Complex Symmetric Matrices

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Outline

- Why complex symmetry?
- Properties of complex symmetric matrices
- Projection of complex symmetric matrices
- Structure preservation and the QEP connection

Why complex symmetry?

Complex symmetric matrices appear in complex analysis:

- Grunsky inequality (Horn and Johnson):
  - f regular analytic on unit disk, normalized
  - Define $B(z) = B(z)^T$, $A(z) = A(z)^H$ s.t. f is 1-1 iff

$$x^H A(z)x \geq |x^T B(z)x|$$

  for all $x, z \in \mathbb{C}^n$ s.t. $|z_i| < 1$.

- Moment problems (Horn and Johnson)
  - Given $\{a_0, a_1, \ldots\} \in \mathbb{C}$
  - Define complex symm Hankel matrices $A_{2n} \in \mathbb{C}^{2n \times 2n}$
  - a_i are Fourier coeff for a bounded function iff for all n

$$|x^T A_{2n} x| \leq c x^H x \text{ all } x \in \mathbb{C}^{2n}$$

Why complex symmetry?

... and in data fitting and quadrature applications:

- Exponential fitting (Vandevoore; Luk and Qiao)
  - Given signals $s_0, \ldots, s_n$
  - Find $\{a_i\}$, $\{z_i\}$, and smallest r so

$$s_k = \sum_{i=1}^{r} a_i z_i^k$$

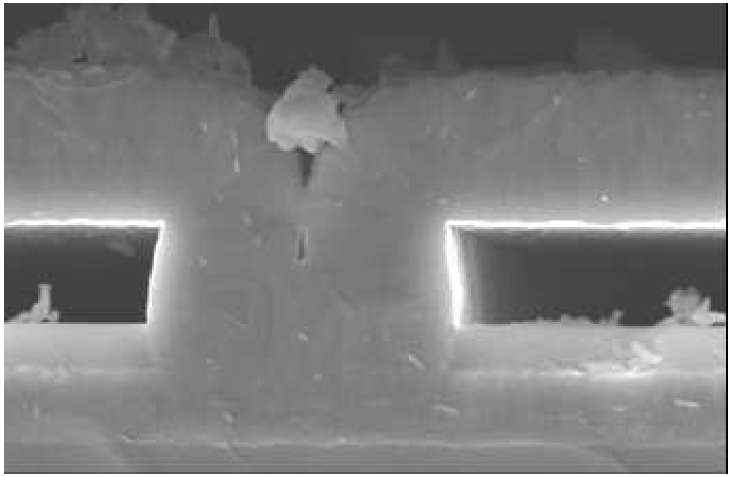
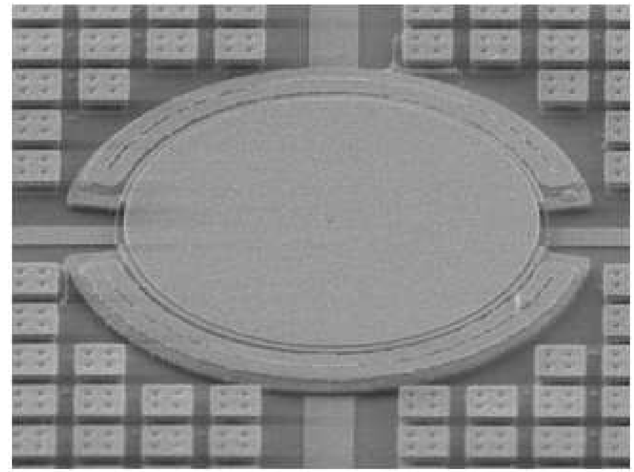
  - Turns into a complex-symm tridiagonal eigenproblem
- Quadrature (Ammar, Calvetti, Reichel)
  - Real symm tridiagonal eigenproblem $\Longrightarrow$ Gauss quadrature rules (Golub-Welsch algorithm)
  - Complex-symm tridiagonal eigenproblem $\Longrightarrow$ Gauss-Kronrod with complex nodes or neg weights

Why complex symmetry?

... and in physical problems with damped resonances:

- Problems with material loss:
  - Viscoelasticity via the Correspondence Principle (e.g. Christensen)
  - EM waveguide simulation in the presence of conductors (e.g. Arbenz and Hochstenbach)
- Infinite domain models:
  - Perfectly matched layer (PML)
    - First in electromagnetics (Berengér 95)
    - Then acoustics, elasticity, etc.
  - Exterior complex scaling in quantum mechanics
    - Invented earlier than PMLs (Simon 79)
    - Same idea, little mutual awareness

Why complex symmetry?

My interest: damping in high-freq MEMS resonators

- Want to minimize losses in RF MEMS
- Physics isn't always well-understood
- Want to compute:
  - Damped mode shapes and frequencies
  - Reduced-order models of freq response

Damped MEMS resonances

- Material losses
  - Low intrinsic losses in silicon, diamond, germanium
  - Terrible material losses in metals
- Anchor loss
  - Elastic waves radiate from structure
- Thermoelastic damping
  - Volume changes induce temperature change
  - Diffusion of heat leads to mechanical loss
- Fluid damping
  - Air is a viscous fluid ($\mathrm{Re} \ll 1$)
  - Can operate in a vacuum
  - Shown not to dominate in many RF designs

