Linear Algebra, Invariant Circles, and Fusion Plasmas

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Who?

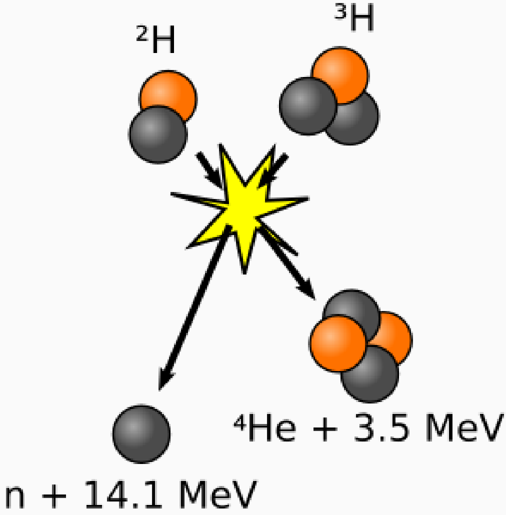
Simons Collaboration: “Hidden Symmetries and Fusion Energy”

`https://hiddensymmetries.princeton.edu/`

Princeton, NYU, Maryland, IPP Greifswald, Warwick, CU Boulder, UW Madison, EPFL, ANU, UT Austin, U Arizona.

- Phase 1: Sep 2017-Aug 2022
- Phase 2: Sep 2022-Aug 2025

D-T fusion



Lawson criterion

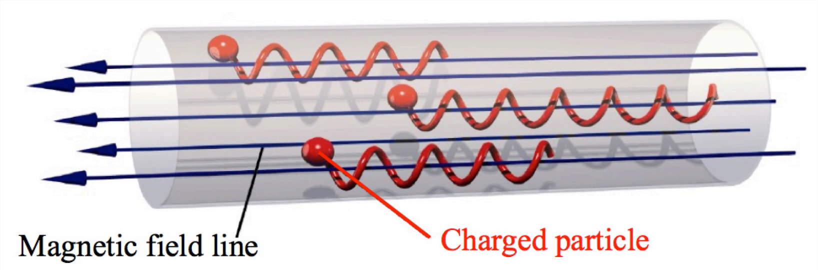
Figure of merit: $nT\tau_E$ where

- n is number density
- T is temperature
- τ_E is energy confinement time

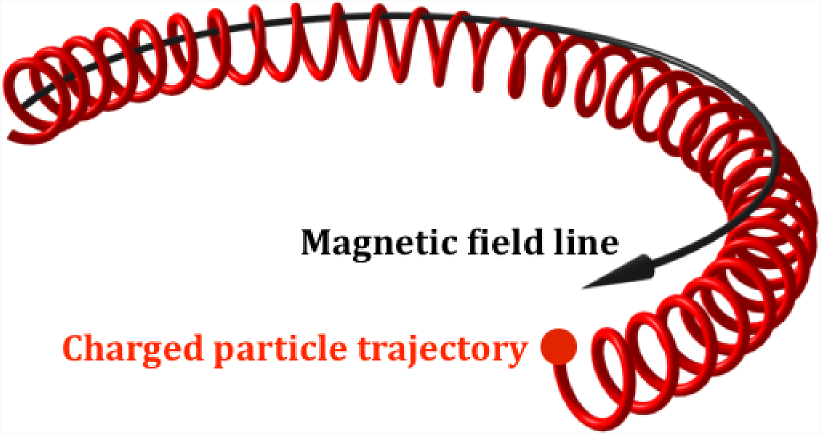
Min value required at $T = 14$ keV (about 162×10^6 K) is

