmetric, $B = p(L)$, $C = q(L)$, and

$$\max_{s \neq 0} \frac{s^T B s}{s^T C s} = \max_{\lambda \in \Lambda(L)} \frac{p(\lambda)}{q(\lambda)} \leq \max_{t \geq 0} \frac{p(t)}{q(t)} = \frac{9}{8}.$$

Directed graphs: still an eigenvalue problem.

Mean opinion and influence

Let e be the vector of all ones. Mean opinion at Nash is:

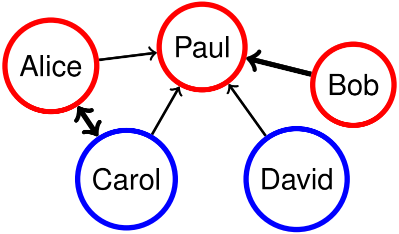
$$\bar{x} = \frac{1}{n} e^T x = \frac{1}{n} e^T (L + I)^{-1} s = f^T s$$

The *influence vector*

$$f = (L + I)^{-T} e$$

tells how much each node influences the mean opinion.

Influence in the model network



Bold lines are weight 2, regular are weight 1:

$$f = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{19}{6} \end{bmatrix}^T$$

Distribution of influence

The *uniform influence* case $f = e/n$ occurs when

$$L^T e = 0,$$

i.e. graph is *Eulerian* (in-degree weight = out-degree weight).
Uniform influence always true for socially optimal opinion!

In general, max influence is

$$\max_i f_i = \|(L + I)^{-1}\|_1$$

Note: $\|(L + I)^{-1}\|_\infty = 1$. What about other norms?

Variance of opinion

Assume s is normalized to mean zero. Variance in intrinsic opinion:

$$\mathrm{Var}[s] = \frac{1}{n} s^T s$$

What about the variance in the expressed opinion at Nash?

$$\frac{\mathrm{Var}[x]}{\mathrm{Var}[s]} = \frac{s^T (L+I)^{-1} (I - ee^T/n)(L+I)^{-1} s}{s^T s}$$

So if s is not identically zero, then

$$\frac{\sigma_x}{\sigma_s} \leq \|(I - ee^T/n)(L+I)^{-1}\|_2 \leq \sqrt{\max_i f_i}$$

Variance can only increase if influence is non-uniform.

More entertainment

Can add edges to improve social cost (by $\leq \mathrm{PoA}$):

1. Figuring out where to add a little weight is easy
2. Figuring out best edge additions is NP-hard

Changing from quadratic cost is interesting, but harder.

David S. Bindel, Sigal Oren, and Jon Kleinberg. "How Bad is Forming Your Own Opinion?," FOCS 2011.