Elastic PMLs for Resonator Anchor Loss Simulation

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Outline

- Electromechanical resonators and RF MEMS
- Damping and quality of resonance
- Anchor losses and Perfectly Matched Layers
- Analysis of the discretized PMLs
- Complex symmetry and structured model reduction
- Analysis of a disk resonator
- Conclusions

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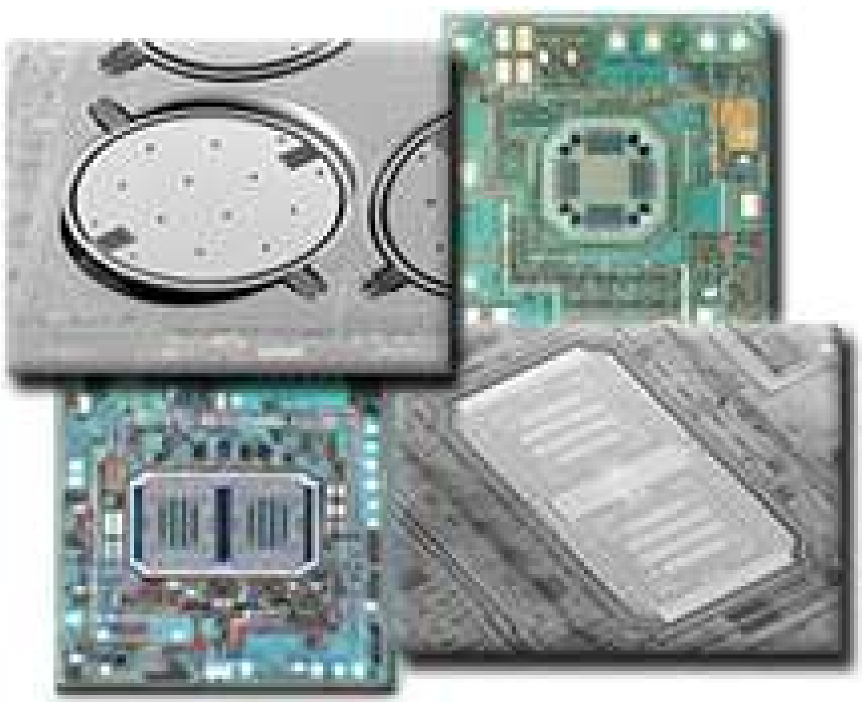
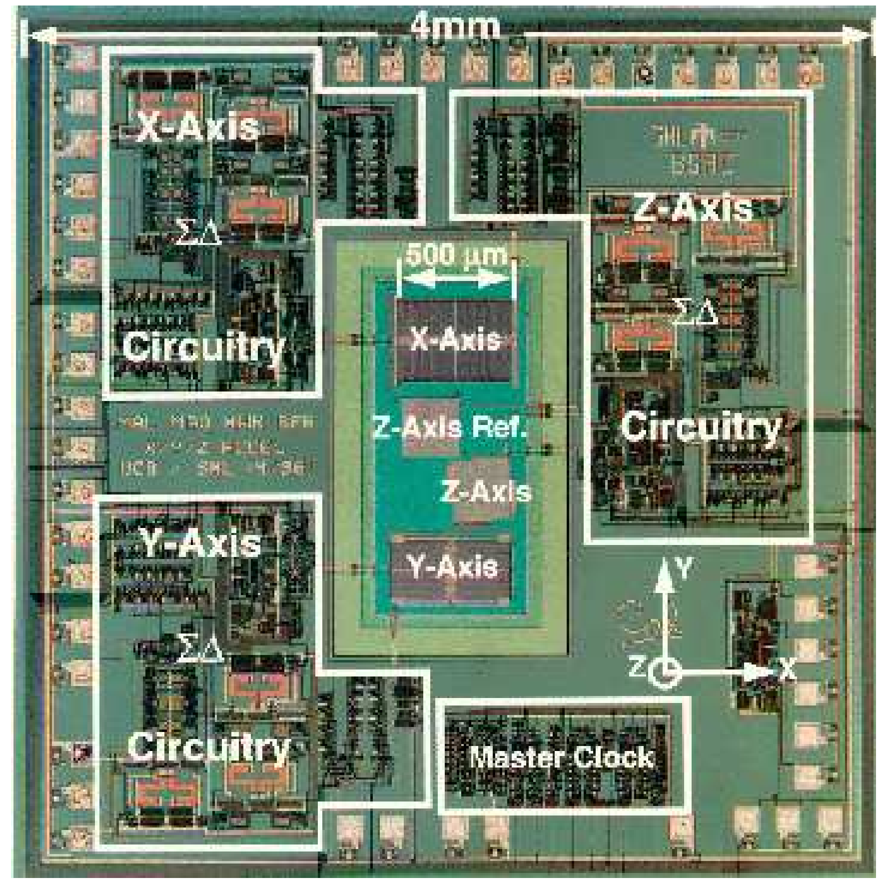
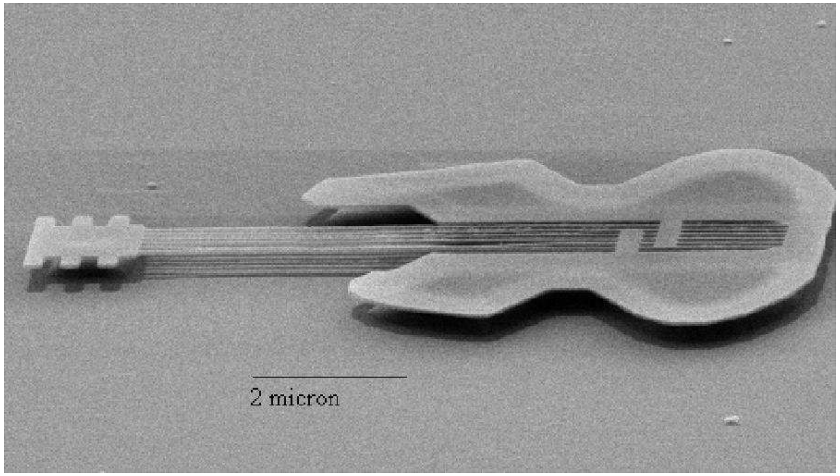
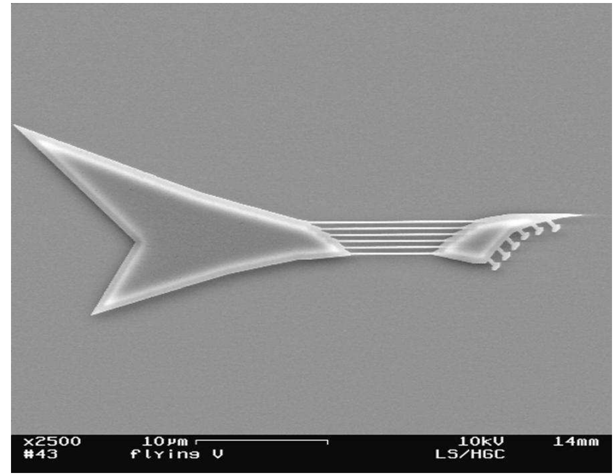
How many MEMS?





Why resonant MEMS?

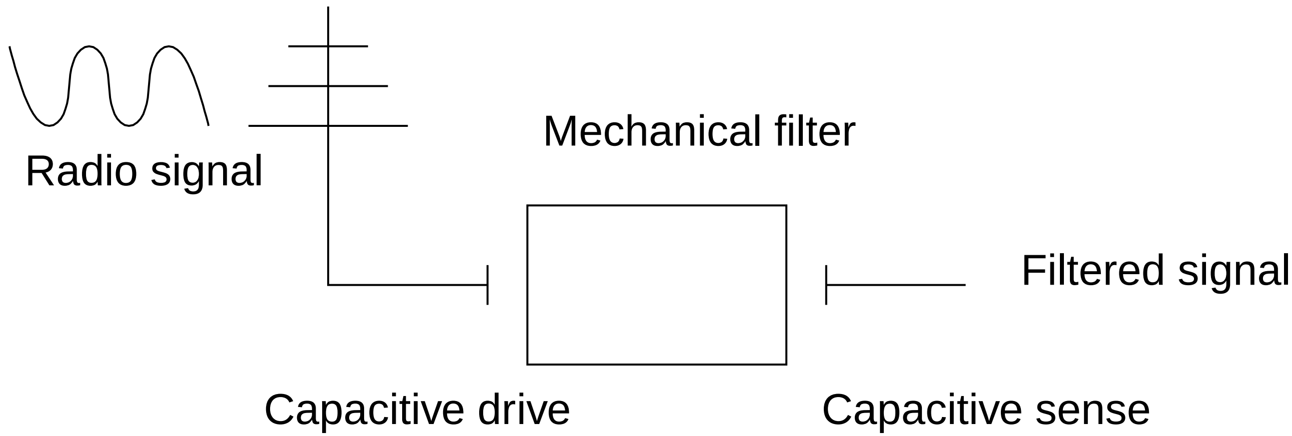




Microguitars from Cornell University (1997 and 2003)

- Sensing elements (inertial, chemical)
- Frequency references
- Filter elements
- Neural networks
- Really high-pitch guitars

Micromechanical filters



- Mechanical high-frequency (high MHz-GHz) filter
  - Your cell phone is mechanical!
- Advantage over quartz surface acoustic wave filters
  - Integrated into chip
  - Low power

Success $\Longrightarrow$ "Calling Dick Tracy!"

Designing transfer functions

Time domain:

$$\begin{aligned} Mu'' + Cu' + Ku &= b\phi(t) \\ y(t) &= p^T u \end{aligned}$$

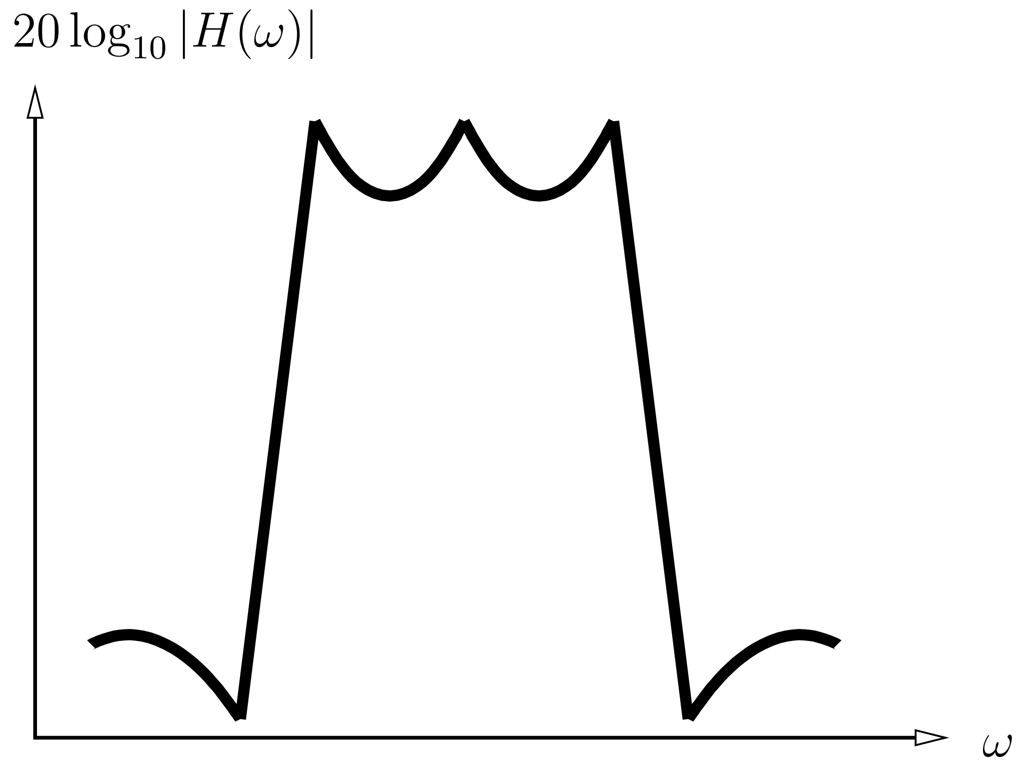
Frequency domain:

$$\begin{aligned} -\omega^2 M\hat{u} + i\omega C\hat{u} + K\hat{u} &= b\hat{\phi}(\omega) \\ \hat{y}(\omega) &= p^T u \end{aligned}$$

Transfer function:

$$\begin{aligned} H(\omega) &= p^T(-\omega^2 M + i\omega C + K)^{-1} b \\ \hat{y}(\omega) &= H(\omega)\hat{\phi}(\omega) \end{aligned}$$

Damping and filters



- Want “sharp” poles for narrowband filters
- $\implies$ Want to minimize damping
  - Electronic filters have too much
  - Understanding of damping in MEMS is lacking

Damping and Q

- Designers want high *quality of resonance* (Q)
  - Dimensionless damping in a one-dof system:

$$\frac{d^2u}{dt^2} + Q^{-1}\frac{du}{dt} + u = F(t)$$

  - For a resonant mode with frequency $\omega \in \mathbb{C}$:

$$Q := \frac{|\omega|}{2\,\mathrm{Im}(\omega)} = \frac{\text{Stored energy}}{\text{Energy loss per radian}}$$

Sources of damping

- Fluid damping
  - Air is a viscous fluid ($\mathrm{Re} \ll 1$)
  - Can operate in a vacuum
  - Shown not to dominate in many RF designs
- Material losses
  - Low intrinsic losses in silicon, diamond, germanium
  - Terrible material losses in metals
- Thermoelastic damping
  - Volume changes induce temperature change
  - Diffusion of heat leads to mechanical loss
- Anchor loss
  - Elastic waves radiate from structure