$$nT\tau_E \geq 3.5 \times 10^{28} \text{ K s/m}^3$$

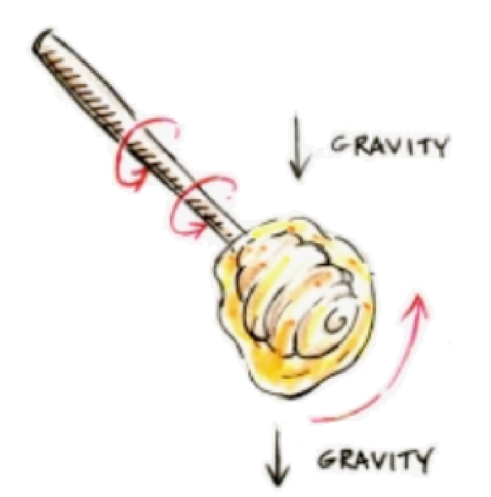
Magnetic confinement basics



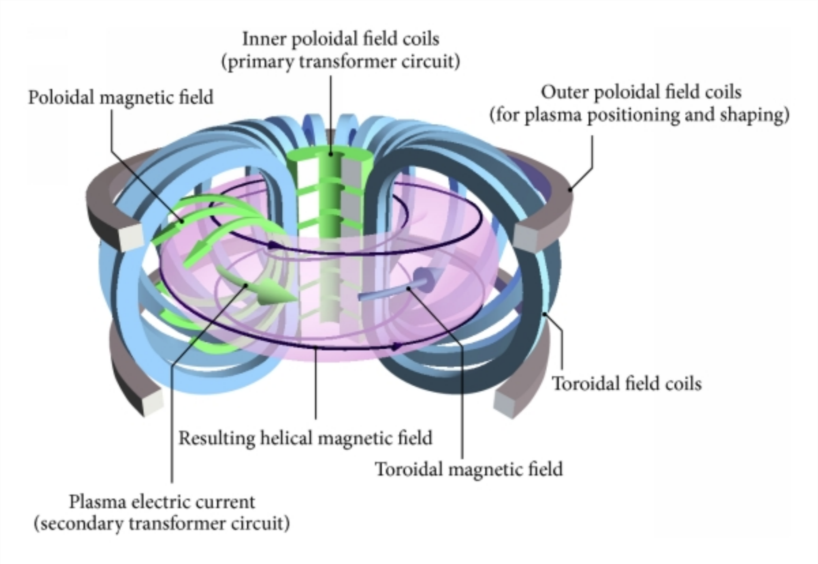
Magnetic confinement basics



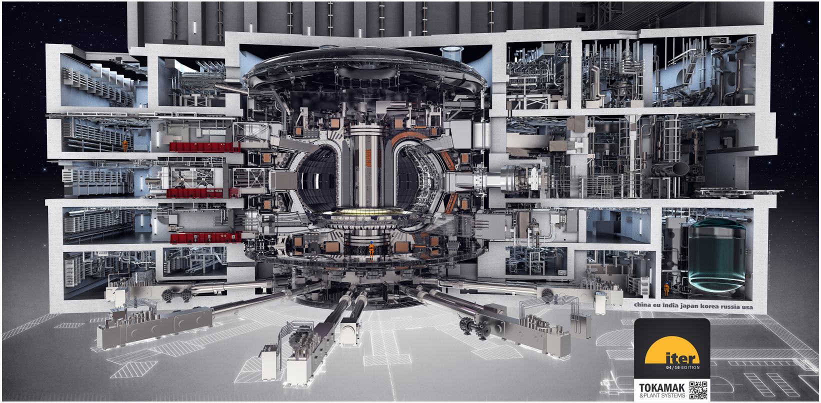
Magnetic confinement basics



The big name: Tokamaks



