LOCALIZATION THEOREMS FOR NONLINEAR EIGENVALUE PROBLEMS*

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Abstract. Let $T : \Omega \to \mathbb{C}^{n\times n}$ be a matrix-valued function that is analytic on some simply-connected domain $\Omega \subset \mathbb{C}$. A point $\lambda \in \Omega$ is an eigenvalue if the matrix $T(\lambda)$ is singular. In this paper, we describe new localization results for nonlinear eigenvalue problems that generalize Gershgorin's theorem, pseudospectral inclusion theorems, and the Bauer-Fike theorem. We use our results to analyze three nonlinear eigenvalue problems: an example from delay differential equations, a problem due to Hadeler, and a quantum resonance computation.

Key words. nonlinear eigenvalue problems, pseudospectra, Gershgorin's theorem, perturbation theory

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1. Introduction.

In this paper, we study the nonlinear eigenvalue problem of finding a complex number λ and nonzero vector v such that

$$T(\lambda)v = 0, \tag{1.1}$$

where $T : \Omega \to \mathbb{C}^{n\times n}$ is analytic on a simply-connected domain $\Omega \subset \mathbb{C}$. One way in which (1.1) arises is in finding special solutions $u(t) = v\exp(\lambda t)$ of homogeneous differential equations and functional-differential equations. Common examples include

	Time-domain equation	$T(z)$
First-order system	$\dot{u} - Au = 0$	$zI - A$
Second-order system	$M\ddot{u} + C\dot{u} + Ku = 0$	$z^2M + zC + K$
Viscoelasticity	$M\ddot{u} + Ku + \int_{-\infty}^{t} \phi(t-\tau)\dot{u}(\tau)\,d\tau = 0$	$z^2M + K + z\Phi(z)$
Delay equations	$\dot{u}(t) - Au(t) - Bu(t-1) = 0$	$z - A - Be^{-z}$

where the function $\Phi(z) = \int_0^\infty \phi(t)e^{-zt}dt$ appearing in the viscoelastic example is the Laplace transform of the history kernel ϕ. The best-studied nonlinear eigenvalue problems are those for which T is polynomial in z [12, 19, 27], and particularly those that are quadratic in z [45, 51], as occurs in the analysis of the second-order ODE system above. In models of hysteretic phenomena such as viscoelasticity, the history kernel ϕ often involves sums of exponentially decaying terms; this results in rational eigenvalues problems [1, 39]. More general nonlinear eigenvalue problems with algebraic or transcendental dependence on z are prevalent in models with delay [42] or radiation [28, 48, 57, 58]. Nonlinear eigenvalue problems also arise in diverse applications beyond dynamics: in solving PDE eigenvalue problems by the method of particular solutions [9] or boundary element methods [46], in camera pose estimation problems from computer vision [33], in finding intersections of curves and surfaces in computational geometry [36, 37], and in many other situations [8, 39].

A standard approach to the analysis and numerical solution of polynomial and rational eigenvalue problems is *linearization*: that is, the conversion of the original nonlinear eigenvalue problem to a linear eigenvalue problem in a higher-dimensional

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state space. For example, the quadratic eigenvalue problem is frequently solved via the companion linearization

$$\hat{T}(z) = z \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix}. \tag{1.2}$$

In fact, we can often relate even a non-rational nonlinear eigenvalue problem to an ordinary eigenvalue problem involving a higher-dimensional operator; for instance, boundary integral equation formulations of PDE eigenvalue problems trade nonlinear dependence on a frequency parameter in return for a problem posed on a lower-dimensional set, while delay differential equations can be posed as abstract ordinary differential equations on an infinite-dimensional state space [42]. Moving in the opposite direction, we may also write eigenvalues of (1.1) as zeros of the scalar analytic function $\det T(z)$. Thus, nonlinear eigenvalue problems can provide intermediate formulations between large linear eigenvalue problems and zero-finding for scalar analytic functions, and the analysis methods employed for these problems draw heavily on both linear algebra and analytic function theory.