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- Electromechanical resonators and RF MEMS
- Damping and quality of resonance
- Anchor losses and Perfectly Matched Layers
  - Anchor losses and infinite domains
  - Idea of the perfectly matched layer
  - Elastic PMLs and finite elements
- Analysis of the discretized PMLs
- Complex symmetry and structured model reduction
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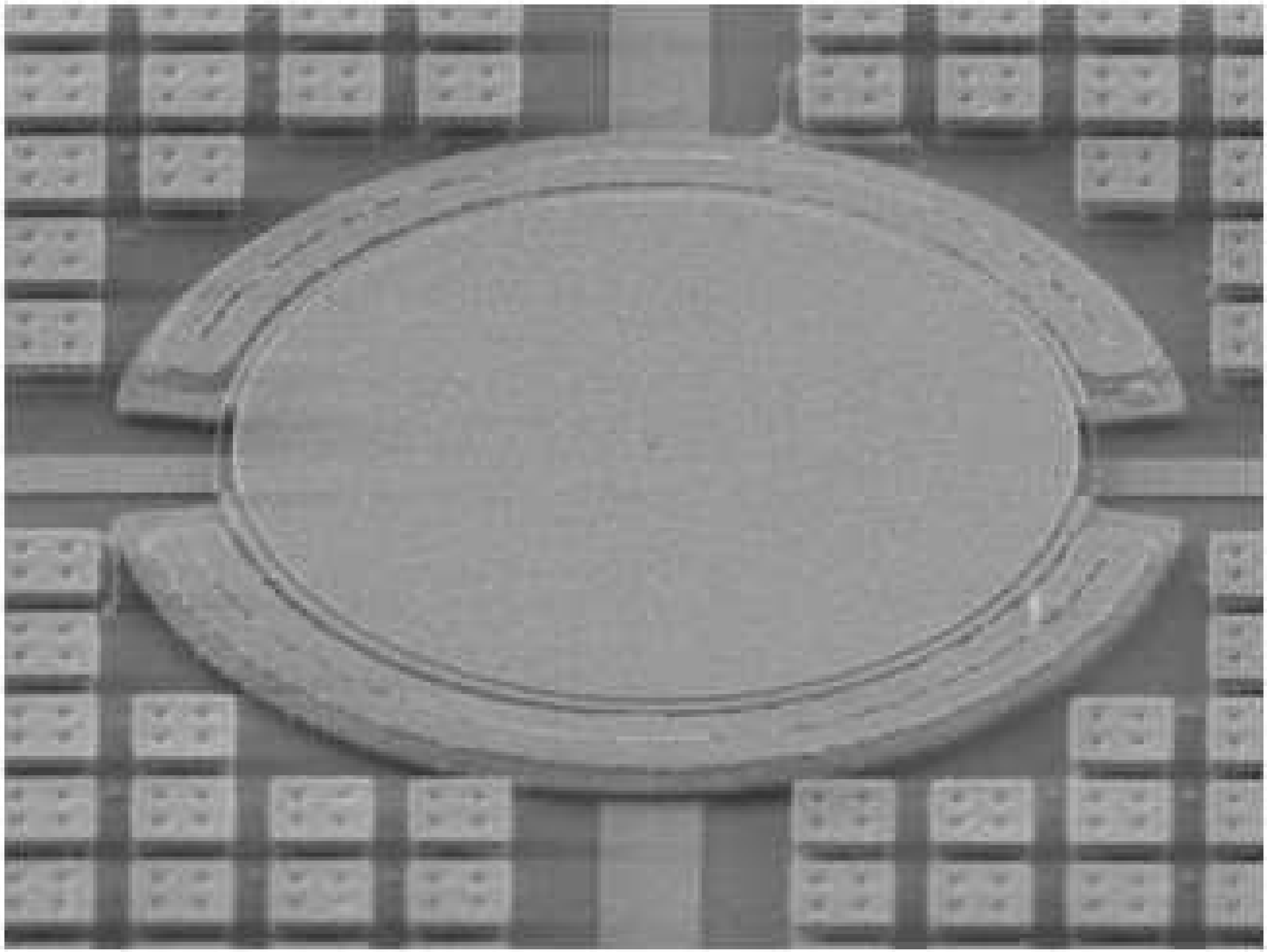
Example: Disk resonator



SiGe disk resonators built by E. Quévy

Substrate model

Goal: Understand energy loss in disk resonator

- Dominant loss is elastic radiation from anchor
- Resonator size $\ll$ substrate size
  - Substrate appears semi-infinite
- Possible semi-infinite models
  - Matched asymptotic modes
  - Dirichlet-to-Neumann maps
  - Boundary dampers
  - Higher-order local ABCs
  - Perfectly matched layers

Perfectly matched layers

- Apply a complex coordinate transformation
- Generates a non-physical absorbing layer
- No impedance mismatch between the computational domain and the absorbing layer
- Idea works with general linear wave equations
  - First applied to Maxwell's equations (Berengér 95)
  - Similar idea earlier in quantum mechanics (*exterior complex scaling*, Simon 79)
  - Applies to elasticity in standard FEM framework (Basu and Chopra, 2003)

1-D model problem

- Domain: $x \in [0, \infty)$
- Governing eq:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

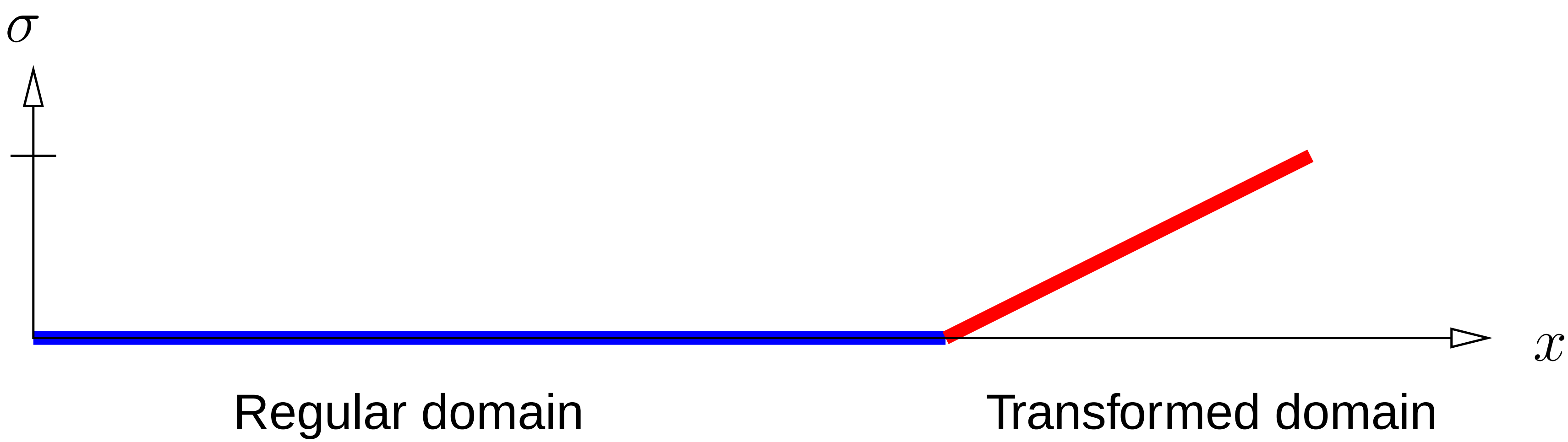
- Fourier transform:

$$\frac{d^2 \hat{u}}{dx^2} + k^2 \hat{u} = 0$$

- Solution:

$$\hat{u} = c_{\mathrm{out}} e^{-ikx} + c_{\mathrm{in}} e^{ikx}$$

1-D model problem with PML

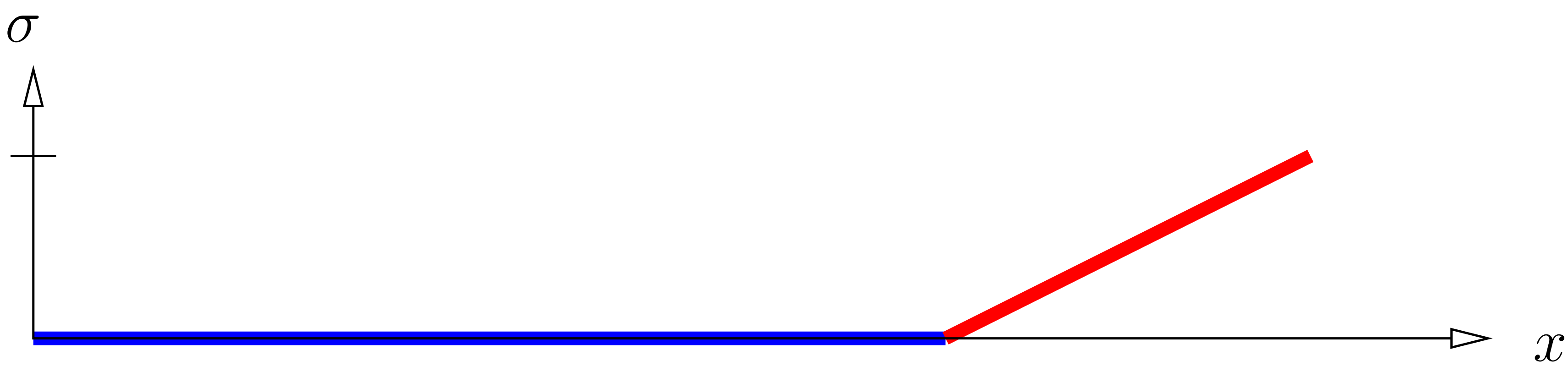


$$\frac{d\tilde{x}}{dx} = \lambda(x) \text{ where } \lambda(s) = 1 - i\sigma(s)$$

$$\frac{d^2\hat{u}}{d\tilde{x}^2} + k^2\hat{u} = 0$$

$$\hat{u} = c_{\text{out}}e^{-ik\tilde{x}} + c_{\text{in}}e^{ik\tilde{x}}$$

1-D model problem with PML



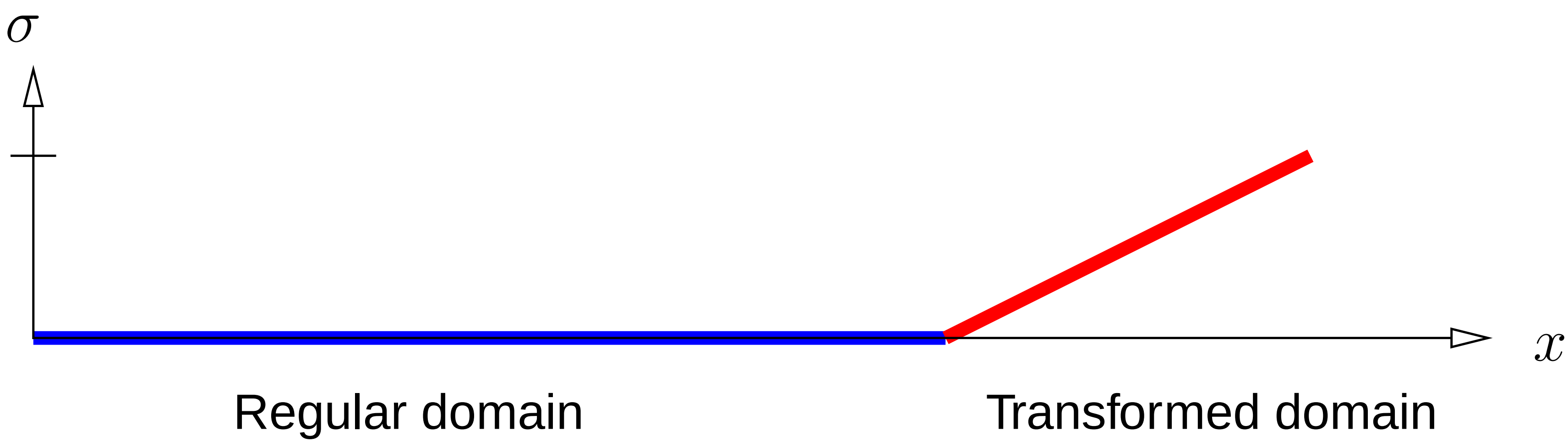
Regular domain   Transformed domain

$$\frac{d\tilde{x}}{dx} = \lambda(x) \text{ where } \lambda(s) = 1 - i\sigma(s)$$

$$\frac{1}{\lambda}\frac{d}{dx}\left(\frac{1}{\lambda}\frac{d\hat{u}}{dx}\right) + k^2\hat{u} = 0$$

$$\hat{u} = c_{\text{out}} \exp\left(-k\int_0^x \sigma(s)\,ds\right) e^{-ikx} + c_{\text{in}} \exp\left(k\int_0^x \sigma(s)\,ds\right) e^{ikx}$$