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Viscoelastic losses

Start with time-harmonic elasticity (weak form):

$$-\omega^2 \int_\Omega w \cdot \rho u \, d\Omega + \int_\Omega \epsilon(w) : \sigma \, d\Omega = \int_\Gamma w \cdot t \, d\Gamma$$

where

- u is time-harmonic displacement ($u^0 = ue^{i\omega t}$)
- $\epsilon = (\nabla u)^s$ is time-harmonic strain
- $\sigma = \mathsf{C} : \epsilon$ is time-harmonic stress
- t is time-harmonic surface traction

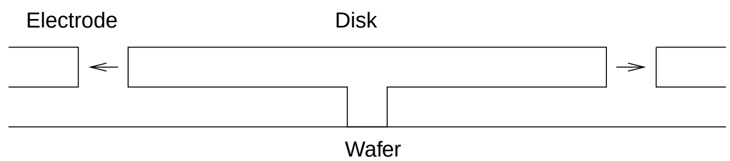
Finite element discretization is real symmetric:

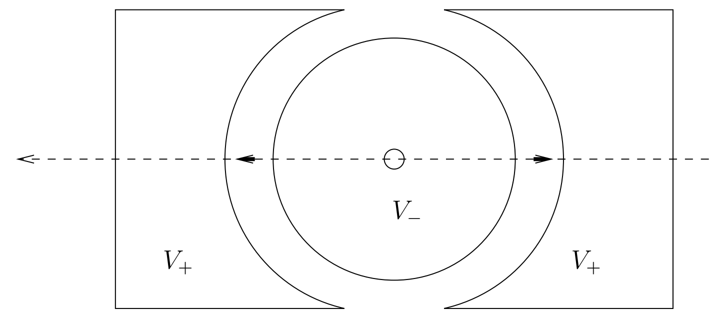
$$-\omega^2 Mu + Ku = F$$

Viscoelastic losses

- Viscoelasticity: $\sigma = \hat{\mathsf{C}}(\omega) : \epsilon$
  - $\hat{\mathsf{C}}(\omega)$ = Fourier transform of relaxation kernel
  - *Correspondence principle*: hysteresis described through complex-valued material properties
  - Similar principle for acoustics, electromagnetics
- Simplest case: $\hat{\mathsf{C}} = \mathsf{C} + i\omega\eta\mathsf{I}$
  - Corresponds to adding a shear viscosity η
- Finite element: $-\omega^2 Mu + K(\omega)u = F$
  - $K(\omega)$ is complex symmetric
  - Simplest case: $K(\omega) = K_0 + i\omega D$