ITER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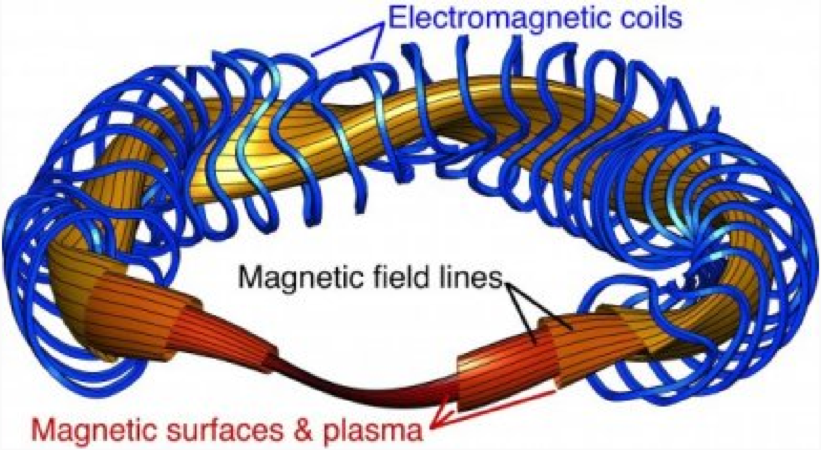
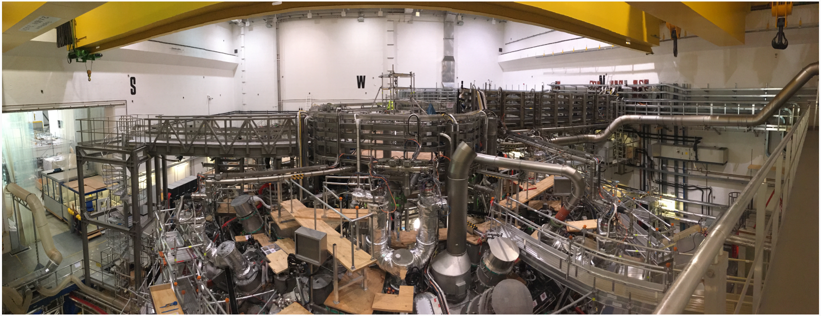
04/16 EDITION
TOKAMAK
&PLANT SYSTEMS

Stellarator Concept

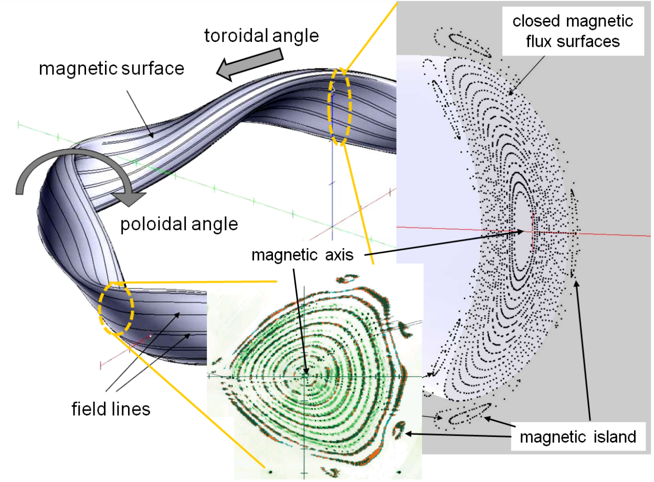


Wendelstein 7-X Machine



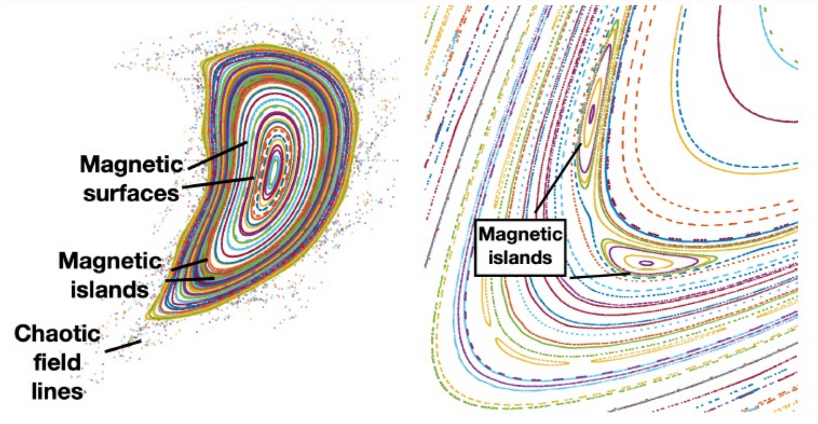
Operating since 2015-12-10;
plasma discharges lasting several min.