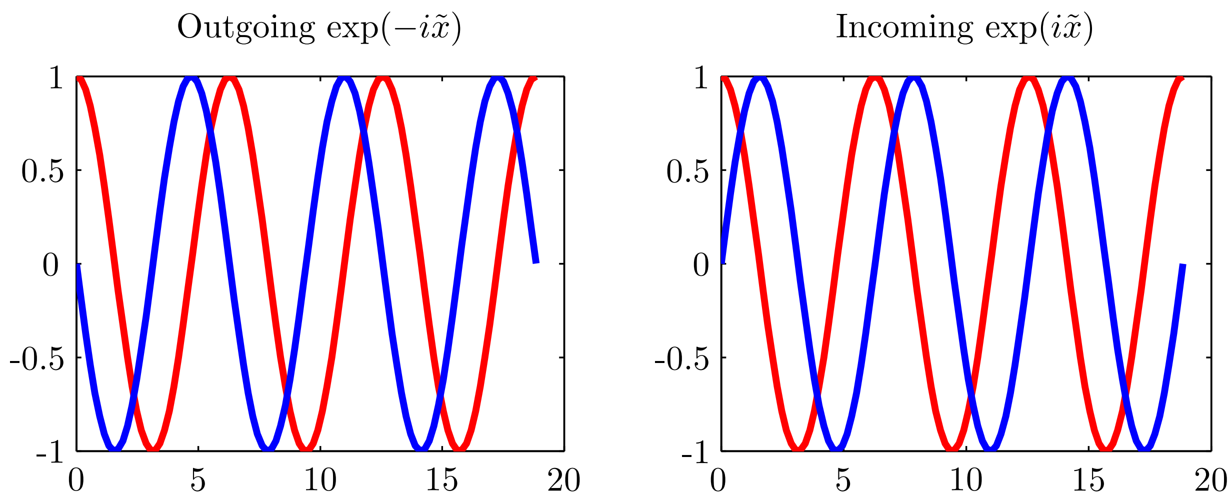
1-D model problem with PML

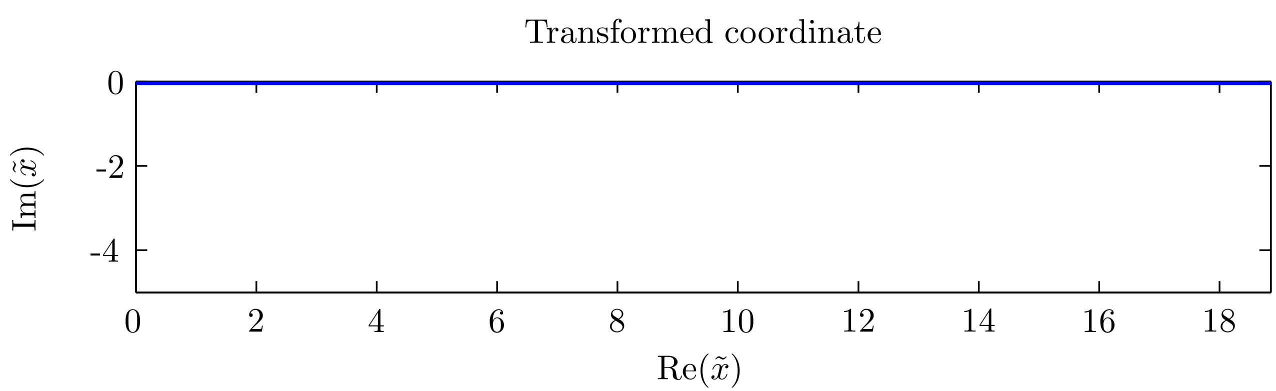


If solution clamped at $x = L$ then

$$\frac{c_{\text{in}}}{c_{\text{out}}} = O(e^{-k\gamma}) \text{ where } \gamma = \int_0^L \sigma(s)\, ds$$

1-D model problem illustrated





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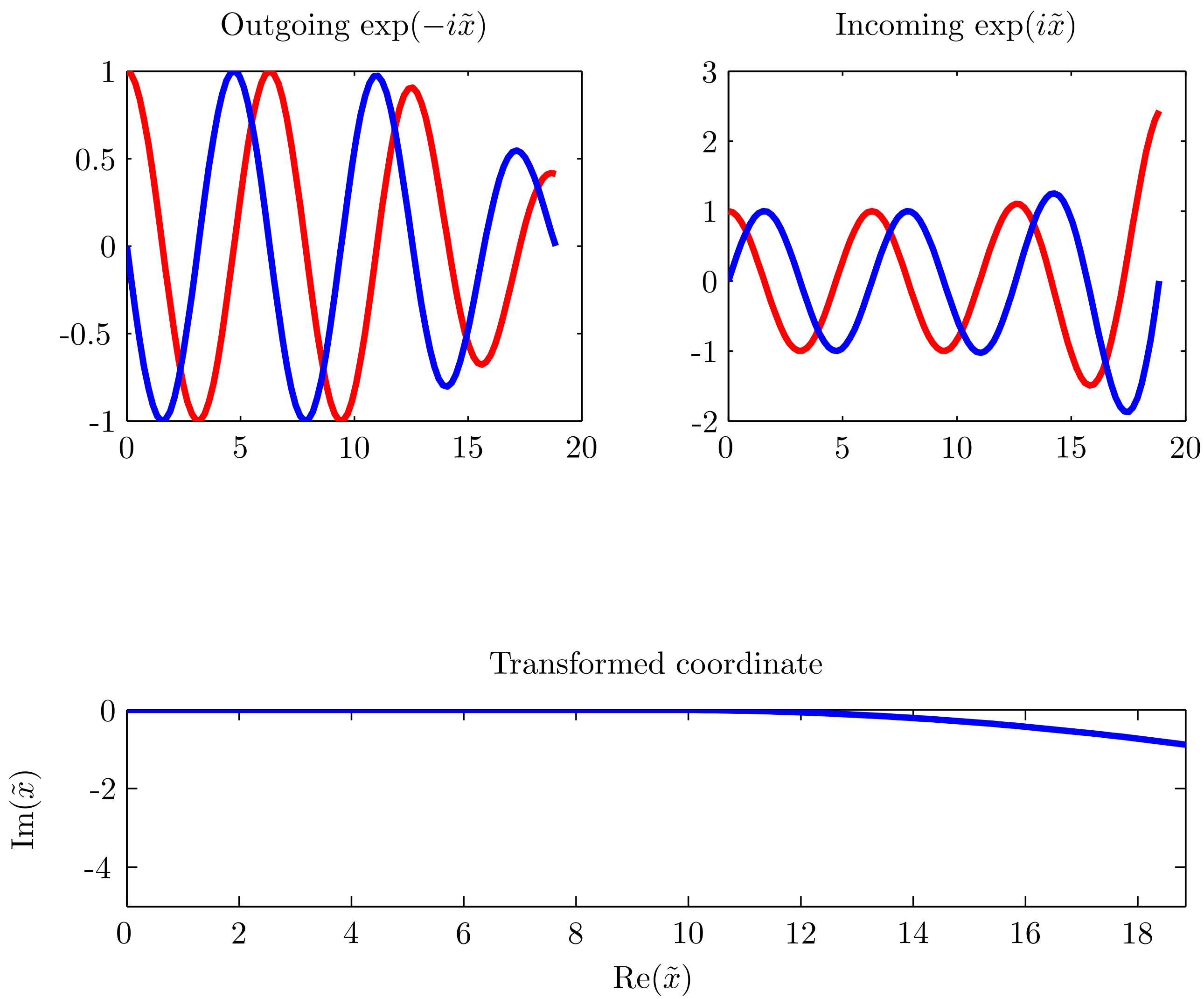

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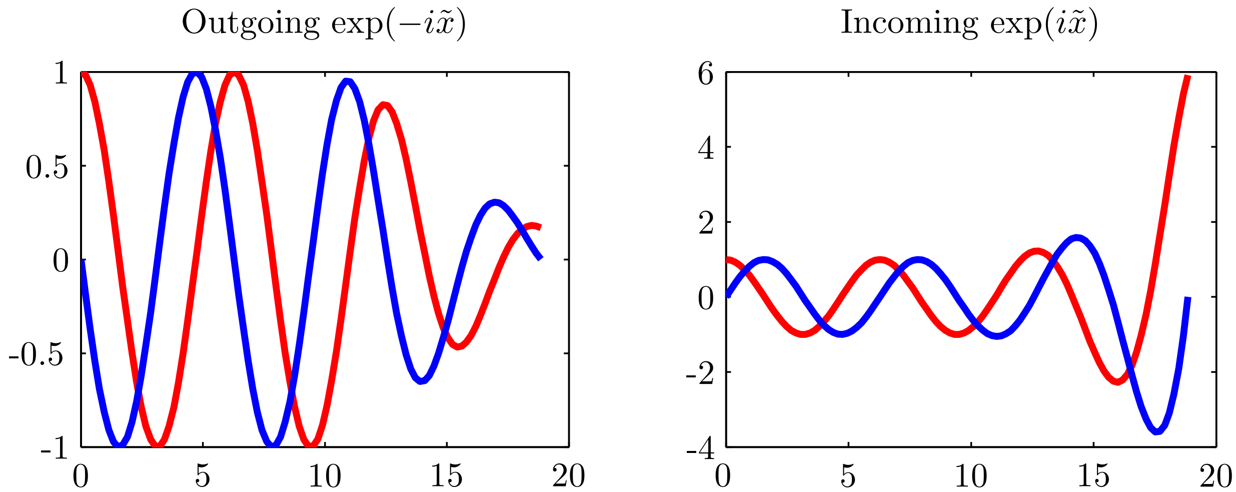
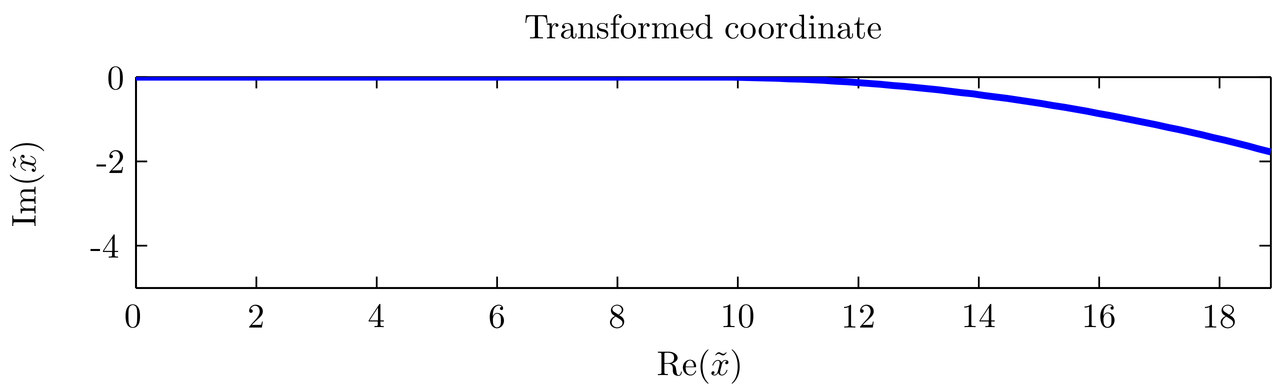
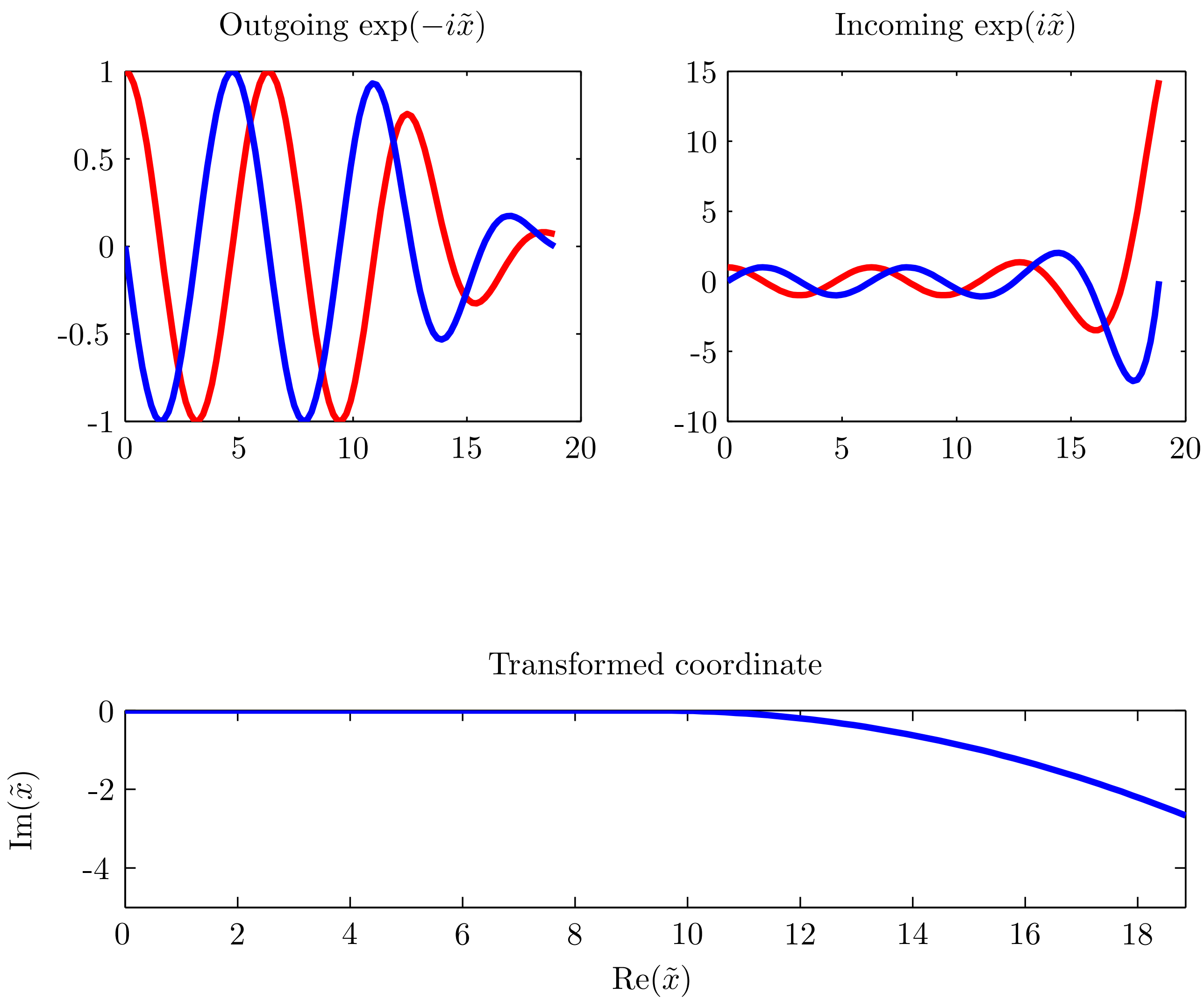