Anchor loss and PMLs

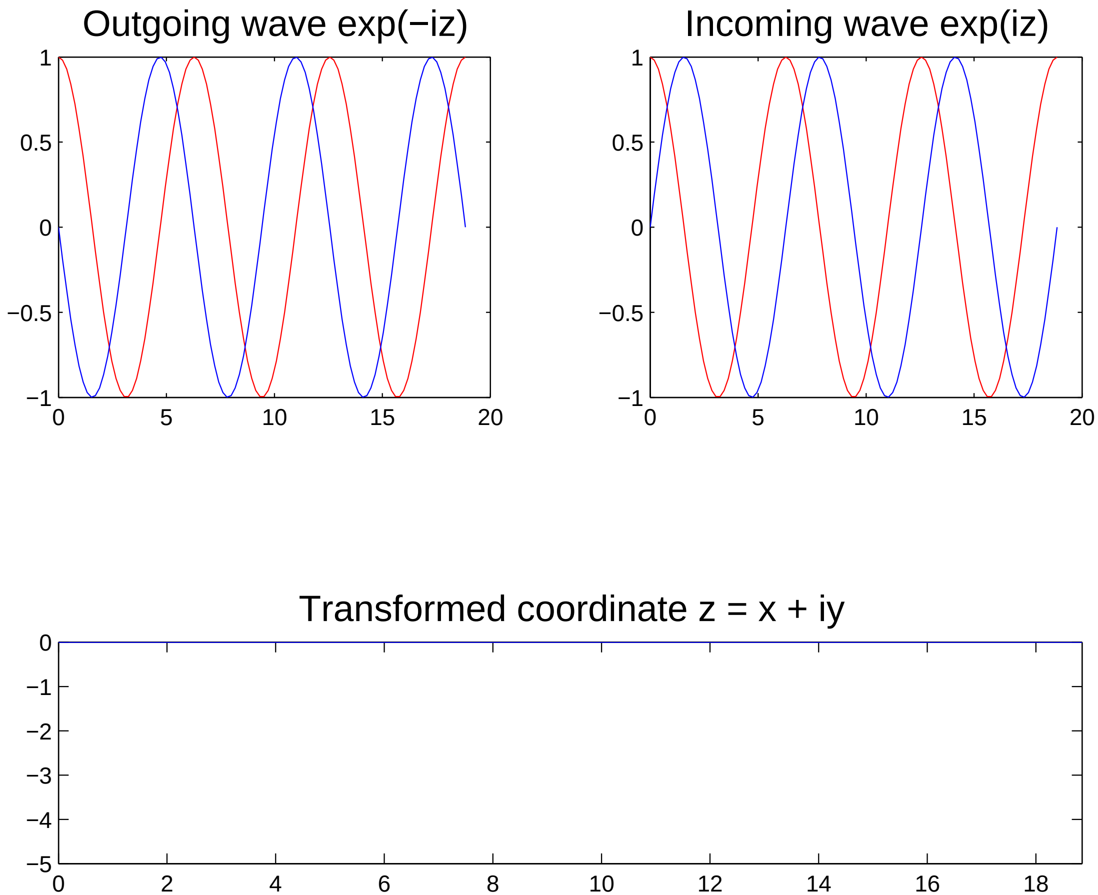




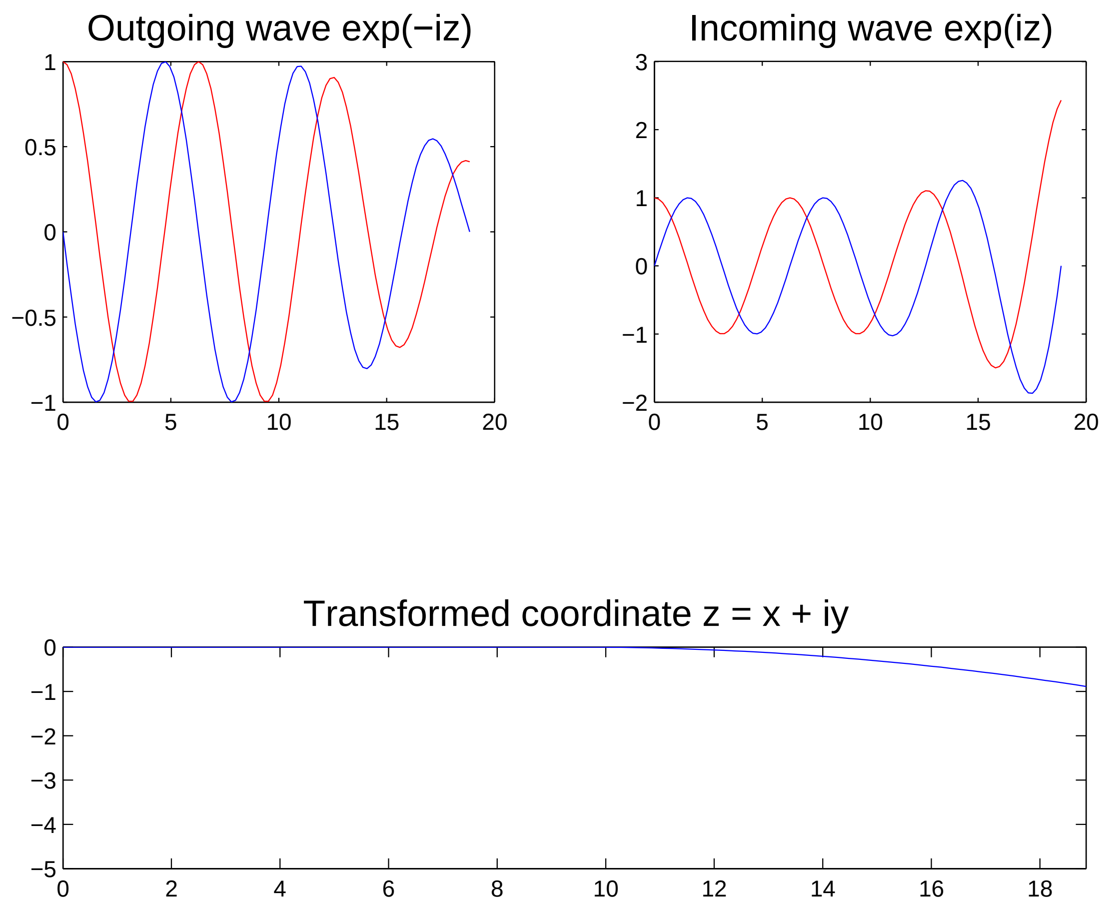
Want to model elastic radiation from resonator to substrate

- Apply a complex coordinate transformation
- Generates a non-physical absorbing layer
- No impedance mismatch between the computational domain and the absorbing layer
- Idea works with general linear wave equations

