Refining Approximate Invariant Subspaces

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Overview

- Problem and motivation
- Representing invariant subspaces
- The Riccati equation
- Eigensolvers as equation solvers
- Summary and concluding thoughts

Problem statement

- Input:
  - Matrix A
  - Approximate invariant subspace V_0
- Output:
  - Improved approximation V

Judging approximate subspaces

- Eigenpair residual equation is

$$R(v, \lambda) = Av - v\lambda = 0$$

- Invariant subspace residual equation is

$$R(V, L) = AV - VL = 0$$

Motivating examples

Numerically solve $x'(t) = f(x)$ at $t_1, t_2, \ldots$.
Interested in modes of linearized system

$$\begin{aligned} \frac{dx}{d\tau}(t_i + \tau) &\approx A_i(x(t_i + \tau) - x(t_i)) \\ A_i &:= \frac{\partial f}{\partial x}(x(t_i)) \end{aligned}$$

about a sequence of operating points. Useful for

- Local reduced models
- Superimposed vibrations (sound)
- Stabilization of integrators

Motivating examples

Following path of equilibrium solutions $\hat{x}(p)$ of parameterized ODE

$$x'(t) = f(x; p)$$

Interested in stability of the equilibrium at a sequence of p_i. Follow eigenvalues of

$$A_i = \frac{\partial f}{\partial x}(\hat{x}(p_i))$$

to see if any cross imaginary axis.

Motivating examples

- Approximating a PDE with multigrid
- Interested in a few modes
- $A_1, A_2, \ldots$ are increasingly fine discretizations.

Summary: Residual equations

Wrote two residual equations:

$$
\begin{aligned}
R(V, L) &= AV - VL \text{ where } V_0^* V = I \\
R(U) &= A_{12} + A_{22}U - UA_{11} - UA_{12}U
\end{aligned}
$$

Applied standard nonlinear solver techniques.
Residual functions also exist for generalized and polynomial problems.

Classifying eigensolvers

- Which residual? Unknown is (V, L) or U?
- What constraint isolates solutions?
- Symmetric or nonsymmetric?
- Standard, generalized, or nonlinear?
- Dense or sparse (projection based)?
- Choice of projection space(s)?
  - How to expand and contract?
  - Preconditioning?
- What nonlinear solver iteration?