1-D model problem illustrated



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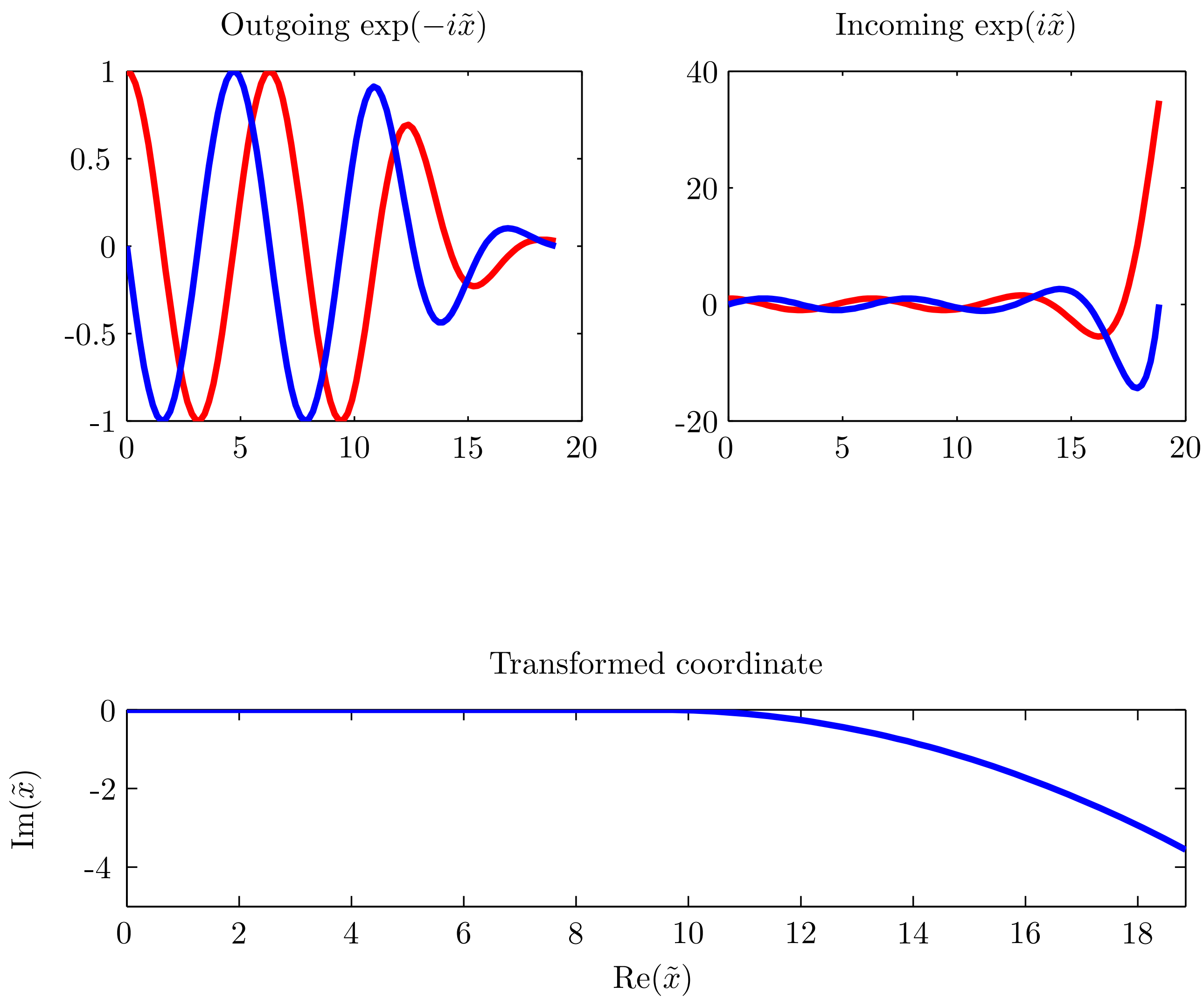

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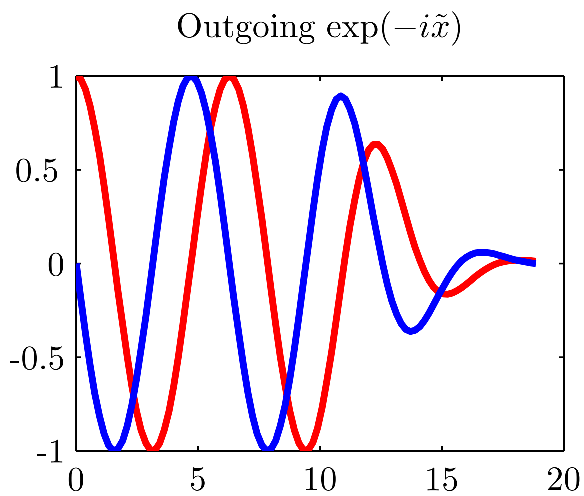

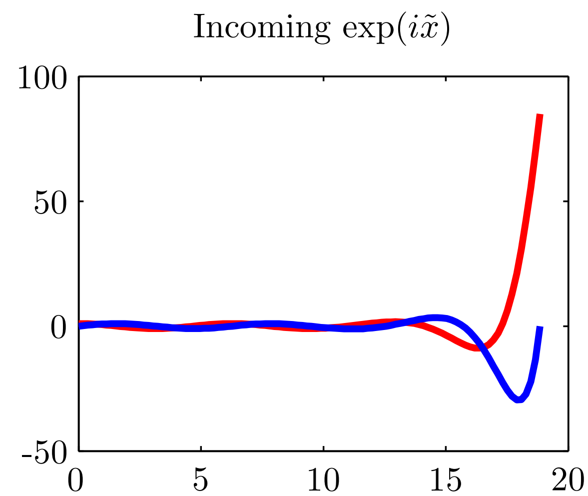

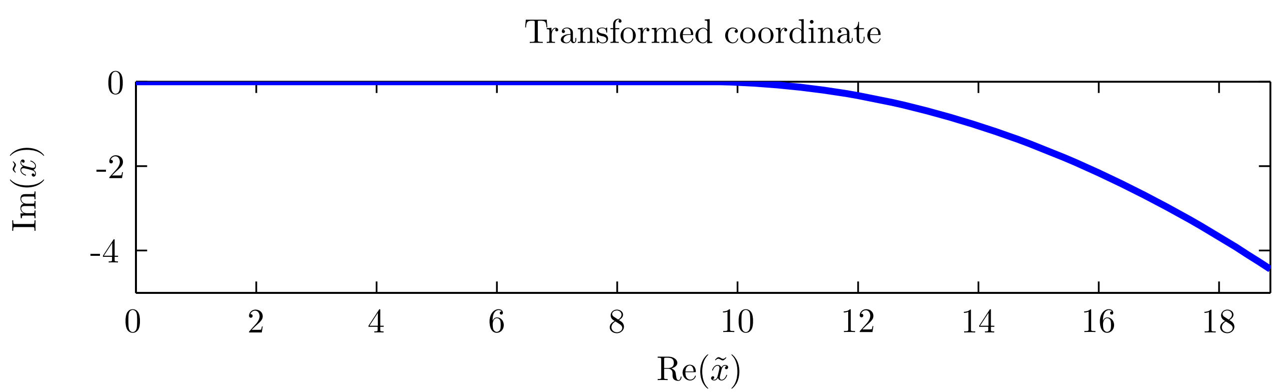

Clamp solution at transformed end to isolate outgoing wave.

Elastic PMLs

$$\int_\Omega \epsilon(w) : \mathsf{C} : \epsilon(u)\, d\Omega - \omega^2 \int_\Omega \rho w \cdot u\, d\Omega = \int_\Gamma w \cdot t_n d\Gamma$$

$$\epsilon(u) = \left(\frac{\partial u}{\partial x}\right)^s$$

- Start from standard weak form

Elastic PMLs

$$\int_{\tilde{\Omega}} \tilde{\epsilon}(w) : \mathsf{C} : \tilde{\epsilon}(u)\, d\tilde{\Omega} - \omega^2 \int_{\tilde{\Omega}} \rho w \cdot u\, d\tilde{\Omega} = \int_{\Gamma} w \cdot t_n d\Gamma$$

$$\tilde{\epsilon}(u) = \left(\frac{\partial u}{\partial \tilde{x}}\right)^s = \left(\frac{\partial u}{\partial x}\Lambda^{-1}\right)^s$$

- Start from standard weak form
- Introduce transformed $\tilde{x}$ with $\frac{\partial \tilde{x}}{\partial x} = \Lambda$

Elastic PMLs

$$\int_\Omega \tilde{\epsilon}(w) : \mathsf{C} : \tilde{\epsilon}(u)\, J_\Lambda \, d\Omega - \omega^2 \int_\Omega \rho w \cdot u \, J_\Lambda \, d\Omega = \int_\Gamma w \cdot t_n \, d\Gamma$$

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- Map back to reference system ($J_\Lambda = \det(\Lambda)$)

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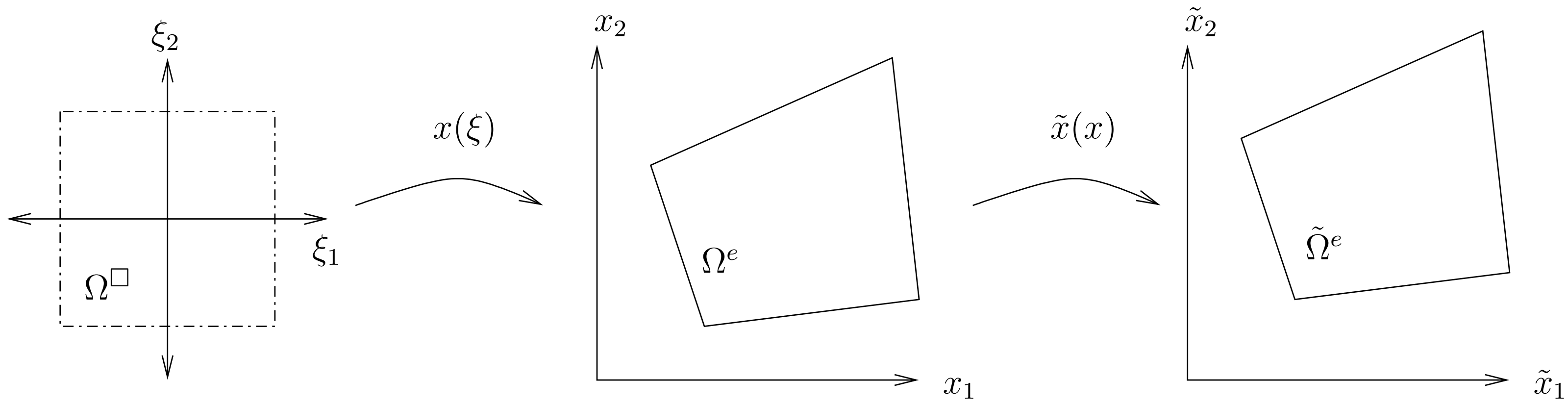
- Start from standard weak form
- Introduce transformed $\tilde{x}$ with $\frac{\partial \tilde{x}}{\partial x} = \Lambda$
- Map back to reference system ($J_\Lambda = \det(\Lambda)$)
- All terms are symmetric in w and u

Finite element implementation



- Combine PML and isoparametric mappings

$$\begin{aligned}
\mathbf{k}^e &= \int_{\Omega^\Box} \tilde{\mathbf{B}}^T \mathbf{D} \tilde{\mathbf{B}} \tilde{J} \, d\Omega^\Box \\
\mathbf{m}^e &= \left(\int_{\Omega^\Box} \rho \mathbf{N}^T \mathbf{N} \tilde{J} \, d\Omega^\Box \right)
\end{aligned}$$

- Matrices are *complex symmetric*

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  - A two-dimensional model problem
  - Analysis of discrete reflection
  - Choice of PML parameters
- Complex symmetry and structured model reduction
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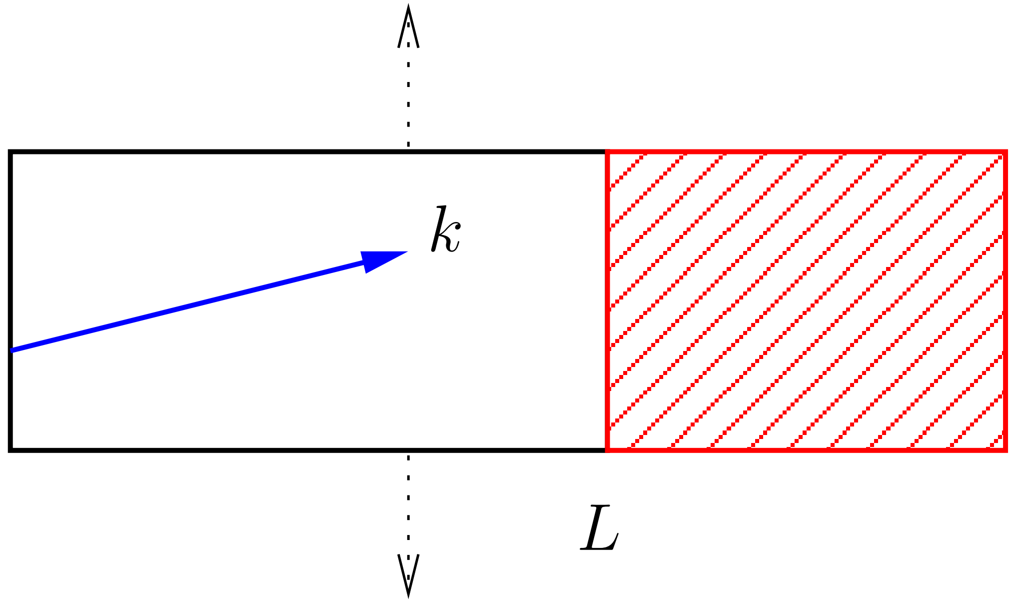
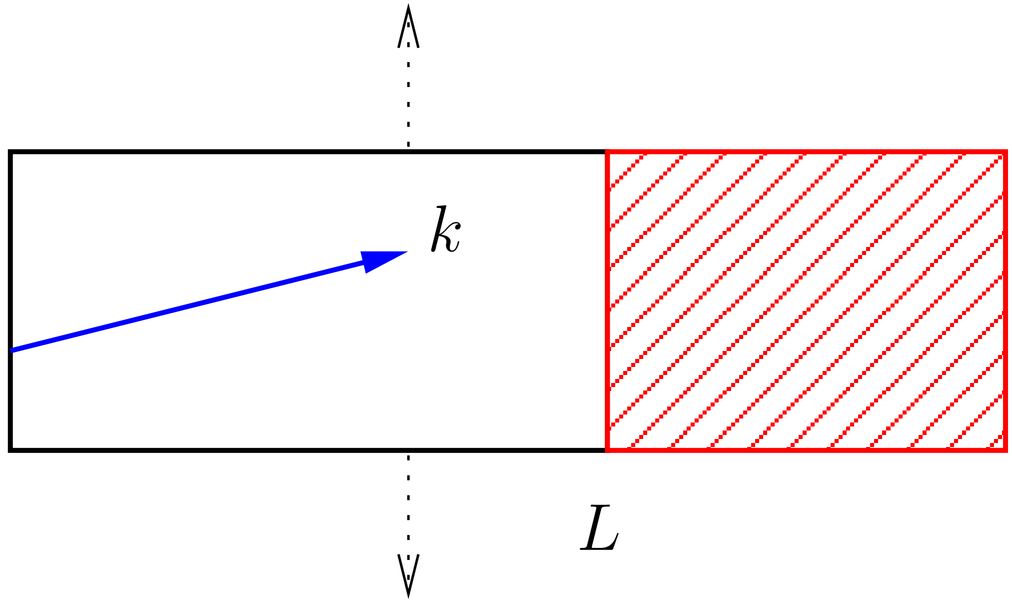
Continuum 2D model problem