Scalar wave example



Scalar wave example



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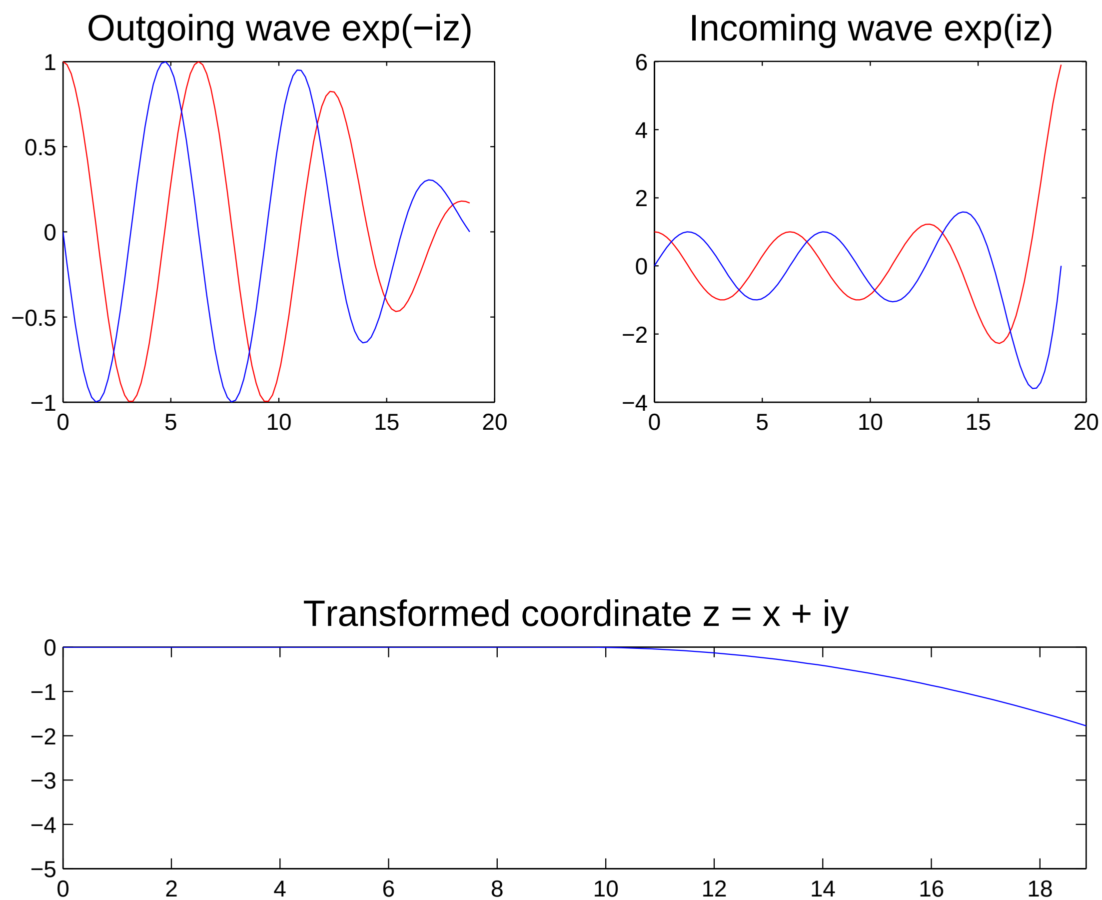

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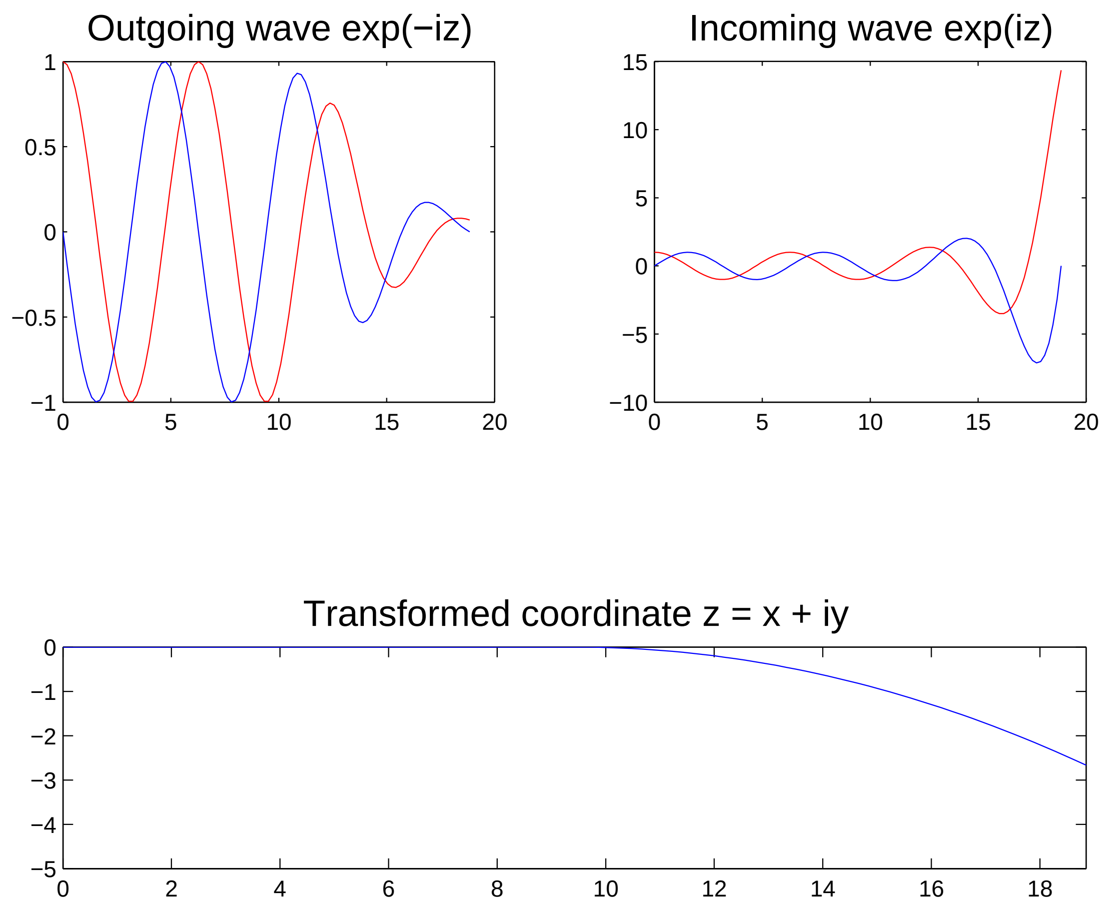