In this paper, we consider *localization results* that define regions in which any eigenvalues must lie. Our results represent a generalization of two types of classic results: localization theorems for scalar analytic functions, such as Rouché's theorem, and theorems involving localization regions such as pseudospectra [54] and Gershgorin disks [55] that are widely used in the analysis of ordinary eigenvalue problems. Such results are broadly useful in several settings:

- *Stability analysis.* To test stability of equilibrium solutions of differential and difference equations, it suffices to localize eigenvalues to the left half plane or to the unit circle. Gershgorin's theorem plays a particularly significant role in the design and analysis of decentralized controllers for multivariable feedback systems [44, Section 10.8]
- *Choosing parameters for numerical methods.* For symmetric eigenvalue solvers based on bisection, Gershgorin bounds establish upper and lower bounds on the spectrum for the start of the iteration, and for large linear eigenvalue solvers based on Krylov subspaces, Gershgorin's theorem is sometimes used to select shifts for spectral transformations [38]. Knowing roughly where to find eigenvalues is also critical for methods that generalize linear eigenvalue solvers to nonlinear problems [31], and for initializing Newton-based nonlinear eigenvalue solvers such as [32]. And for nonlinear eigenvalue solvers based on contour integrals, such as [4, 10], it is important to have prior information about eigenvalue locations so as to choose a contour that is not too close to any eigenvalues.
- *Bounding error in numerical methods.* Localization results serve to bound the error in eigenvalue computations, whether due to roundoff, termination of iterative methods, or error inherited from uncertain inputs. As we show in Section 6, localization results can be equally useful in the nonlinear setting for analyzing the error in approximating a nonlinear eigenvalue problem by a linear one.

But though localization is as useful for nonlinear eigenvalue problems as for linear problems, little has been done to adapt localization results for linear eigenvalue problems to the general nonlinear case.

Apart from some work in the context of delay differential equations [29], we are not aware of any efforts to extend localization results that work directly with

the matrix, such as Gershgorin's theorem or the Bauer-Fike theorem, to the general nonlinear case. However, related work has been done for certain instances of (1.1). For polynomial eigenvalue problems in particular, several researchers have explored perturbation theory [2, 13, 14, 17, 25, 49] and localization theorems that generalize results for scalar polynomials (e.g. Pellet's theorem) [12, 40], though these results are of limited use outside the polynomial case. Similarly, research into pseudospectra for nonlinear problems [16, 22, 26, 41, 42, 50, 56] has primarily focused on specific types of eigenvalue problems, such as polynomial problems or problems arising from delay differential equations.

The rest of this paper is organized as follows. In Section 2, we recall a useful result from the theory of analytic matrix-valued functions and some background on subharmonic functions. In Section 3, we describe a generalized Gershgorin theorem for nonlinear eigenvalue problems, and in Section 4, we introduce and discuss a nonlinear generalization of pseudospectra. We then turn to the useful special case of linear functions with nonlinear perturbations in Section 5, where we describe analogues of Gershgorin's theorem and the Bauer-Fike theorem for this case. We illustrate the usefulness of our bounds through some examples in Section 6, and conclude in Section 7.

2. Preliminaries. We assume throughout this paper that $\Omega \subset \mathbb{C}$ is a simply-connected domain and $T : \Omega \to \mathbb{C}^{n\times n}$ is analytic and *regular*, i.e. $\det(T(z)) \not\equiv 0$. For T regular, the zeros of $\det(T(z))$ are a discrete set with no accumulation points in Ω. We call $\lambda \in \Omega$ an eigenvalue with multiplicity m if $\det(T(z))$ has a zero of order m at λ. The set of all eigenvalues of the matrix-valued function T is the spectrum $\Lambda(T)$. Note that, for simplicity, we have deliberately restricted our attention to finite eigenvalues. As with standard eigenvalue problems, when we count eigenvalues in a region, we always count multiplicity. If $\Gamma \subset \mathbb{C}$ is a simple closed contour and $T(z)$ is nonsingular for all $z \in \Gamma$, the number of eigenvalues inside Γ is given by the winding number

$$W_\Gamma(\det T(z)) = \frac{1}{2\pi i}\int_\Gamma \left[\frac{d}{dz}\log\det(T(z))\right] dz = \frac{1}{2\pi i}\int_\Gamma \operatorname{tr}\left(T(z)^{-1}T'(z)\right) dz.$$

The following counting argument based on the winding number underpins most of the results in this paper.