$$\lambda(x) = \begin{cases} 1 - i\beta|x-L|^p, & x > L \\ 1 & x \leq L. \end{cases}$$

$$\frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

Continuum 2D model problem



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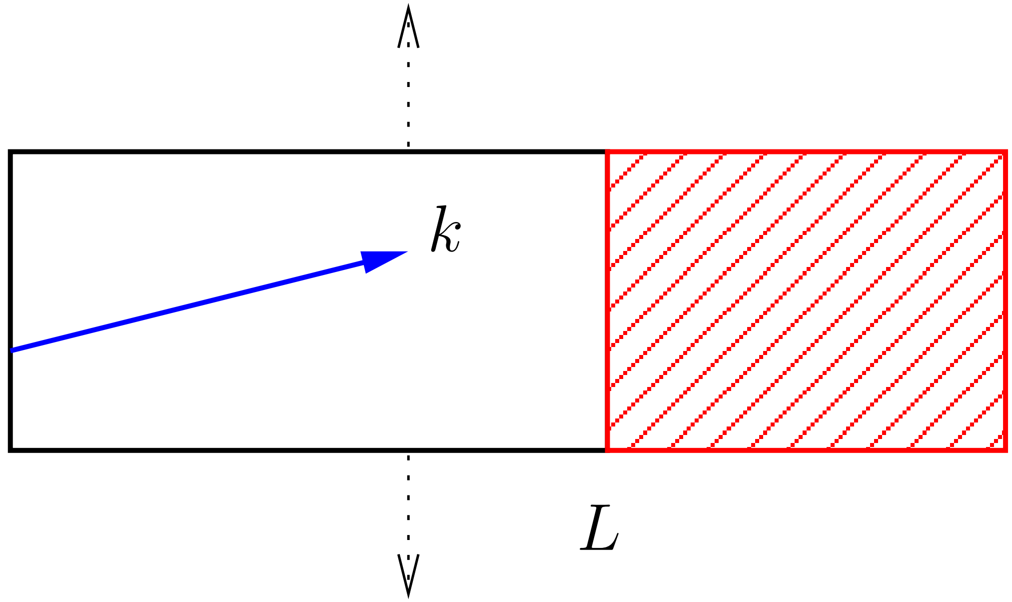
$$\frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial u}{\partial x}\right) - k_y^2 u + k^2 u = 0$$

Continuum 2D model problem



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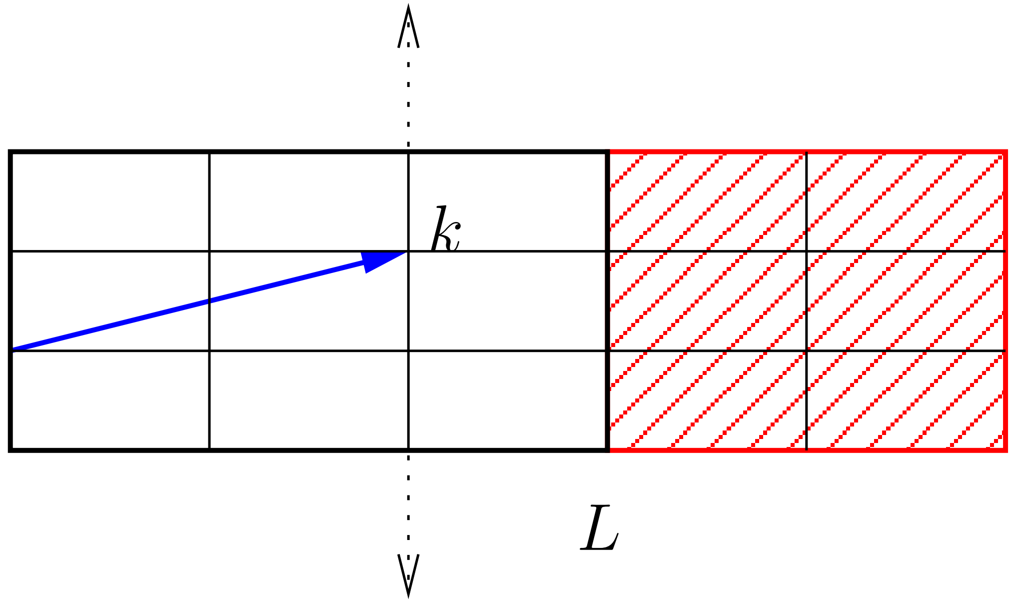
$$\frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial u}{\partial x}\right) + k_x^2 u = 0$$

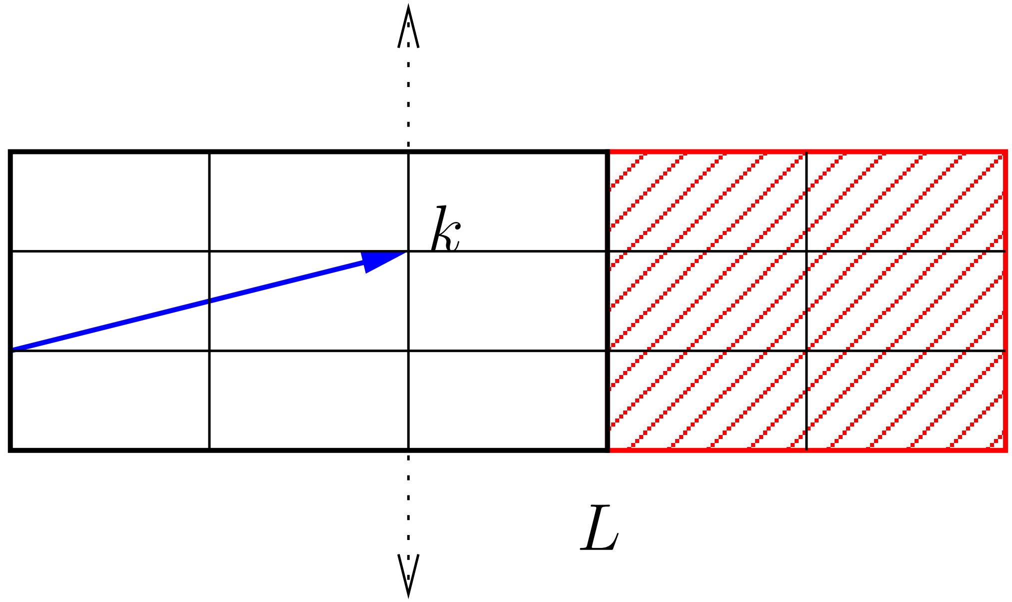
1D problem, reflection of $O(e^{-k_x \gamma})$

Discrete 2D model problem



- Discrete Fourier transform in y
- Solve numerically in x
- Project solution onto infinite space traveling modes
- Extension of Collino and Monk (1998)

Nondimensionalization



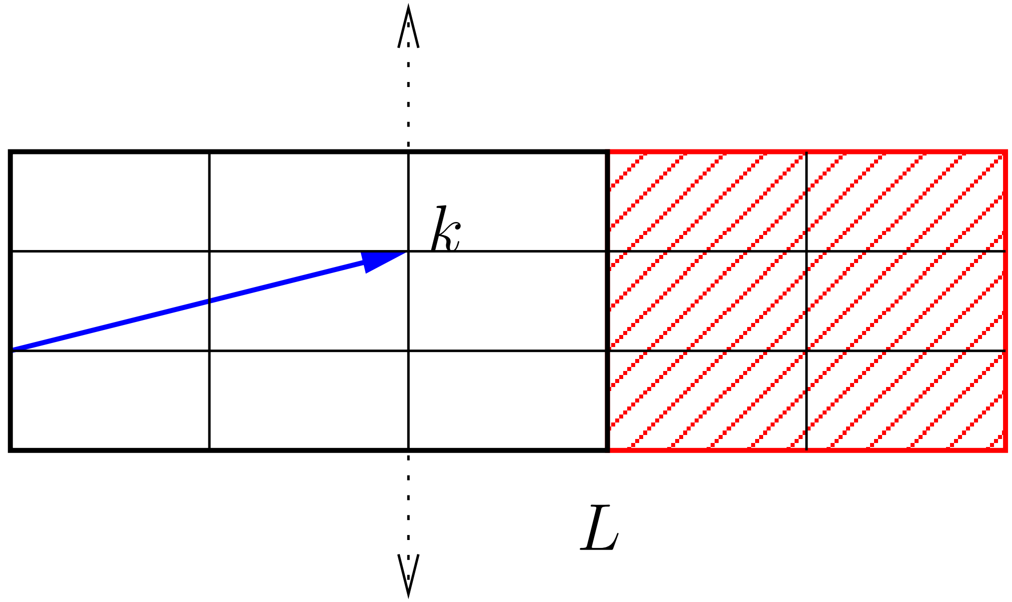
$$\lambda(x) = \begin{cases} 1 - i\beta|x-L|^p, & x > L \\ 1 & x \leq L. \end{cases}$$

Rate of stretching: βh^p

Elements per wave: $(k_x h)^{-1}$ and $(k_y h)^{-1}$

Elements through the PML: N

Nondimensionalization



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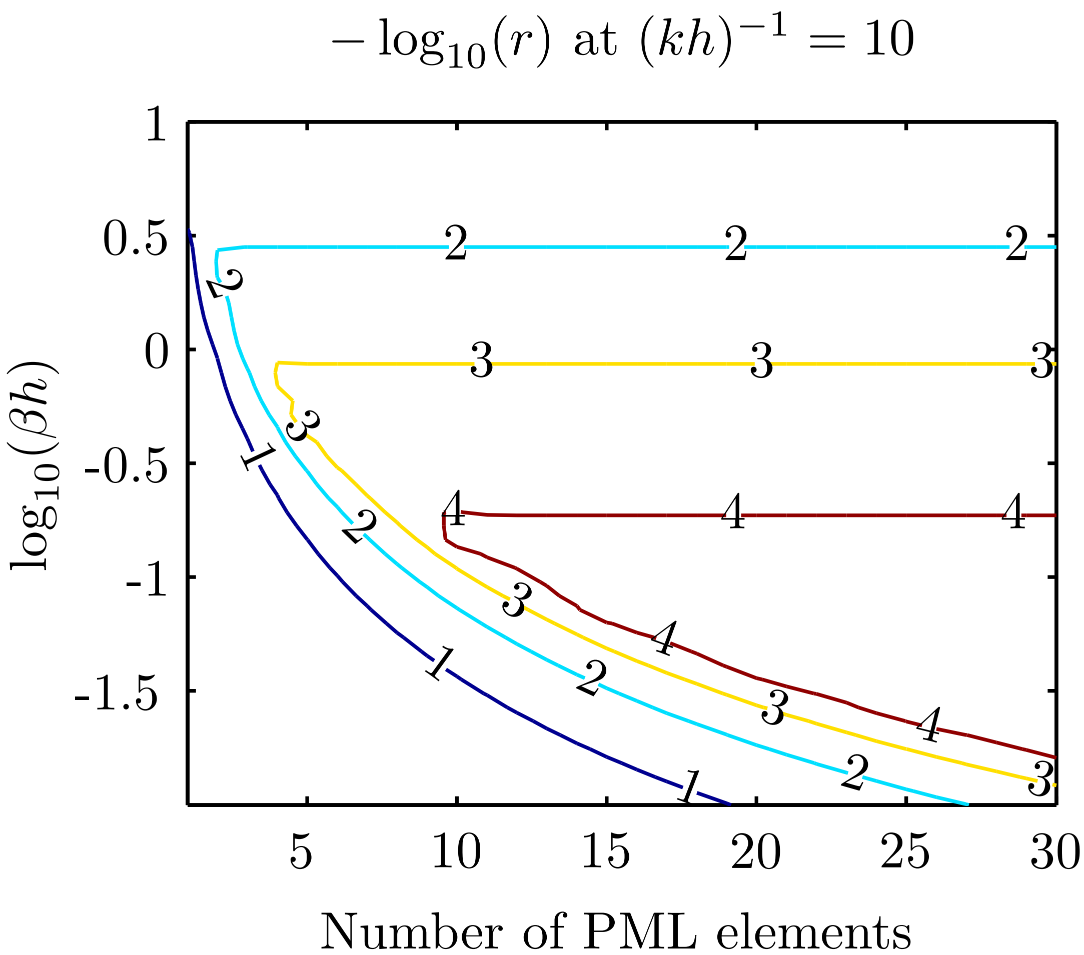
Discrete reflection behavior



