

Week 12: Monday, Apr 11

Logistics

- Prelim 2 solutions are posted
- HW 6 is posted, due Monday, April 18 at 11:59 by CMS

Special behaviors

Consider the integral

$$I(z) = \int_{-z}^z \frac{\cos(x)}{\sqrt{|x|}} dx$$

How could we compute $I(\pi/2)$ numerically?

First, note that the function is even, so we can compute

$$I(z) = 2 \int_0^z x^{-1/2} \cos(x) dx.$$

This function is now awkward because the integrand diverges at $x = 0$. We can deal with this in a few different ways:

1. **Subtracting off the singularity:** Write

$$I(z) = 2 \left[\int_0^z x^{-1/2} dx + \int_0^z x^{-1/2} (\cos(x) - 1) dx \right].$$

The first term can be handled analytically:

$$\int_0^z x^{-1/2} dx = 2\sqrt{z}.$$

The second term has a removable singularity at the origin ($O(x^{3/2})$ as $x \rightarrow 0$), and we can treat the integrand as zero at that point.

2. **Integration by parts:** If we integrate by parts, we have

$$\int_0^z x^{-1/2} \cos(x) dx = 2\sqrt{z} \cos(z) + \int_0^z 2\sqrt{x} \sin(x) dx$$

3. **Change of variables:** If we let $t^2 = x$, then we can use the change of variables formula to recast the integral with as

$$I(z) = 2 \int_0^{z^2} t^{-1} \cos(t^2) (2t dt) = 4 \int_0^{z^2} \cos(t^2) dt$$

As it happens, we could at this point declare victory, since this integral is closely related to the Fresnel integral

$$C(x) = \int_0^x \cos(\pi t^2/2) dt,$$

and there are libraries that will compute Fresnel integrals for you.

4. **Special quadrature rules:** The *Gauss-Jacobi* family of quadrature rules approximates integrals of the form

$$\int_{-1}^1 f(x) (1-x)^\alpha (1+x)^\beta dx$$

If we set $\alpha = 0$, and $\beta = -1/2$, then this gives us a rule for computing something with an inverse-square-root singularity at $x = -1$. If we apply the change of variables

$$y = \frac{z}{2}(1+x),$$

we have

$$\int_{-1}^1 f(y(x)) (1+x)^{-1/2} dx = \left(\frac{2}{z}\right)^{1/2} \int_0^z y^{-1/2} f(y) dy,$$

so we have a quadrature rule that deals with integrals with this sort of singularity. Let $f(y) = \cos(y)$ and we're all set.

These are most of the tricks I know for dealing with integrals with singularities or unbounded domains. Fortunately, these tricks tend to work well for a variety of problems.

Ordinary differential equations

Consider ordinary differential equations of the form

$$y' = f(t, y)$$

together with the initial condition $y(0) = y_0$. These are equivalent to integral equations of the form

$$y(t) = y_0 + \int_0^t f(s, y(s)) ds.$$

Only the simplest sorts of differential equations can be solved in closed form, so our goal for the immediate future is to try to solve these ODEs numerically.

Euler's method can be written in terms of finite difference approximation of the derivative, Hermite interpolation, Taylor series, numerical quadrature using the left-hand rule, or the method of undetermined coefficients. For the moment, we will start with a derivation by Taylor series. If $y(t)$ is known and $y' = f(t, y)$, then

$$y(t+h) = y(t) + hf(t, y(t)) + O(h^2)$$

Now, drop the $O(h^2)$ term to get a recurrence for $y_k \approx y(t_k)$:

$$y_{k+1} = y_k + h_k f(t_k, y_k)$$

where $t_{k+1} = t_k + h_k$.

We can derive another method based on first-order Taylor expansion about the point $t+h$ rather than t :

$$y(t) = y(t+h) - hf(t+h, y(t+h)) + O(h^2).$$

If we drop the $O(h^2)$ term, we get the approximation $y(t_k) = y_k$ where

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}).$$

This is the backward Euler method, sometimes also called implicit Euler. Notice that in the backward Euler step, the unknown y_{k+1} appears on both sides of the equations, and in general we will need a nonlinear equation solver to take a step.

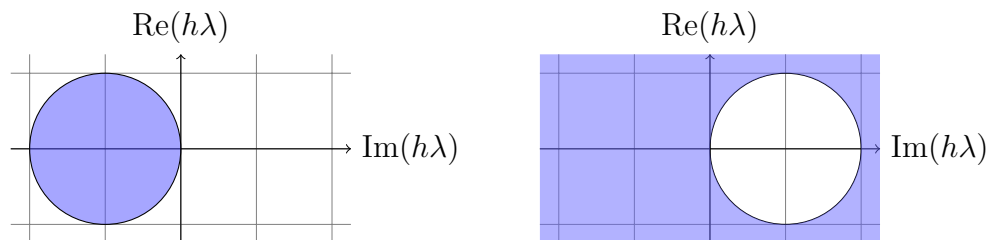


Figure 1: Regions of absolute stability for Euler (left) and backward Euler (right). For values of $h\lambda$ in the colored region, the numerical method produces decaying solutions to the test problem $y' = \lambda y$.

Stability regions

Consider what happens when we apply Euler and backward Euler to a simple linear test problem

$$y' = \lambda y$$

with a fixed step size h . Note that the solutions $y(t) = y_0 \exp(\lambda t)$ decay to zero whenever $\text{Re}(\lambda) < 0$. This is a qualitative property we might like our numerical methods to reproduce. Euler's method yields

$$y_{k+1} = (1 + h\lambda)y_k,$$

which gives decaying solutions only when $|1 + h\lambda| < 1$. The set of values $h\lambda$ where Euler produces a decaying solution is called the *region of absolute stability* for the method. This region is shown in Figure 1.

Backward Euler produces the iterates

$$y_{k+1} = (1 - h\lambda)^{-1}y_k$$

Therefore, the discrete solutions decay whenever $|(1 - h\lambda)^{-1}| < 1$ — or, equivalently, whenever $|1 - h\lambda| > 1$. Thus, the region of absolute stability includes the entire left half plane $\text{Re}(\lambda) < 0$ (see Figure 1), and so backward Euler produces a decaying solution when $\text{Re}(\lambda) < 0$, no matter how large or small h is. This property is known as *A-stability*.