LEMMA 2.1. *Suppose $T : \Omega \to \mathbb{C}^{n\times n}$ and $E : \Omega \to \mathbb{C}^{n\times n}$ are analytic, and that $\Gamma \subset \Omega$ is a simple closed contour. If $T(z) + sE(z)$ is nonsingular for all $s \in [0, 1]$ and all $z \in \Gamma$, then T and $T + E$ have the same number of eigenvalues inside Γ, counting multiplicity.*

Proof. Define $f(z; s) = \det(T(z) + sE(z))$. The winding number of $f(z; s)$ around Γ is the number of eigenvalues of $T + sE$ inside Γ. For $z \in \Gamma$ and $s \in [0, 1]$, by hypothesis, $T(z) + sE(z)$ is nonsingular, and so $f(z; s) \neq 0$. Hence, the winding number is continuously defined (and thus constant) for $s \in [0, 1]$. ∎

REMARK 1. Lemma 2.1 is almost a special case of an operator generalization of Rouché's theorem due to Gohberg and Sigal [20]. However, where Gohberg and Sigal ensured nonsingularity of $T(z) + sE(z)$ for $z \in \Gamma$ by requiring $\|T(z)^{-1}E(z)\| < 1$ for some operator norm, in this paper we consider other tests of nonsingularity.

In Theorem 3.1 and Proposition 4.2, we also make use of the theory of subharmonic functions; see [43, Ch. 17]. Recall that an upper semicontinuous function

$\phi : \Omega \to \mathbb{R}$ is subharmonic at z if for any small enough $r > 0$,

$$\phi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z + re^{i\theta}) \, d\theta.$$

It immediately follows that subharmonic functions obey a maximum principle: if ϕ is subharmonic on a compact set, the maximum occurs on the boundary. If f is holomorphic at z, then $|f|$ and $\log |f|$ are subharmonic at z; if ϕ and ψ are subharmonic, then so are $\phi + \psi$ and $\max(\phi, \psi)$; and if ϕ_j is a sequence of subharmonic functions that converges uniformly to a limit ϕ, then ϕ is also subharmonic. We can write any vector norm as $\|v(z)\| = \max_{l^* \in \mathcal{B}^*} |l^* v(z)|$ where $\mathcal{B}^*$ is an appropriate unit ball in the dual space; hence, if v is a vector-valued holomorphic function, then $\|v\|$ and $\log \|v\| = \max_{l^* \in \mathcal{B}^*} \log |l^* v|$ are also subharmonic.

3. Gershgorin bounds for nonlinear problems. Lemma 2.1 provides a template for constructing inclusion regions to compare the spectra of two related problems. The following is a nonlinear generalization of Gershgorin's theorem that allows us to compare the spectrum of a general matrix-valued function to the zeros of a list of scalar-valued functions.

THEOREM 3.1. *Suppose $T(z) = D(z) + E(z)$ where $D, E : \Omega \to \mathbb{C}^{n \times n}$ are analytic and D is diagonal. Then for any $0 \leq \alpha \leq 1$,*

$$\Lambda(T) \subset \bigcup_{j=1}^{n} G_j^\alpha,$$

where G_j^α is the jth generalized Gershgorin region

$$G_j^\alpha = \{z \in \Omega : |d_{jj}(z)| \leq r_j(z)^\alpha c_j(z)^{1-\alpha}\}$$

and r_j and c_j are the jth absolute row and column sums of E, i.e.

$$r_j(z) = \sum_{k=1}^{n} |e_{jk}(z)|, \quad c_j(z) = \sum_{i=1}^{n} |e_{ij}(z)|.$$

Moreover, suppose that $\mathcal{U}$ is a bounded connected component of the union $\bigcup_j G_j^\alpha$ such that $\overline{\mathcal{U}} \subset \Omega$. Then $\mathcal{U}$ contains the same number of eigenvalues of T and D; and if $\mathcal{U}$ includes m connected components of the Gershgorin regions, it must contain at least m eigenvalues.