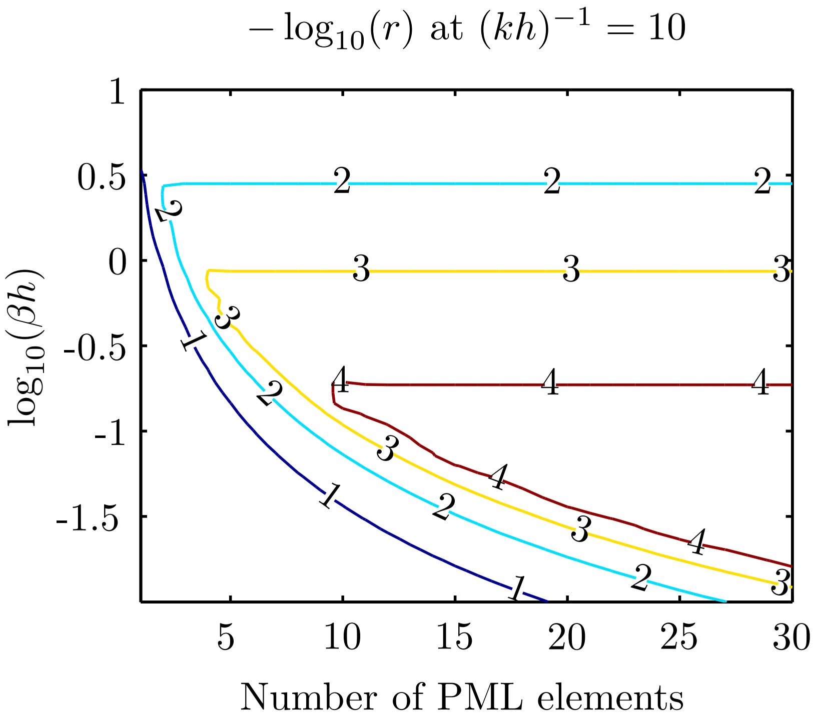
Quadratic elements, $p = 1$, $(k_x h)^{-1} = 10$

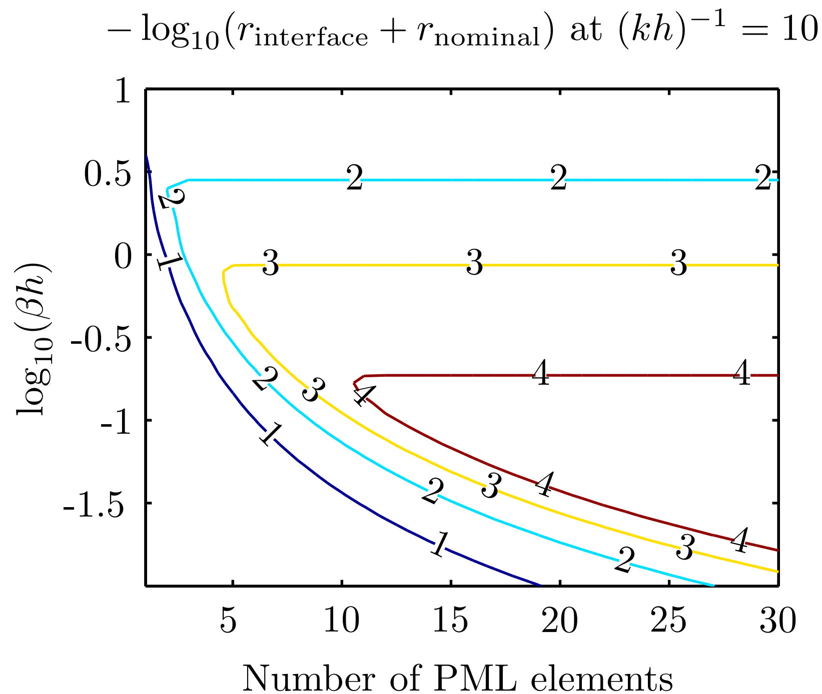
Discrete reflection decomposition

Model discrete reflection as two parts:

- Far-end reflection (clamping reflection)
  - Approximated well by continuum calculation
  - Grows as $(k_x h)^{-1}$ grows
- Interface reflection
  - Discrete effect: mesh does not resolve decay
  - Does not depend on N
  - Grows as $(k_x h)^{-1}$ shrinks

Discrete reflection behavior





Quadratic elements, $p = 1$, $(k_x h)^{-1} = 10$

- Model does well at predicting actual reflection
- Similar picture for other wavelengths, element types, stretch functions

Choosing PML parameters

- Discrete reflection dominated by
  - Interface reflection when k_x large
  - Far-end reflection when k_x small
- Heuristic for PML parameter choice
  - Choose an acceptable reflection level
  - Choose β based on interface reflection at $k_x^{\max}$
  - Choose length based on far-end reflection at $k_x^{\min}$

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  - Krylov subspace projections
  - Structure-preserving eigencomputations
  - Structure-preserving model reduction
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Eigenvalues and model reduction

Want to know about the transfer function $H(\omega)$:

$$H(\omega) = p^T (K - \omega^2 M)^{-1} b$$

Can either

- Locate poles of H (eigenvalues of (K, M))
  - Determine $Q = \frac{|\omega|}{2\,\mathrm{Im}(\omega)}$
- Plot H in a frequency range (Bode plot)

Solve both problems with the same tool:

Krylov subspace projections

Projecting via Arnoldi

Build a Krylov subspace basis by shift-invert Arnoldi

- Choose shift σ in frequency range of interest
- Form and factor $K_{\text{shift}} = K - \sigma^2 M$
- Use Arnoldi to build an orthonormal basis V for

$$\mathcal{K}_n = \text{span}\{u_0, K_{\text{shift}}^{-1} u_0, \ldots, K_{\text{shift}}^{-(n-1)} u_0\}$$

Compute eigenvalues and reduced models from projection

- Compute eigenvalues from $(V^* K V, V^* M V)$
- Approximate $H(\omega)$ by Galerkin projection

$$H(\omega) \approx (V^* p)^* (V^* K V - \omega^2 V^* M V)^{-1} (V^* b)$$

Accurate eigenvalues

- *Hermitian* systems: Rayleigh-Ritz is optimal
  - Raleigh quotient is stationary at eigenvectors

$$\rho(v) = \frac{v^* K v}{v^* M v}$$

  - First-order accurate eigenvectors $\Longrightarrow$ second-order accurate eigenvalues
- Can we obtain optimal accuracy for PML eigenvalues?

Accurate eigenvalues

- PML matrices are *complex symmetric*
  - Modified RQ is stationary at eigenvectors

$$\theta(v) = \frac{v^T K v}{v^T M v}$$

  - $\implies$ second-order accurate eigenvalues
  - Hochstenbach and Arbenz, 2004

Accurate model reduction

- Accurate eigenvalues from v and $\bar{v}$ together
- Accurate model reduction in the same way
  - Build new projection basis from V:

$$W = \text{orth}[\text{Re}(V), \text{Im}(V)]$$

  - $\text{span}(W)$ contains both $\mathcal{K}_n$ and $\bar{\mathcal{K}}_n$
    - Double convergence vs projection with V
  - W is a real-valued basis
    - Projected system remains complex symmetric

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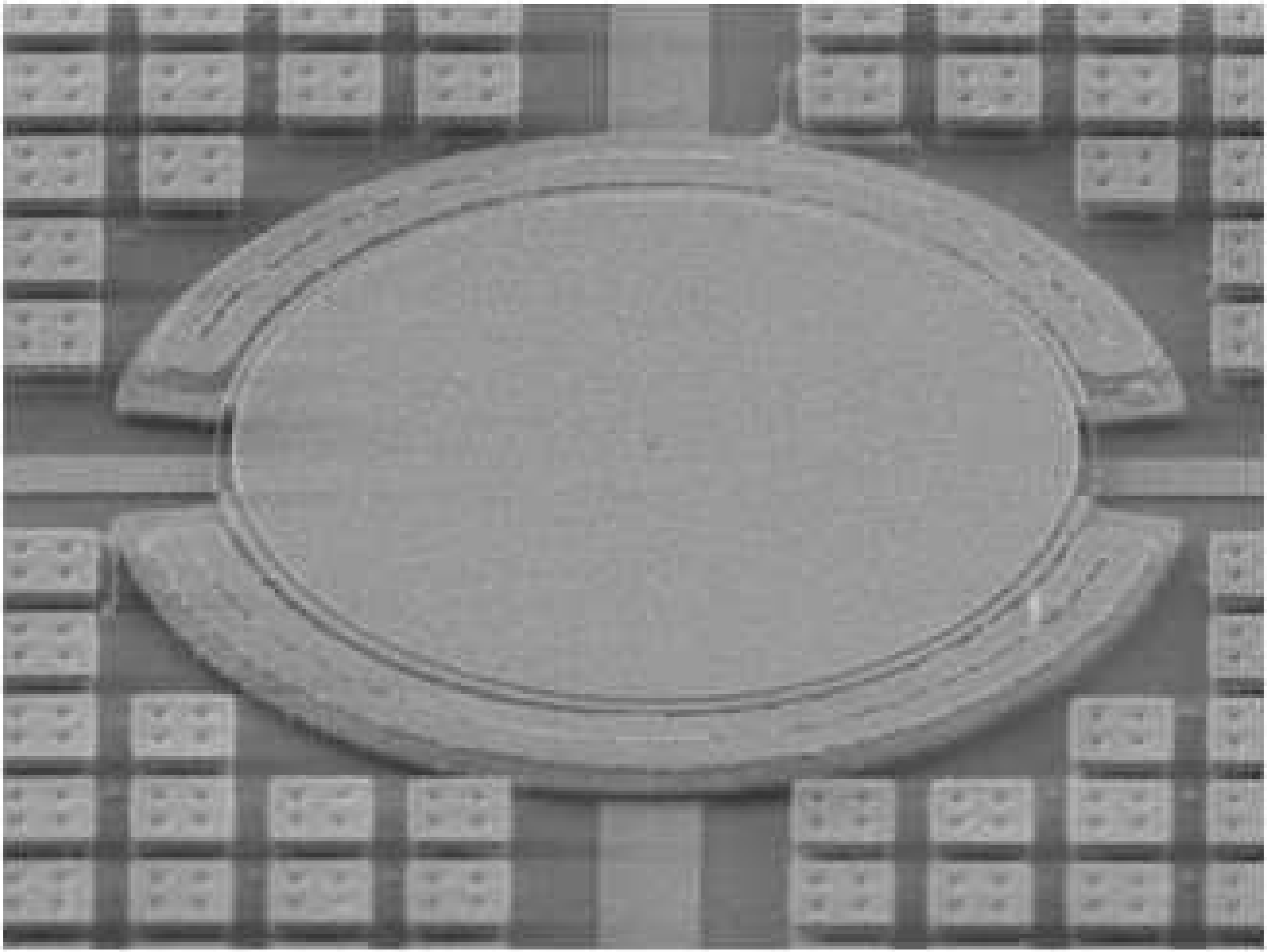
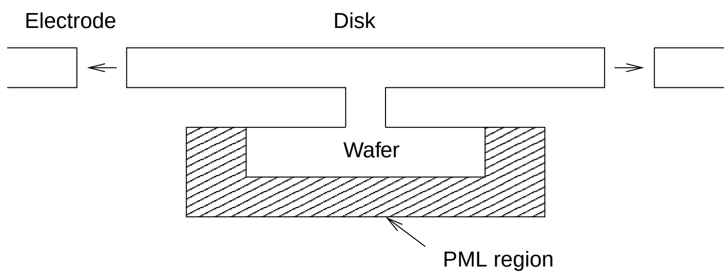
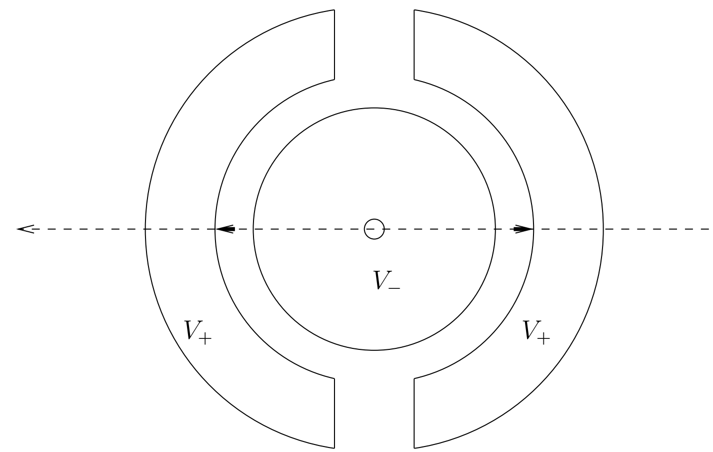
  - Accuracy of the numerics
  - Description of the loss mechanism
  - Sensitivity to fabrication variations
- Conclusions