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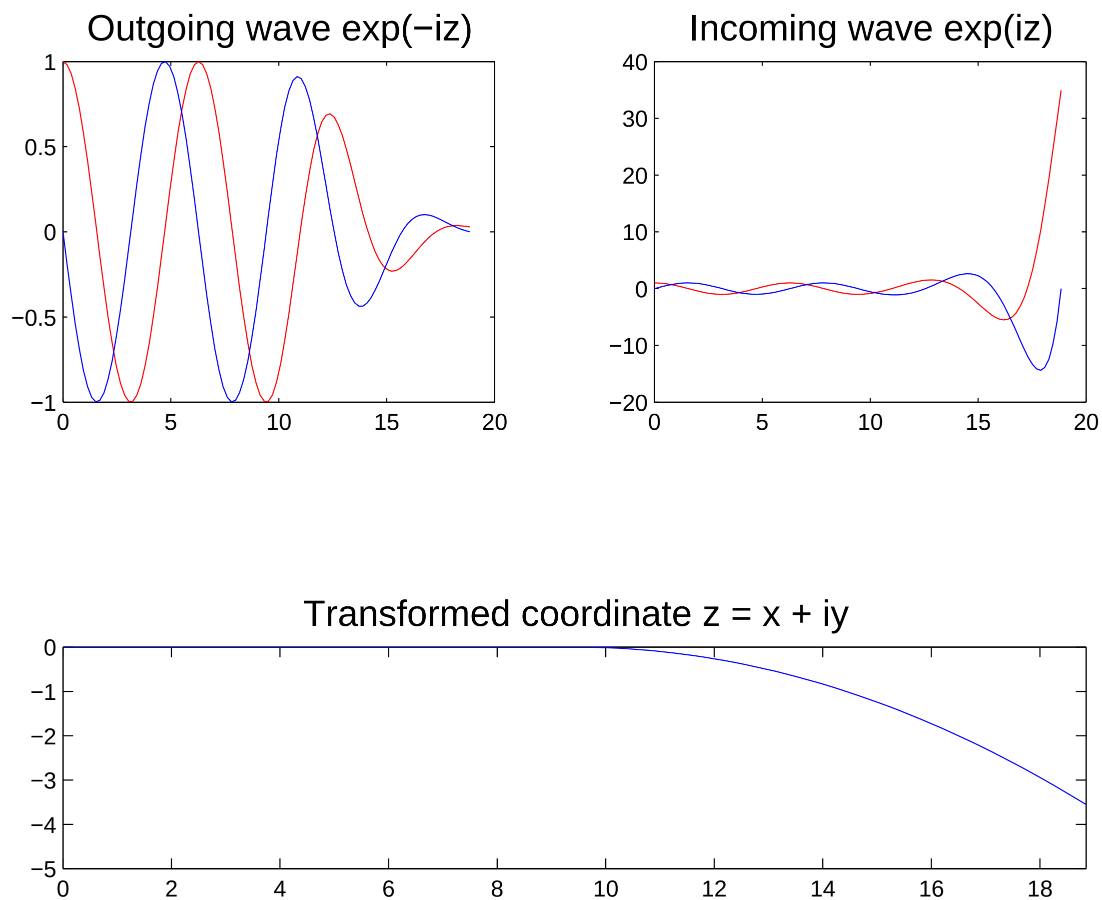

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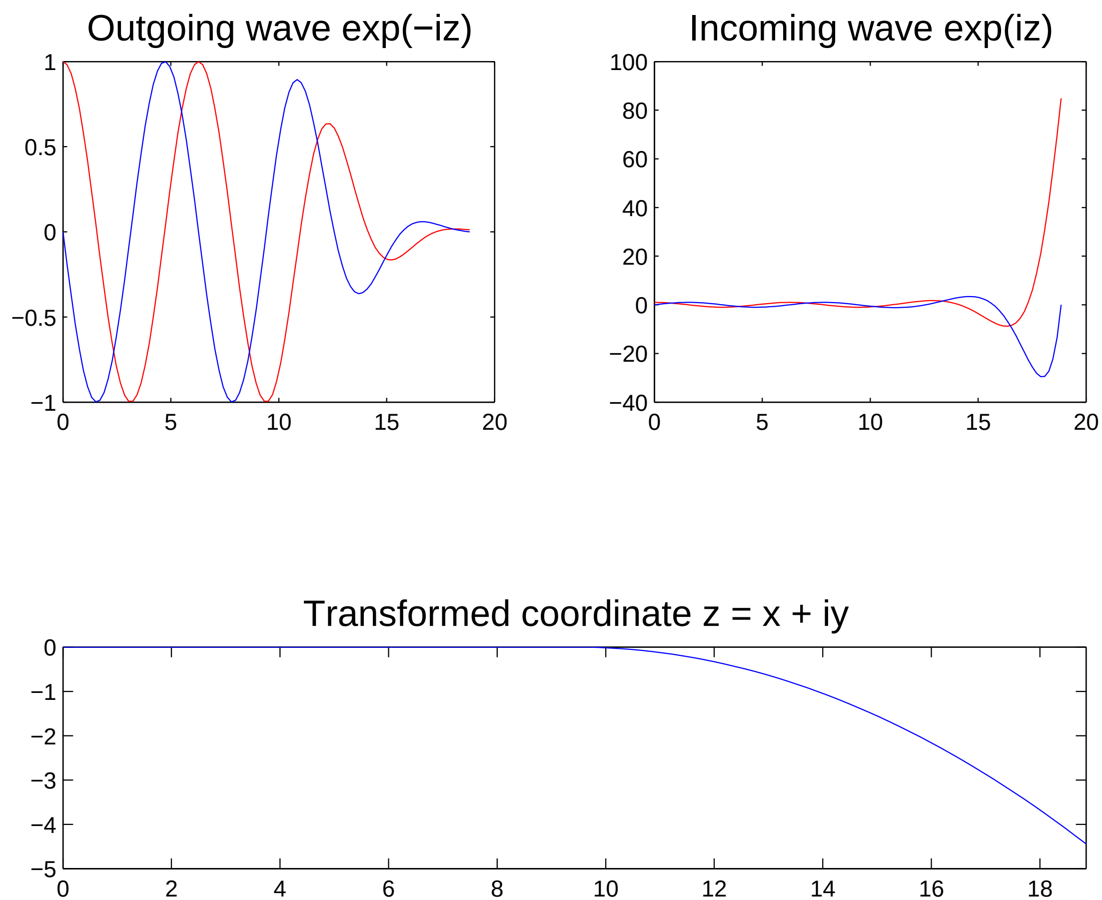