Proof. If $z \in \Omega$ is not in $\bigcup_j G_j^\alpha$, then, for each j,

$$|d_{jj}| > r_j^\alpha c_j^{1-\alpha} = (\hat{r}_j + |e_{jj}|)^\alpha \, (\hat{c}_j + |e_{jj}|)^{1-\alpha},$$

where

$$\hat{r}_j = \sum_{k \neq j} |e_{jk}|, \quad \hat{c}_j = \sum_{i \neq j} |e_{ij}|$$

are the deleted absolute row and column sums of E. Applying Hölder's inequality with $p = 1/\alpha$ and $q = 1/(1-\alpha)$, we have

$$|d_{jj}| \quad > \quad (\hat{r}_j^{\alpha p} + |e_{jj}|^{\alpha p})^{1/p} \, (\hat{c}_j^{(1-\alpha)q} + |e_{jj}|^{(1-\alpha)q})^{1/q} \quad \geq \quad \hat{r}_j^\alpha \hat{c}_j^{1-\alpha} + |e_{jj}|,$$

and by the triangle inequality,

$$|d_{jj} + e_{jj}| \geq |d_{jj}| - |e_{jj}| > \hat{r}_j^{\alpha} \hat{c}_j^{1-\alpha}.$$

Therefore, for each j,

$$|t_{jj}| > \left(\sum_{k \neq j} |t_{jk}| \right)^{\alpha} \left(\sum_{i \neq j} |t_{ij}| \right)^{1-\alpha},$$

and so by a nonsingularity test of Ostrowski [55, Theorem 1.16], $T(z)$ is nonsingular. The same argument shows that $D(z) + sE(z)$ is nonsingular for any $0 \leq s \leq 1$.

Because $D+sE$ is nonsingular outside the Gershgorin regions, Lemma 2.1 implies that any closed contour in Ω that does not pass through $\bigcup_j G_j^{\alpha}$ contains the same number of eigenvalues from $\Lambda(T)$ and $\Lambda(D)$, counting multiplicity. Thus, if $\mathcal{U}$ is a bounded connected component of $\bigcup_j G_j^{\alpha}$, $\bar{\mathcal{U}} \subset \Omega$, then D and T must have the same number of eigenvalues inside $\mathcal{U}$.

To establish the final counting result, we now show that d_{jj} has at least one zero in each bounded connected component of G_j^{α} whose closure is in Ω. Define vector-valued functions v and w by $v_k = e_{jk}/d_{jj}$ and $w_k = e_{kj}/d_{jj}$, and note that

$$G_j^{\alpha} = \{z \in \Omega : \phi(z) \geq 0\}, \quad \phi(z) \equiv \alpha \log \|v(z)\|_1 + (1-\alpha) \log \|w(z)\|_1.$$

Let $K \subset \Omega$ be the closure of a connected component of G_j^{α}, and define

$$K_{\epsilon} = \bigcup_{z \in K} \{z' \in \mathbb{C} : |z - z'| \leq \epsilon\}.$$

For small enough ϵ, we know that K_{ϵ} lies within Ω and does not intersect any other connected components, so the maximum value of $\phi(z)$ on K_{ϵ} does not occur on the boundary. Therefore, $\phi(z)$ cannot be subharmonic on K_{ϵ}; but it would be subharmonic on K_{ϵ} if d_{jj} had no zeros inside K_{ϵ}. Thus, there must be at least one zero of d_{jj} inside K_{ϵ}, and hence in $K = \bigcap_{\epsilon} K_{\epsilon}$. □

The usual statement of Gershgorin's theorem corresponds to the special case when $T(z) = A - zI = (D - zI) + E$, where D is the diagonal part of A and α is set to zero or one. Then the Gershgorin regions are simply disks, and a component consisting of m disks contains m eigenvalues. However, Theorem 3.1 involves some hypotheses that are not needed for the standard version of Gershgorin's theorem. We illustrate the role of these hypotheses through three examples.