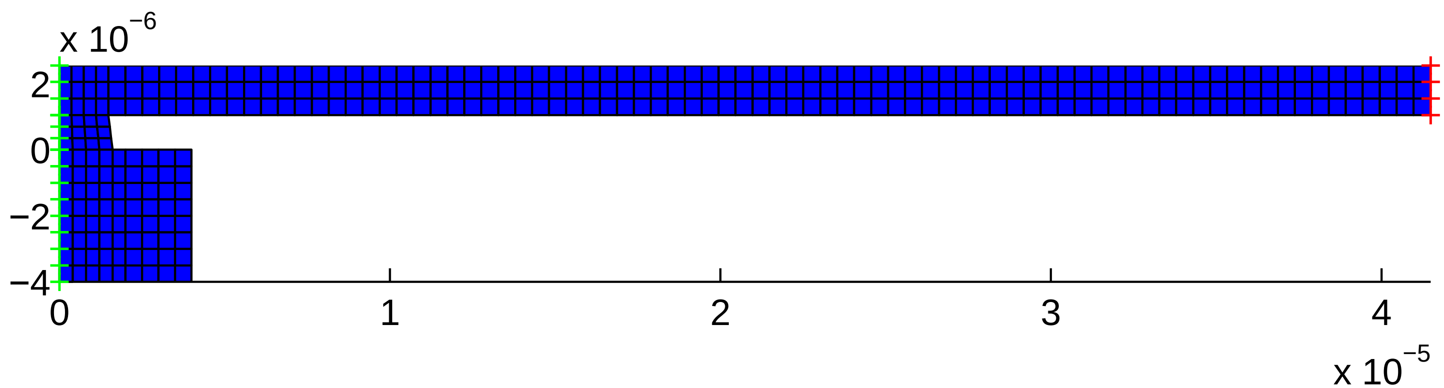
Disk resonator simulations



Disk resonator mesh

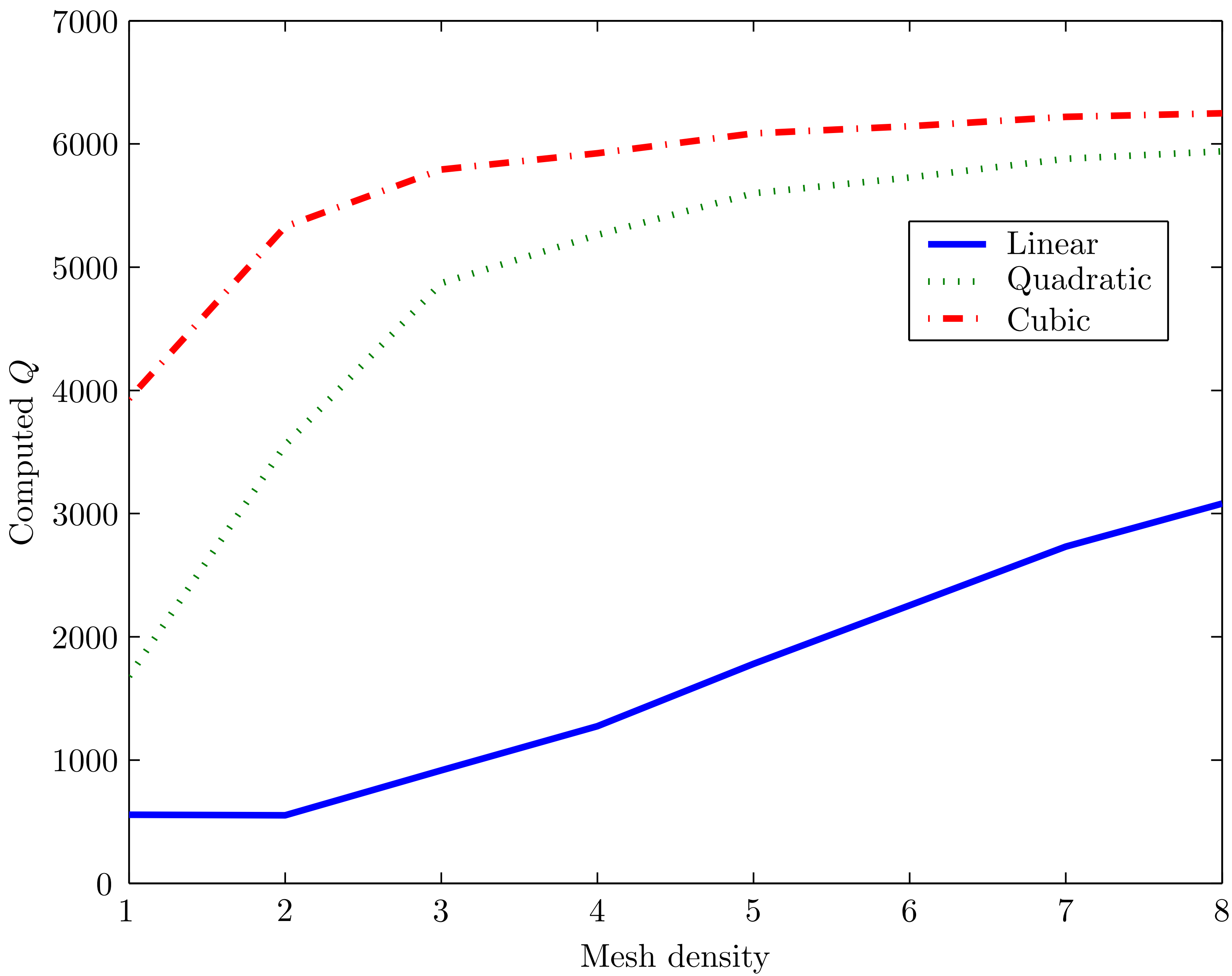




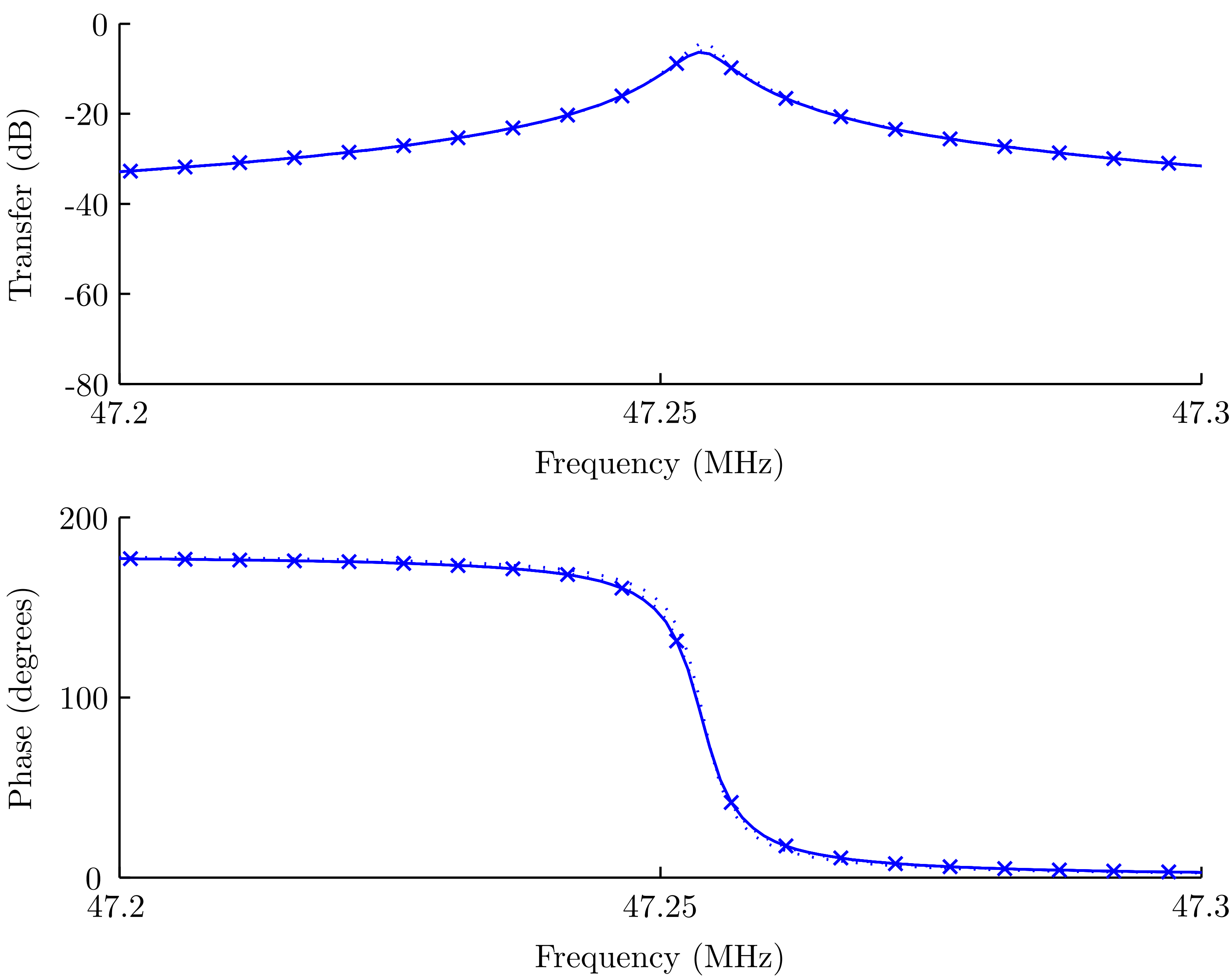


- Axisymmetric model with bicubic mesh
- About 10K nodal points in converged calculation