Clamp solution at transformed end to isolate outgoing wave.

Elastic PML

Weak form of time-harmonic PML equation:

$$-\omega^2 \int_\Omega w \cdot \rho u \, J \, d\Omega + \int_\Omega \tilde{\nabla} w : \mathsf{C} : \tilde{\nabla} u \, J \, d\Omega = \int_\Omega w \cdot t \, d\Gamma$$

- J is the Jacobian of the transformation, $\tilde{\nabla}$ is the transformed gradient operator
- Coordinate transform typically depends on ω
  - Needed to make decay frequency-independent
- Finite element: $-\omega^2 M(\omega) u + K(\omega) u = F$
  - M and K now *both* complex symmetric

Eigenvalue problems

Consider the eigenvalue problem

$$\left(-\omega^2 M(\omega) + K(\omega)\right) u = 0$$

where M and K are complex and may depend on ω.

Can get a complex symmetric *linear* eigenproblem by

- Linearizing: $\omega = \omega_0 + \delta$, discard $O(\delta^2)$ terms
- Using $M(\omega_0)$ and $K(\omega_0)$
  - Makes sense for PML (damping usually adequate for ω the same order of magnitude as ω_0)
- Good idea when good shift is available

Eigenvalue problems

Not always approximating nonlinear eigenproblems.
Can get eigenvalue problems from separation of variables.

- Continuous translational symmetry
  - Infinite guide, constant cross-section
  - Fixed forcing frequency
- Discrete translational symmetry (Bloch-Floquet waves)
  - SAW filter arrays (Zaglymayr, Sch oberl, Langer)
  - Electromagnetic filters

Complex symmetric eigenproblem

Thm: Every matrix is similar to a complex symmetric matrix.

- Can have arbitrary Jordan structure
- Complex symmetry is still useful

Analogues exist for many statements about Hermitian matrices (see Horn and Johnson, section 4.4).

Complex symmetric eigenproblem

- If z is a column eigenvector, then z^T is a row eigenvector
- The modified Rayleigh quotient

$$\theta(z) = \frac{z^T K z}{z^T M z}$$

  is stationary at eigenvectors (assuming $z^T M z \neq 0$); at an eigenvector, θ equals the eigenvalue.
- Eigenvectors for distinct eigenvalues are *complex orthogonal*: $z^T M w = 0$.
- But the nice minimax results of the Hermitian case lack analogues here.

Ordinary and modified RQ

- $\rho(z) = \frac{z^H K z}{z^H M z}$
  - $\{z^H M z = 1\}$ is compact (for M pos def) $\Longrightarrow$ ρ has bounded range (field of values)
  - Only first-order accurate eigenvalue estimate
- $\theta(z) = \frac{z^T K z}{z^T M z}$
  - $\{z^T M z = 1\}$ is non-compact, θ can generally go wild
  - Second-order accurate eigenvalue estimate when z is near an eigenvalue

Physics of $z^T z = 0$

The bad case $z^T M z = 0$ (or ≈ 0) *can* happen

- Mimicking infinite domain means we approximate the essential spectrum
- Propogating waves give $z^T M z \approx 0$ (Olyslager 04)
- Same occurs in quantum mechanical computations
- Usually interested in the discrete part of the spectrum

Complex-symmetric projection

- Algorithms:
  - Complex-symmetric Lanczos (Cullum and Willoughby)
  - Arnoldi
  - Complex Jacobi-Davidson
  - Splitting bases
- Can do spectral transformations (e.g. shift-invert)
- Can start nonlinear eigencomputation from a linear one
- Projections may be used to build reduced models, too

Complex-symmetric Lanczos

- $u_0 = 0$, $\beta_0 = 0$
- for $j = 1$ to k
  - $v := Ku_j$
  - $\alpha_j := u_j^T Mv$
  - $v := v - \alpha_j u_j - \beta_{j-1} u_{j-1}$
  - $\beta_j := \sqrt{v^T Mv}$
  - $u_{j+1} := v/\beta_j$

Complex-symmetric Lanczos

- Half the work, storage of usual non-symmetric Lanczos
- Used for model-reduction (with proportional drive and sense), gets usual PVL matching in $2n$ moments
- Still has breakdown, near breakdown, woe and doom
- Has been used both for eigenproblems and for solving linear systems (Freund)
- See Eigentemplates section 7.11.