Wendelstein 7-X Poincaré Plots



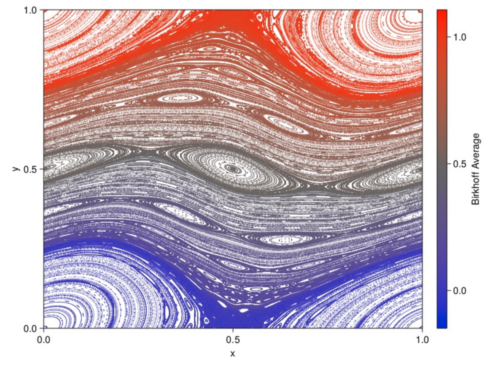
https://commons.wikimedia.org/wiki/File:Stellarator_magnetic_field.png

Poincaré Features (NCSX)



“An Introduction to Stellarators” (2020)
Imbert-Gerard, Paul, and Wright.
`https://arxiv.org/abs/1908.05360`

A Non-Stellarator Test Problem

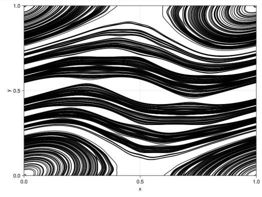


Illustrate with standard (Chirikov-Taylor) map

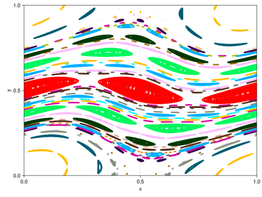
$$x_{t+1} = x_t + y_{t+1} \bmod 1$$
$$y_{t+1} = y_t - \frac{0.7}{2\pi} \sin(2\pi x_t)$$

Plan in Pictures

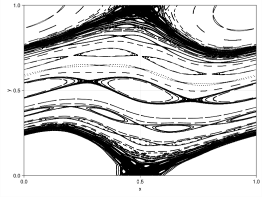
Circles



Islands



Chaos



- Iterating gives a Poincaré plot showing
  - X and O points (hyperbolic and elliptic periodic points)
  - Invariant circles and island chains (quasiperiodic orbits)
  - Chaos
- Goal: Identify these structures cheaply and automatically

Processing Poincaré Plots

1. Make a Poincaré plot and eyeball it
2. Parameterization method
3. Form a function with invariant level sets
    - Birkhoff averaging
    - Weighted Birkhoff averaging
    - Adaptive weighted Birkhoff (*)
    - Learned labels (*)
4. Model dynamics for a field line (*)

Parameterization method

Goal: $z : \mathbb{T} \to \mathbb{R}^2$ s.t.

$$F(z(\theta)) = z(\theta + \omega).$$

Discretize via Fourier:

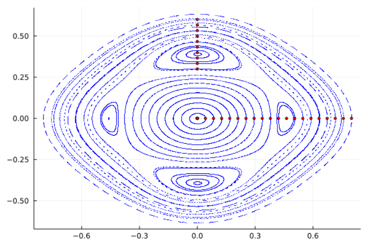
$$\hat{z}(\theta) = \sum_{n=-m}^{m} \hat{z}_n \exp(2\pi i n\theta)$$

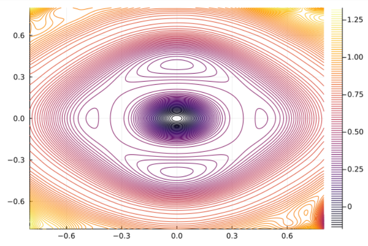
Solve nonlinear least squares problem

$$\min \sum_{i=0}^{N-1} \|z(i/N) - F(z(i/N + \omega))\|^2$$

with two additional constraints (phase + which circle).

Usually combine with continuation (e.g. from fixed point of F).





Goal: Find (non-constant) h s.t. $h \circ F = h$.

Discretize via favorite ansatz, e.g. $h = \sum_{j=1}^{m} c_j \phi(\|x - x_j\|)$.
Define $h(x_j) = y_j$ and $h(F(x_j)) = y_j'$, solve (for example)

$$\text{minimize } \frac{\eta}{2} y^T K^{-1} y + \frac{1}{2}\|y - \tilde{y}\|^2 \text{ s.t. } y_i = y_i'$$

to encourage h smooth, non-constant, invariant under F.

Birkhoff Average

Consider $f : \Omega \to \Omega$ symplectic, $h \in \mathcal{C}^\infty(\Omega)$

Define *Birkhoff average*:

$$\mathcal{B}_K[h](x) = \frac{1}{K+1} \sum_{k=0}^{K} (h \circ F^k)(x).$$

Birkhoff-Khinchin: for $h \in \mathcal{L}^1$, converges a.e. to conditional expectation of an invariant measure on an invariant set.