Mesh convergence



Model reduction performance



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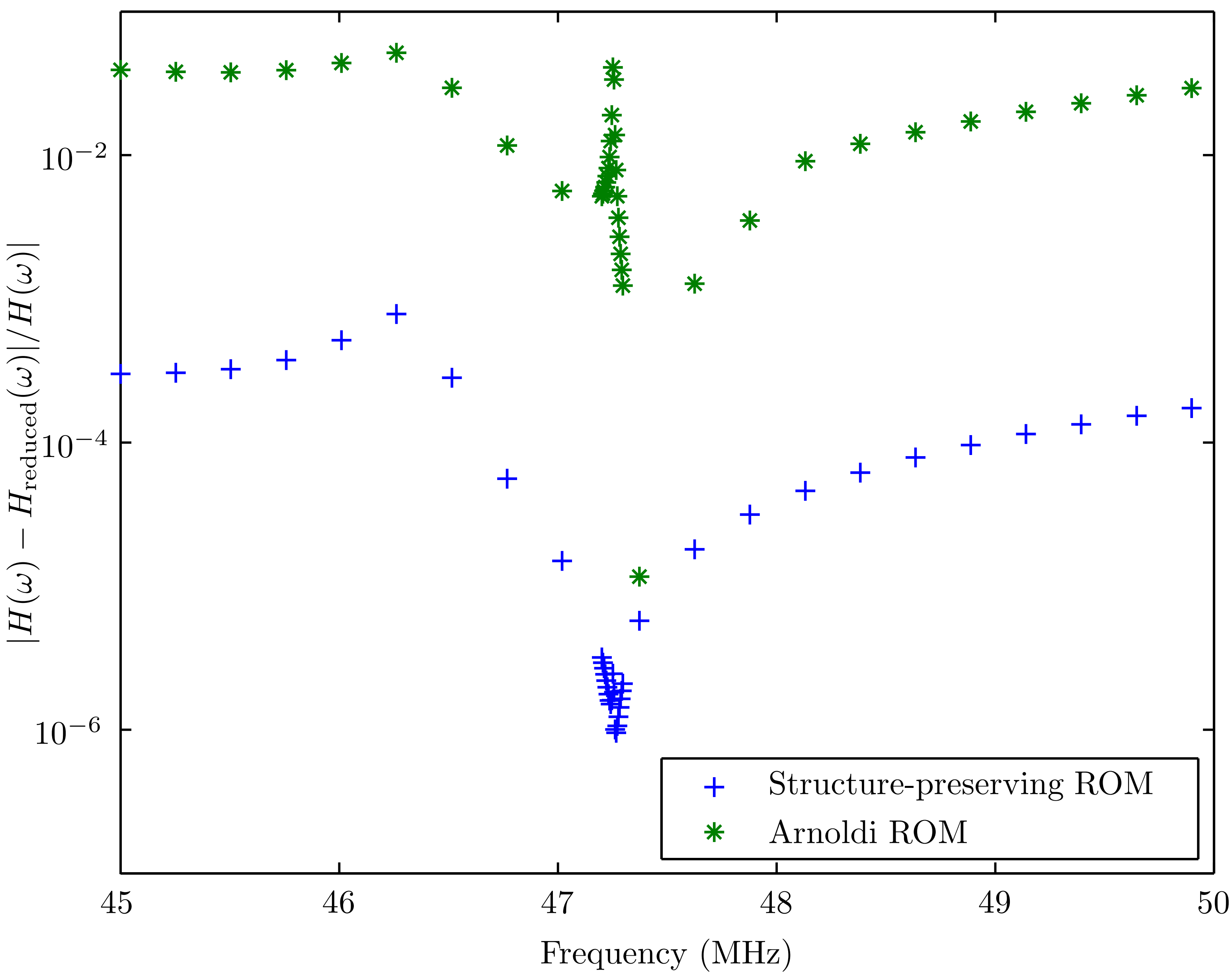
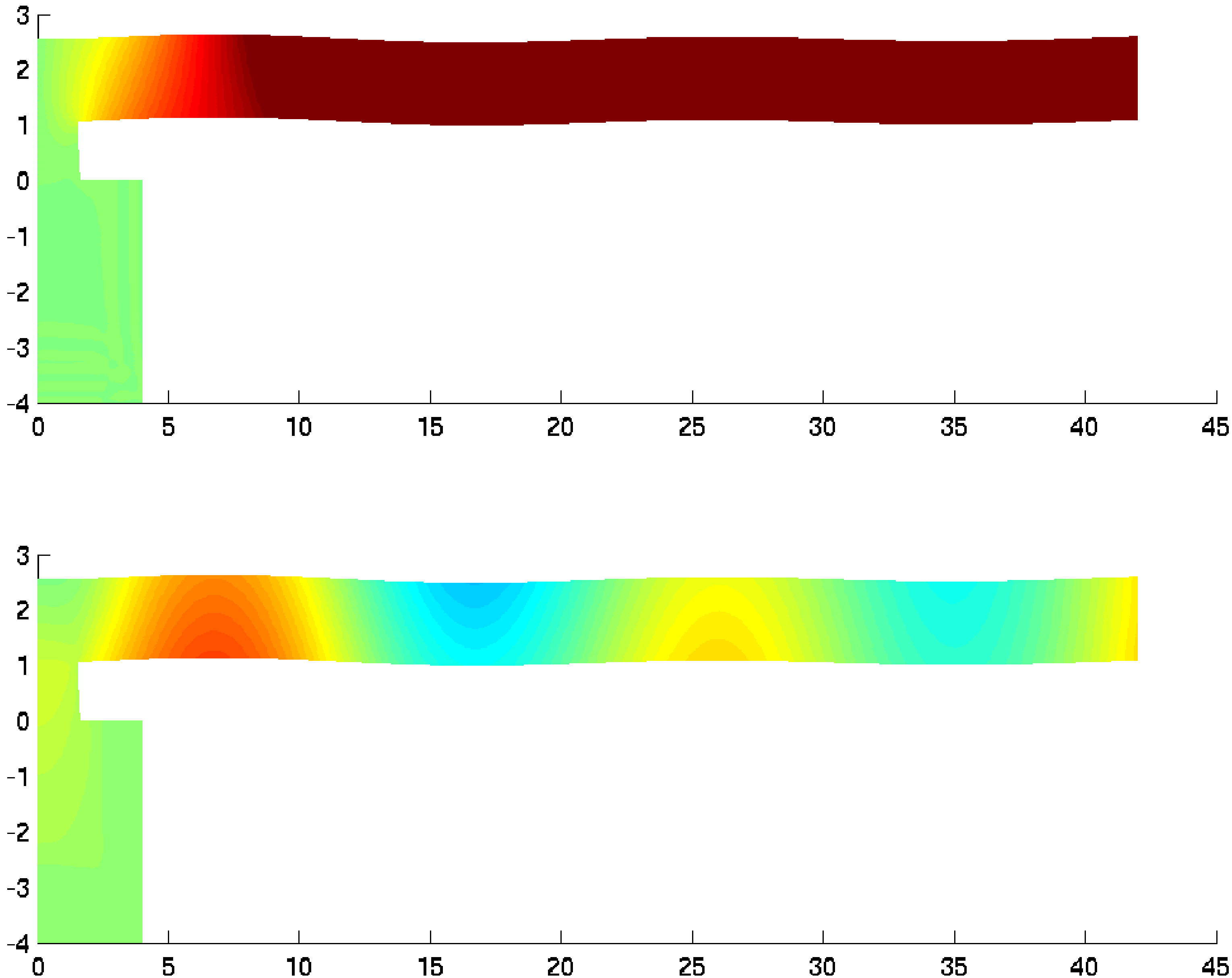
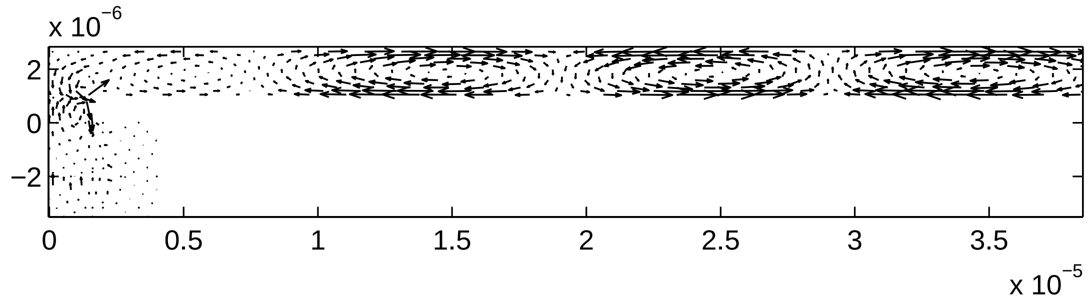
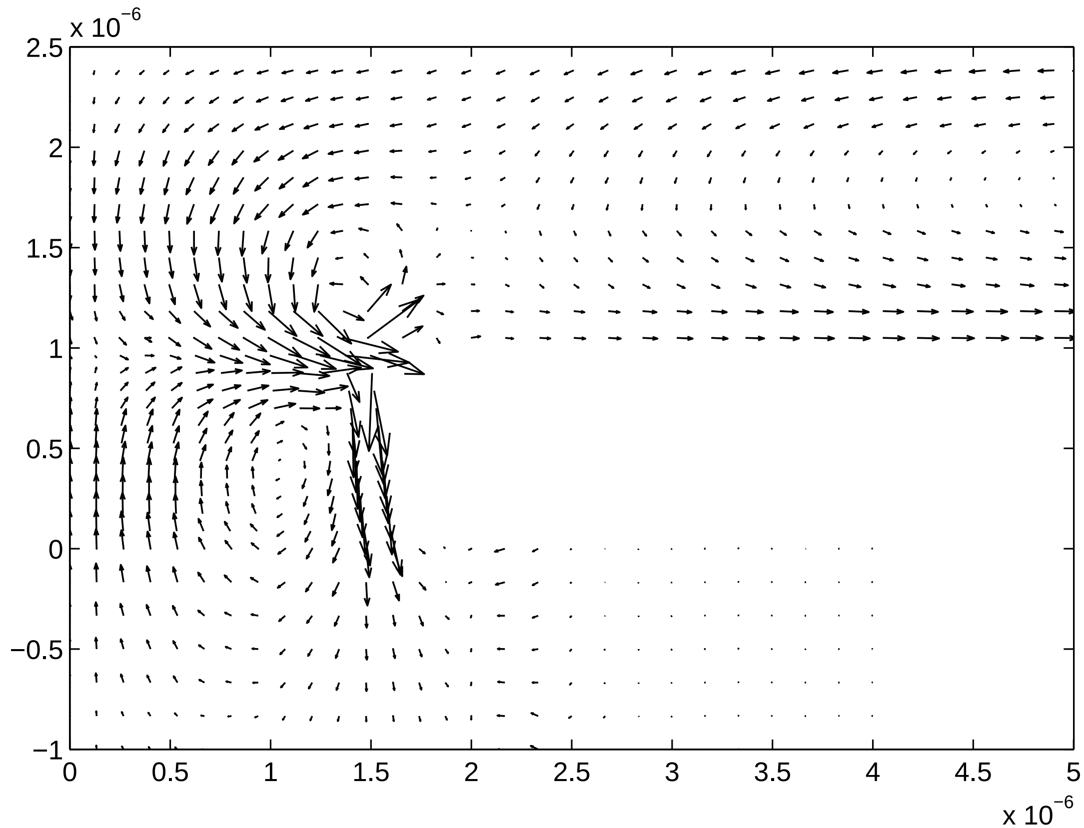


Response of the disk resonator

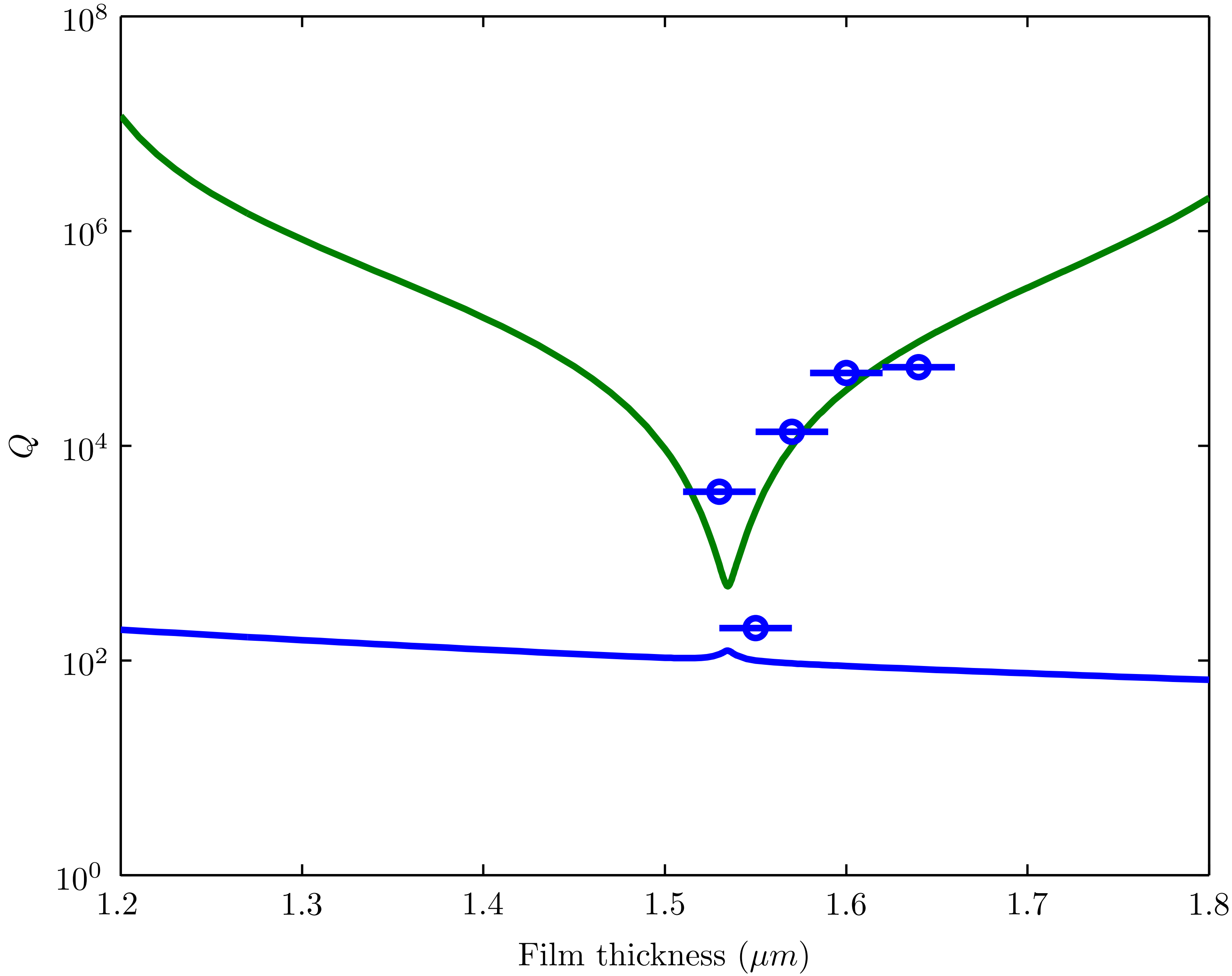


Time-averaged energy flux

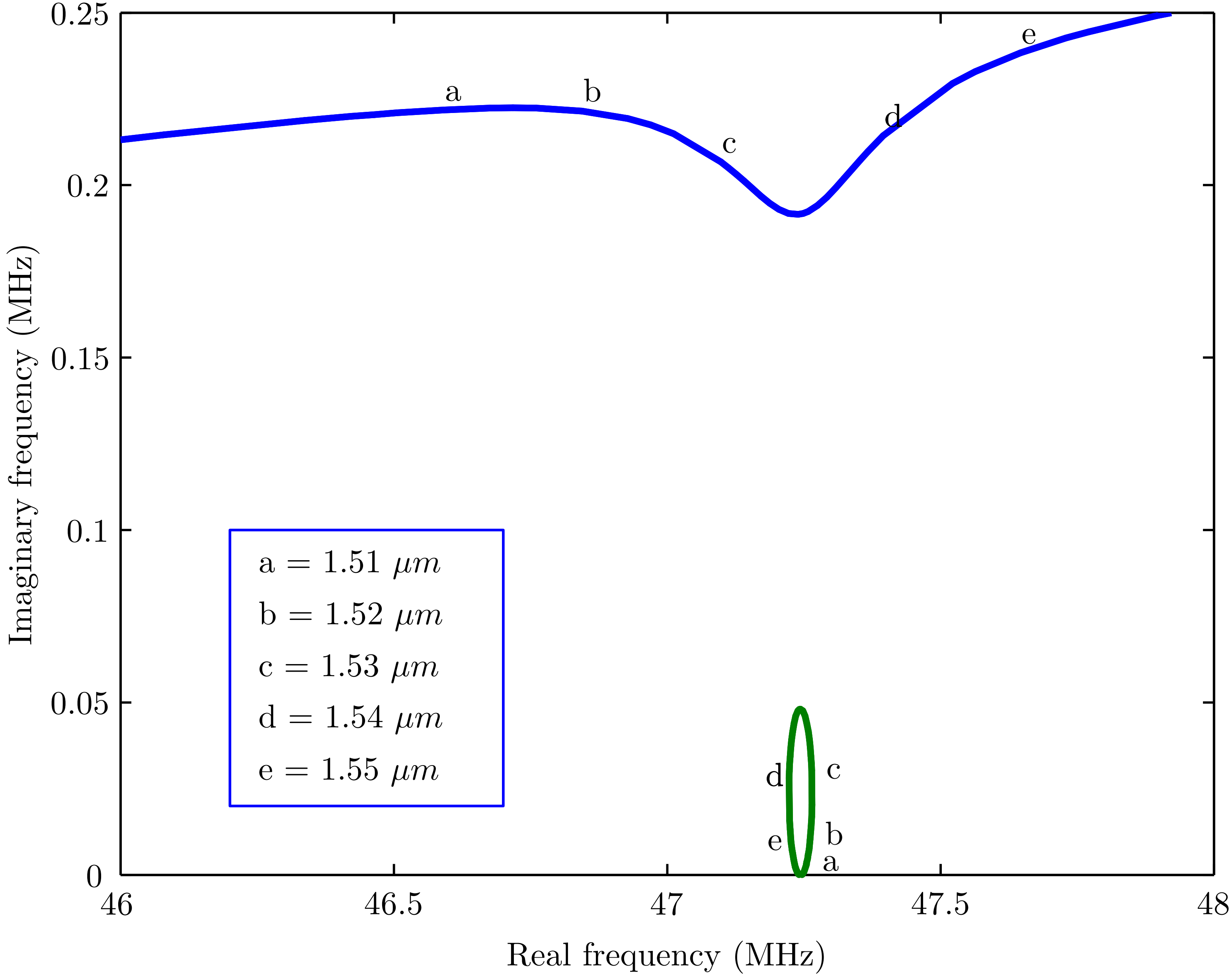




Q variation



Explanation of Q variation



Conclusions

- MEMS damping is important and non-trivial
- Elastic PMLs work well for modeling anchor loss
  - Formulation fits naturally with mapped elements
  - Estimate multi-D performance with simple models
- Use complex symmetry to compute eigenvalues and reduced models
- Simulations show effects that hand analysis misses

Reference:
Bindel and Govindjee, “Elastic PMLs for resonator anchor loss simulation,” *IJNME* (to appear).