Arnoldi

- Can compute a unitary (vs complex orthogonal) Krylov subspace basis W using standard Arnoldi
  - Avoids issues with ill-conditioning in the basis
  - But requires work to orthogonalize against more previous vectors
- Once the basis is in hand:
  - Use eigenvalues of $(W^H K W, W^H M W)$
    - Usual nonsymmetric approach
  - Use eigenvalues of $(W^T K W, W^T M W)$
    - Get second-order accuracy when W contains good eigenvector estimates
    - Identical (in exact arithmetic) to estimates from nonsymmetric Lanczos.
  - Could we combine the two?

Complex-symmetric Jacobi-Davidson

- Proposed by Arbenz and Hochstenbach
- Specializes two-sided JD (half the work, storage)
- Uses modified Rayleigh quotient
- Main problem in examples was preconditioning inner solver

Basis-splitting

Let $W = U + iV \in \mathbb{C}^k$ be a basis (e.g. from Arnoldi)

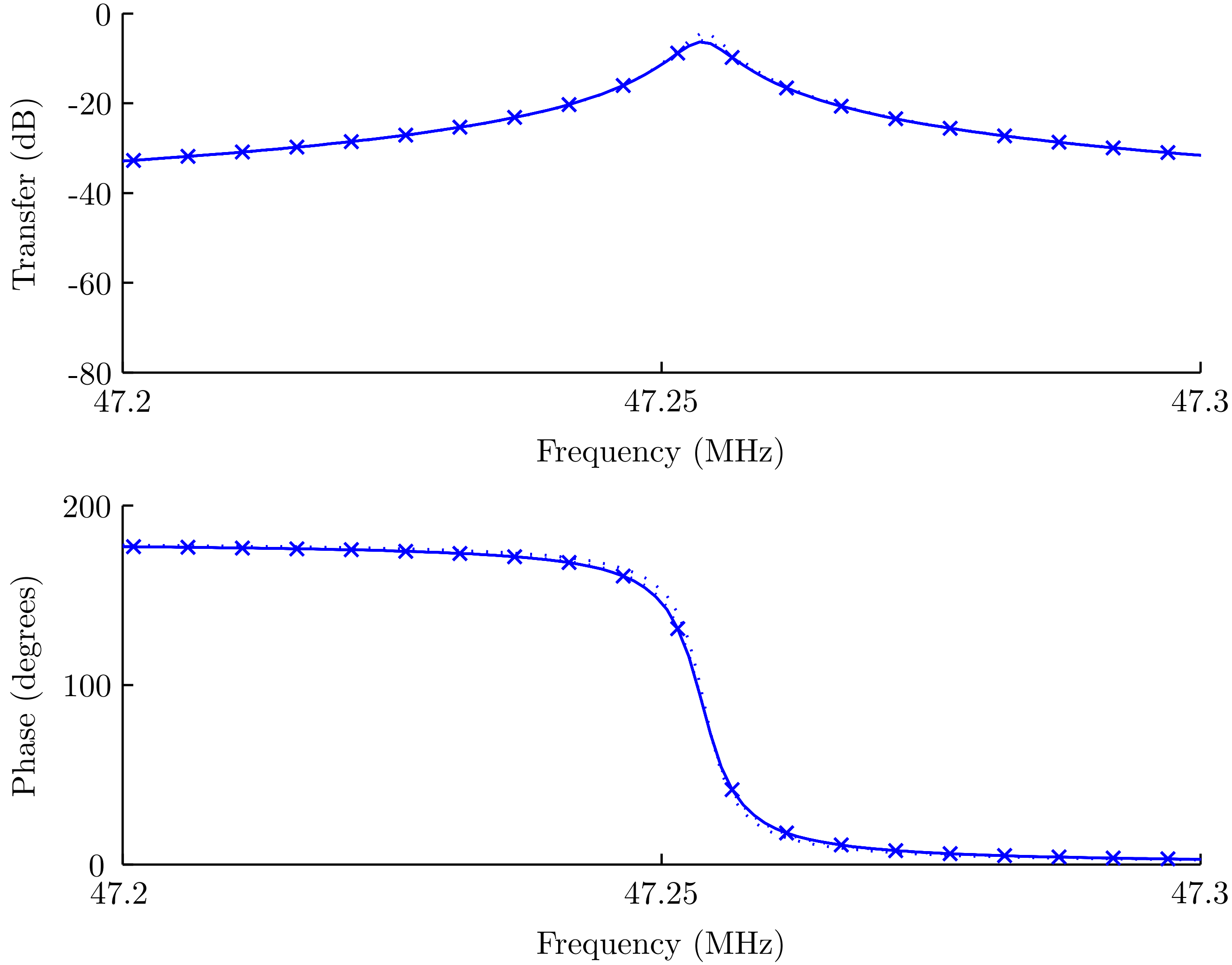
- Form $Q = \mathrm{orth}([U, V]) \in \mathbb{R}^{n \times 2k}$
  - Span of Q contains span of $[W, \bar{W}]$
- Compute eigenvalues of $(Q^T K Q, Q^T M Q)$
  - Forming $(Q^T K Q, Q^T M Q)$ not more expensive than projection with W
  - If M is pos def, Ritz values will remain bounded
  - Maintain accuracy of modified Rayleigh quotient

Basis splitting

Using the split basis preserves several structures:

- Projected system remains complex symmetric
- Projection doesn't mix up real and imaginary parts
  - Real symmetries of mass, damping, stiffness preserved
- Matches Galerkin discretization of PDEs
  - Like choosing real-valued global shape functions
  - Easier to think about physically
  - Provided the original motivation for this splitting

Example: Disk resonator response



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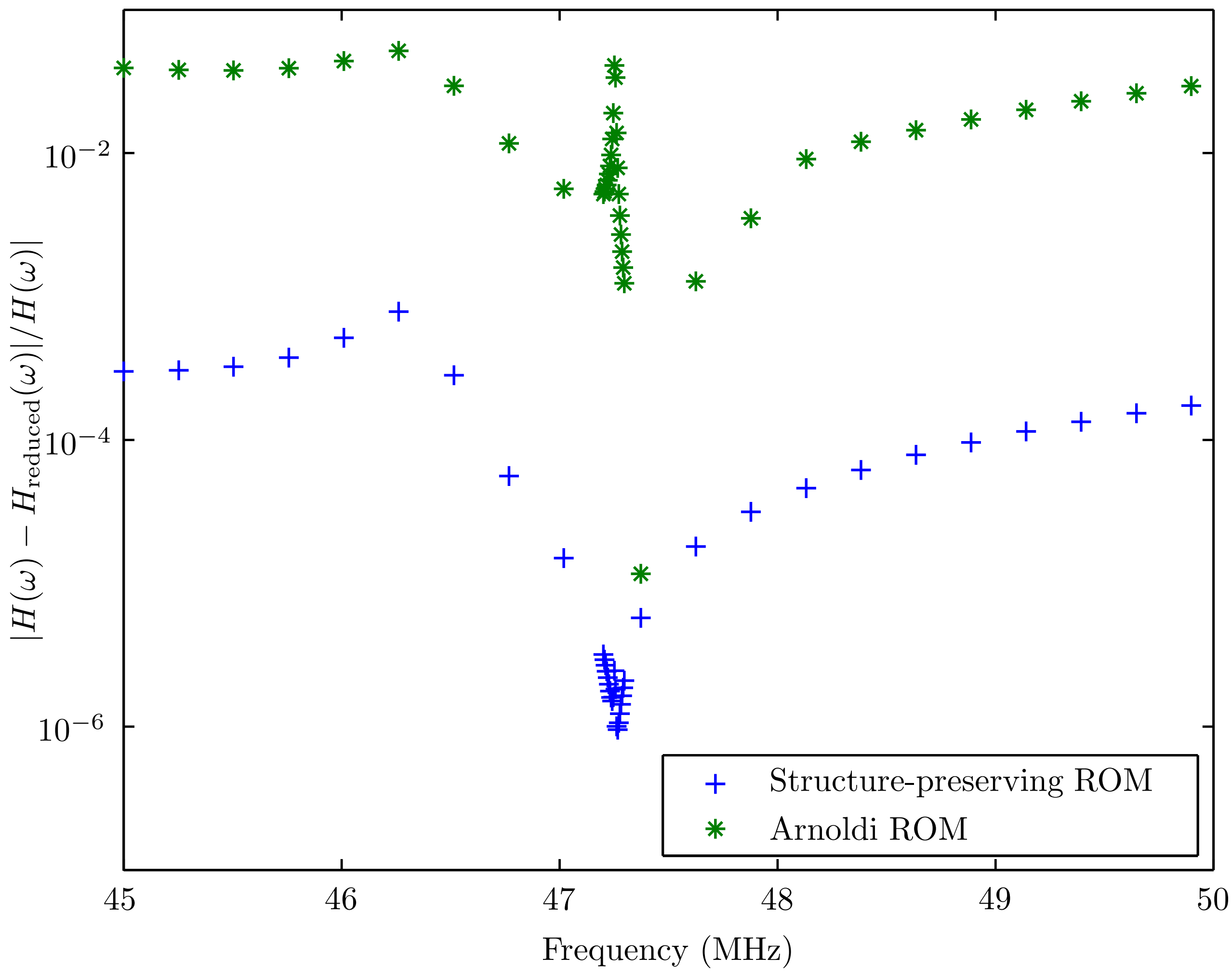

Another relation to the QEP

Linearize the real QEP $(\lambda^2 I + \lambda D + K)v = 0$:

$$\begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \begin{bmatrix} v & \bar{v} \\ \lambda v & \bar{\lambda} v \end{bmatrix} = \begin{bmatrix} v & \bar{v} \\ \lambda v & \bar{\lambda} v \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

Map $\mathbb{C}$ to $\mathbb{R}^{2\times 2}$ in the standard way and consider $C = A + iB$:

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} z & \bar{z} \\ -iz & -\bar{i}z \end{bmatrix} = \begin{bmatrix} z & \bar{z} \\ -iz & -\bar{i}z \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

Another relation to the QEP

- For both real form of complex symmetric eigenproblems and QEP, want to preserve structure under projection
  - Probably best to stay within original form
- For both complex symmetric eigenproblems and QEP, may want to split complex projection bases

Conclusions

- Complex symmetric systems occur in interesting places
  - Particularly in any damped resonant systems
  - Often tangled into nonlinear eigenproblems
- Can pay to exploit complex symmetry when it occurs

Further reading:

- Reduced order models in microsystems and RF MEMS (`www.cs/~dbindel/papers/para04.pdf`)
- Elastic PMLs for resonator anchor loss simulation (`www.cs/~dbindel/papers/pml-tr.pdf`)