Error behavior $\mathcal{B}_K[h](x) - \bar{h}(x)$?

- Invariant circle/island? $O(K^{-1})$
- Chaos? $O(K^{-1/2})$

Rates signal regular vs chaotic ("stochastic") trajectories.

Birkhoff Average

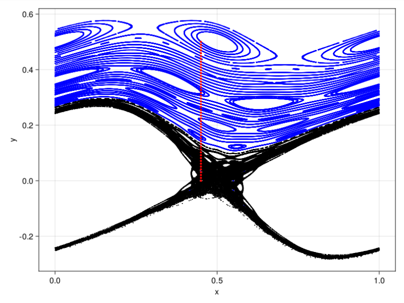
Ideas:

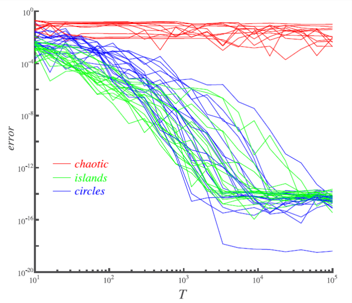
- Invariant sets as level sets of Birkhoff average
- Convergence rates as signal of regularity vs chaos

Converges in the long run – but in the long run, we are all dead. (with apologies to Keynes)

Related: Learn a continuous, nonconstant $\bar{h}$ s.t. $\bar{h} = \bar{h} \circ F$. Can do pretty well with kernel interpolation ansatz – a topic for another talk.

Weighted Birkhoff average





Sander and Meiss, Physica D, 411 (2020) p. 132569;
Das, Sander, and Yorke, Nonlinearity, 30 (2017), pp. 4111-4140

Weighting accelerates regular convergence to super-algebraic:

$$\mathcal{WB}_K[h](x) = \sum_{k=0}^{K} w_{k,K}(h \circ F^k)(x).$$

Signal Processing Perspective

Parameterize $z(\theta)$ for invariant circle

$$F(z(\theta)) = z(\theta + \omega), \quad z(\theta) = \sum_{n\in\mathbb{Z}} \hat{z}_n \exp(2\pi i n\theta)$$

Trajectory $z_t = z(\omega t)$ has series expansion

$$z_t = \sum_{n\in\mathbb{Z}} \hat{z}_n \xi^{nt}, \quad \xi = \exp(2\pi i\omega)$$

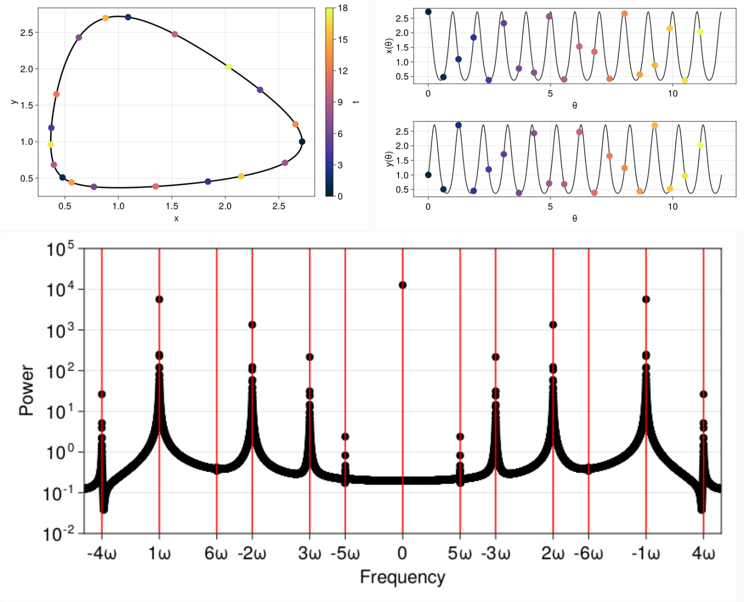
Observables $h_t = h(z_t)$ can be similarly expanded