Example 1. For the matrix

$$T(z) = \begin{bmatrix} 1 & z \\ 0 & z \end{bmatrix},$$

we have Gershgorin regions $G_1^1 = \{0\}$ and $G_2^1 = \{z : |z| \geq 1\}$. The first region contains the sole eigenvalue for the problem. The fact that the second region contains no eigenvalues does not violate the counting result in Theorem 3.1, since the second region is unbounded.

Example 2. Consider the matrix

$$T(z) = \begin{bmatrix} z & 1 & 0 \\ 0 & z^2 - 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

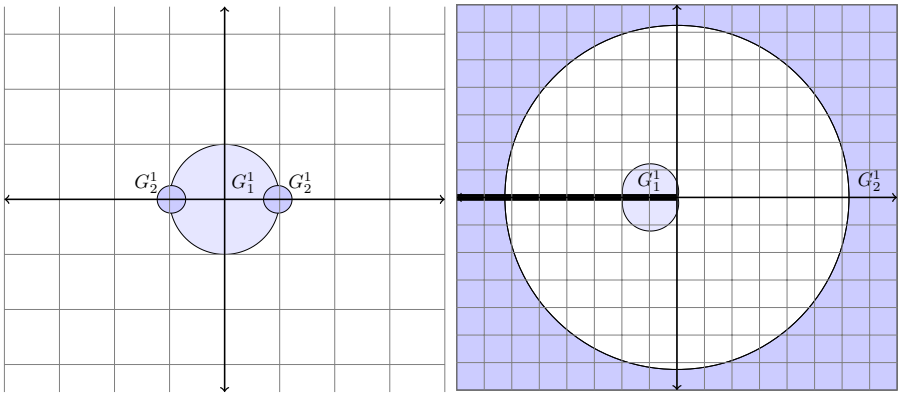

FIG. 3.1. *Gershgorin regions in Example 2 (left) and Example 3 (right). In Example 2, the second Gershgorin region consists of two pieces, and the union of G_1^1 and G_1^2 contains three eigenvalues. In Example 3, neither Gershgorin region contains eigenvalues; but both $\bar{G_1^1}$ and $\bar{G_2^2}$ intersect $(-\infty, 0]$, which is not in the domain Ω for this problem.*

The Gershgorin regions are shown in Figure 3.1 (left). For this problem, G_1^1 is the closed unit disk, G_2^1 consists of two roughly circular components around ± 1, and G_3^1 is empty. The region $\mathcal{U} = G_1^1 \cup G_2^1$ intersects two Gershgorin regions, and contains $3 > 2$ eigenvalues. Unlike Gershgorin disks in the standard problem, each bounded Gershgorin region may contain one or many eigenvalues.

EXAMPLE 3. Consider the matrix

$$T(z) = \begin{bmatrix} z - 0.2\sqrt{z} + 1 & -1 \\ 0.4\sqrt{z} & 1 \end{bmatrix}$$

defined on $\Omega = \mathbb{C} - (-\infty, 0]$, where $\sqrt{z}$ is taken to be the principal branch of the square root function. If we let $D(z)$ be the diagonal of $T(z)$, the Gershgorin regions are as shown in Figure 3.1 (right). Note that

$$\det(D(z)) = z - 0.2\sqrt{z} + 1 = (\sqrt{z} - 0.1 - i\sqrt{0.99})(\sqrt{z} - 0.1 + i\sqrt{0.99})$$

has two solutions on the primary sheet of the square root function, but

$$\det(T(z)) = z + 0.2\sqrt{z} + 1 = (\sqrt{z} + 0.1 - i\sqrt{0.99})(\sqrt{z} + 0.1 + i\sqrt{0.99})$$

only has solutions on the second sheet of definition. Thus, the set G_1^1 contains two eigenvalues of $D(z)$, but no eigenvalues of $T(z)$. This does not violate Theorem 3.1, because the closed set $\bar{G_1^1}$ includes $[-1, 0] \not\subset \Omega$.