$$h_t = \sum_{n\in\mathbb{Z}} \hat{h}_n \xi^{nt}, \quad \bar{h} = \hat{h}_0$$

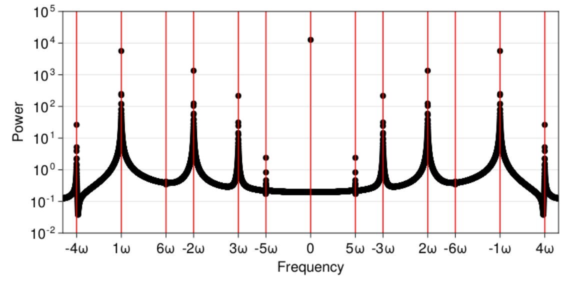
Weighted Birkhoff starting from x_0

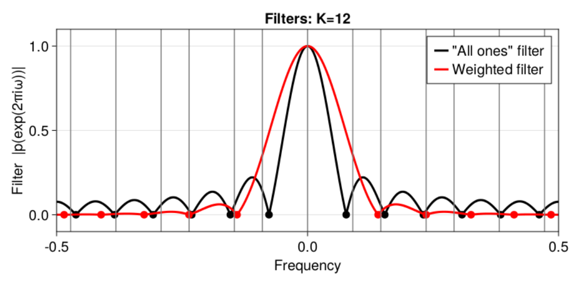
$$\mathcal{B}_K[h](x_0) = \sum_{n\in\mathbb{Z}} \hat{h}_n p_K(\xi^n), \quad p_K(z) = \sum_{k=0}^{K} w_{k,K} z^k$$

Signal Processing Perspective

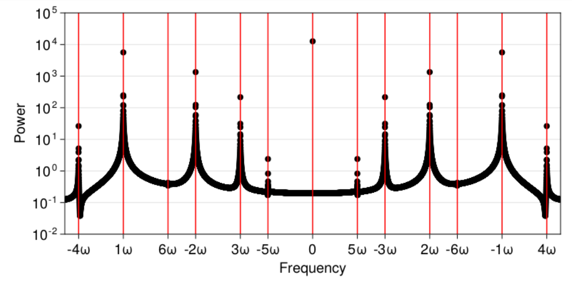


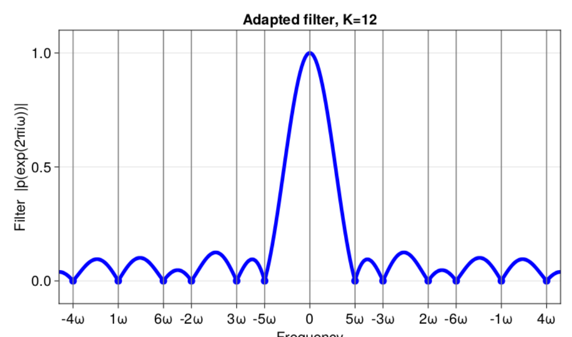
Signal Processing Perspective





Signal Processing Perspective: Adaptive Filtering





Adaptive Filtering

Series for $h_t = h(z_t)$

$$h_t = \sum_{n\in\mathbb{Z}} \hat{h}_n \xi^{nt}$$

Filtered/accelerated series with polynomial p_K:

$$\mathcal{AWB}_K[h](x_t) = \sum_{n\in\mathbb{Z}} \hat{h}_n \xi^{nt} p_K(\xi^n) \to \hat{h}_n$$

How do we adaptively choose the filter polynomial?

Desiderata for this to work:

- Fast enough decay of $\hat{h}_n$
- "Sufficiently irrational" ω (Diophantine condition)

(Vector) Reduced Rank Extrapolation

Assume

$$h_t = \hat{h}_0 + \sum_{n \neq 0} \lambda_n^t \quad (\text{e.g. } \lambda_n = \xi^n)$$

Difference sequence removes mean:

$$u_t = h_{t+1} - h_t = \sum_{n \neq 0} (\lambda_n - 1) \hat{h}_m \lambda_m^t$$

Seek coeffs c_k to minimize

$$\sum_{t=0}^{T-1} \left(\sum_{k=0}^{K} c_k u_{k+t} \right)^2 \text{ s.t. } \sum_{k=0}^{K} c_k = 1.$$

Accelerated series is

$$\tilde{h}_t = \sum_{k=0}^{K} c_k h_{k+t}.$$

Details

- Can (and do) use vector observables
- Rectangular Hankel matrix $\implies$ fast matvecs via FFT
- Solve least squares problem with LSQR
- Constrain for time reversibility $\implies$ palindromic polynomial:
  $$c_j = c_{K-j}$$
  Roots come in inverse pairs (generally on unit circle)
- Measure convergence adaptively via residual

(Vector) Reduced Rank Extrapolation

Standard vector RRE convergence (Sidi, *Vector Extrapolation Methods with Applications*): if $|\lambda_j|$ are in descending order, error for Kth extrapolated average goes like

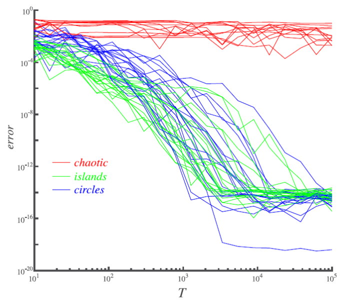
$$\hat{h}_{0,K} - \hat{h}_0 = O(\lambda_{K+1}^{2K}).$$

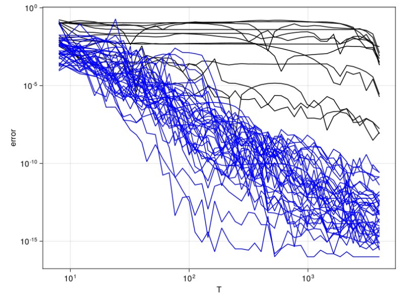
But for us everything is on the unit circle!

Alternate analysis gives super-algebraic convergence given

- Enough smoothness of circle (decay of $|\hat{h}_n|$ with $|n|$)
- "Sufficient irrationality" (Diophantine condition) so ξ_n doesn't get too close to 1 too fast.

Weighted Birkhoff vs RRE

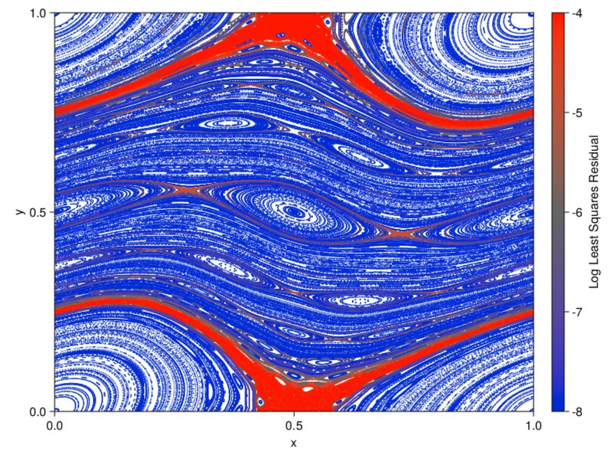




Still good for classification.convergence slightly faster than weighted Birkhoff.

Residuals and Regularity

Use least squares residual to judge "circleness."



(Hard cases near rational rotational transform)

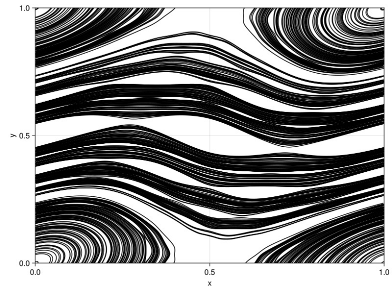
Post-Processing (Filter Diagonalization)

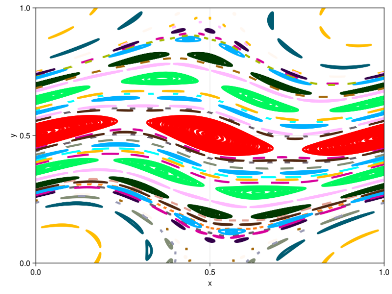
Why use the RRE model just for averaging?

1. Form filter polynomial with coefficients c
2. Find natural frequencies / polynomial roots
3. Sort by contribution to signal
4. Of 10 most contributing frequencies, identify rationals (Sander & Meiss)
    - Yes: island chain — RRE on qth step
    - No: call largest the rotational transform
5. Project signal onto Fourier modes

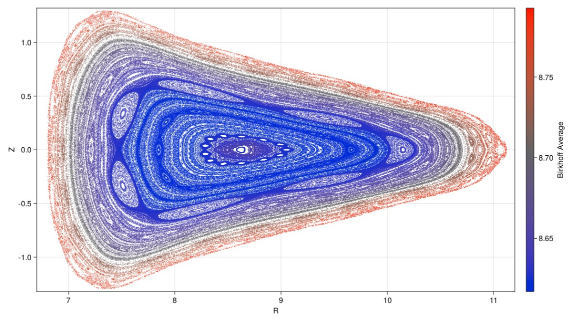
Get shape and characteristics of circles and islands.

Island Identification





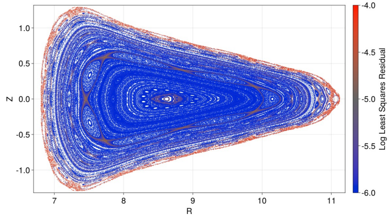
Wistell Stellarator Configuration



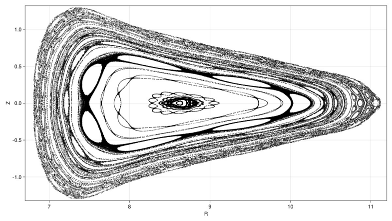
- 1000 random trajectories (via RK4 on interpolated B field)
- $K_{\text{max}} = 300, T_{\text{max}} = 900$
- Residual tolerance = 10^{-6}
- Rational tolerance = 10^{-6}

Wistell Analysis

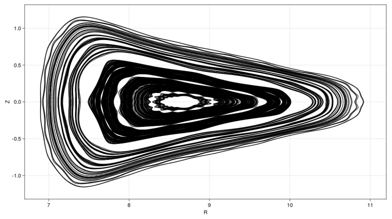
Residual



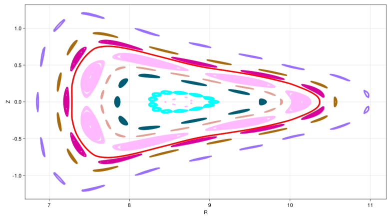
Chaos



Circles



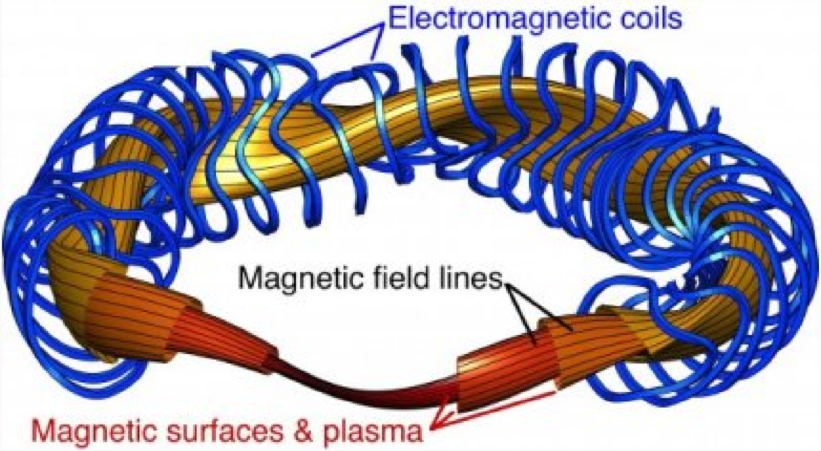
Islands



Concluding Notes

- Extrapolation pros
  - Classifies chaos vs regular trajectories
  - Recovers invariant circles/islands
  - No need for continuation or initial guesses
  - Parallelizable over trajectories
- Cons
  - Problems near low-order rationals
  - Linear algebra adds extra cost vs weighted Birkhoff
- Higher dimensions?
  - Relevant beyond field line flow (guiding center approx)
  - Invariant sets are more complicated
  - The "model the trajectory" philosophy should still work

Fin



https://github.com/maxeruth/SymplecticMapTools.jl
https://hiddensymmetries.princeton.edu/