4. Pseudospectral regions. The spectrum of a matrix A is the complement of the resolvent set, i.e., the set of z such that the resolvent operator $R(z) = (zI - A)^{-1}$ is well-defined. The ϵ-*pseudospectrum* of A is equivalently defined as

$$\Lambda_\epsilon(A) \equiv \{z : \|R(z)\| > \epsilon^{-1}\} \tag{4.1}$$

$$\equiv \bigcup_{\|E\|<\epsilon} \Lambda(A + E), \tag{4.2}$$

with the convention that $\|R(\lambda)\| = \infty$ when $\lambda \in \Lambda(A)$.

Several authors have worked on nonlinear generalizations of pseudospectra [16, 22, 26, 41, 42, 50, 56]. The usual definitions of pseudospectra for nonlinear problems generalize (4.2). Let $\mathcal{F}$ be a space consisting of some set of analytic matrix-valued functions of interest; then the ϵ-pseudospectrum for $T \in \mathcal{F}$ is

$$\Lambda_\epsilon(T) = \bigcup_{E\in\mathcal{F}, \|E\|_{\text{glob}}<\epsilon} \Lambda(T+E). \tag{4.3}$$

where $\|E\|_{\text{glob}}$ is a global measure of the size of the perturbing function E. For polynomial eigenvalue problems and nonlinear eigenvalue problems from the analysis of delay differential equations, many authors use the definition (4.3) with

$$\mathcal{F} \equiv \left\{ \sum_{i=0}^{m} A_i p_i(\lambda) : A_i \in \mathbb{C}^{n\times n} \right\}, \quad \|\cdot\|_{\text{glob}} = \text{function of } A_0, A_1, \ldots, A_m \tag{4.4}$$

where the functions $p_i(\lambda)$ are fixed entire functions [42, Chapter 2]. However, we wish to use our results to compare nonlinear eigenvalue problems with different types of dependencies on λ; for example, we want to compare problems with transcendental dependence on λ to approximations that have polynomial or rational dependence on λ. For this purpose, there may not be a natural formulation in terms of a standard set of coefficient functions.

We take $\mathcal{F}$ to be the space of all analytic matrix-valued functions $C^\omega(\Omega, \mathbb{C}^{n\times n})$, and measure size with

$$\|E\|_{\text{glob}} \equiv \sup_{z\in\Omega} \|E(z)\|. \tag{4.5}$$

Using the general definition (4.3) with the size measure (4.5), we have three equivalent expressions for the pseudospectra, similar to the equivalent definitions for ordinary pseudospectra; see [54, Theorem 2.1].

PROPOSITION 4.1. *Let* $\mathcal{E} = \{E : \Omega \to \mathbb{C}^{n\times n}$ *s.t.* E *analytic,* $\sup_{z\in\Omega} \|E(z)\| < \epsilon\}$ *and* $\mathcal{E}_0 = \{E_0 \in \mathbb{C}^{n\times n} : \|E_0\| < \epsilon\}$. *Then the following definitions are equivalent:*

$$\Lambda_\epsilon(T) = \{z \in \Omega : \|T(z)^{-1}\| > \epsilon^{-1}\} \tag{4.6}$$

$$= \bigcup_{E\in\mathcal{E}} \Lambda(T+E) \tag{4.7}$$

$$= \bigcup_{E_0\in\mathcal{E}_0} \Lambda(T+E_0). \tag{4.8}$$

Proof. Denote the sets in (4.6), (4.7), and (4.8) as $\Lambda_\epsilon^1(T)$, $\Lambda_\epsilon^2(T)$, and $\Lambda_\epsilon^3(T)$. We break the proof into three steps:

$z \in \Lambda_\epsilon^2(T) \iff z \in \Lambda_\epsilon^3(T)$: If $T(z) + E(z)$ is singular for some $E \in \mathcal{E}$, then $T(z) + E_0$ is singular for $E_0 = E(z)$. Since $E_0 \in \mathcal{E}_0$, it follows that $z \in \Lambda_\epsilon^3(T)$. Conversely, if $T(z) + E_0$ is singular for some $E_0 \in \mathcal{E}_0$, then $T(z) + E(z)$ is singular for E the constant function E_0.

$z \notin \Lambda_\epsilon^1(T) \implies z \notin \Lambda_\epsilon^3(T)$: Suppose $\|T(z)^{-1}\| \le \epsilon^{-1}$. Then for any E_0 such that $\|E_0\| < \epsilon$, we have that $\|T(z)^{-1}E_0\| < 1$, so there is a convergent Neumann series for $I + T(z)^{-1}E_0$. Thus, $(T(z) + E_0)^{-1} = (I + T(z)^{-1}E_0)^{-1}T(z)^{-1}$ is well defined.

$z \in \Lambda_\epsilon^1(T) \implies z \in \Lambda_\epsilon^3(T)$: Eigenvalues of T belong to both sets, so we need only consider $z \in \Lambda_\epsilon^1(T)$ not an eigenvalue. So suppose $T(z)$ is invertible and

$s^{-1} = \|T(z)^{-1}\| > \epsilon^{-1}$. Then $T(z)^{-1}u = s^{-1}v$ for some vectors u and v with unit norm; alternately, write $su = T(z)v$. Let $E_0 = -suw^*$, where w^* is a dual vector of v. Then $\|E_0\| = s < \epsilon$, and $T(z) + E$ is singular with v as a null vector. □

The ϵ-pseudospectra clearly contains the ordinary spectrum, but we can say more. The following result is nearly identical to the analogous statement for ordinary pseudospectra [54, Theorem 4.2]:

PROPOSITION 4.2. *Suppose $T : \Omega \to \mathbb{C}^{n\times n}$ is analytic and $\mathcal{U}$ is a bounded connected component of $\Lambda_\epsilon(T)$ with $\bar{\mathcal{U}} \subset \Omega$. Then $\mathcal{U}$ contains an eigenvalue of T.*

Proof. If $T(z)^{-1}$ is analytic on $\bar{\mathcal{U}}$, then $\|T(z)^{-1}\|$ is subharmonic on $\bar{\mathcal{U}}$. Therefore, the maximum of $\|T(z)^{-1}\|$ must be attained on the boundary. But $\|T(z)^{-1}\| = \epsilon^{-1}$ for $z \in \partial\mathcal{U}$, and $\|T(z)^{-1}\| > \epsilon^{-1}$ for $z \in \mathcal{U}$. Therefore, $T(z)^{-1}$ cannot be analytic on $\mathcal{U}$, i.e. there is an eigenvalue in $\mathcal{U}$. □

A useful feature of pseudospectra is the connection with backward error, and this carries over to the nonlinear case:

PROPOSITION 4.3. *Suppose $T(\hat{\lambda})x = r$ and $\|r\|/\|x\| < \epsilon$. Then $\hat{\lambda} \in \Lambda_\epsilon(T)$.*

Proof. Define $E = -\frac{rx^T}{\|x\|^2}$. Then $(T(\hat{\lambda}) + E)x = 0$ and $\|E\| = \|r\|/\|x\| < \epsilon$. □

We can also compare eigenvalue problems via pseudospectra. As discussed in the next section, this is particularly useful in the case when one of the problems is linear.

THEOREM 4.4. *Suppose $T : \Omega \to \mathbb{C}^{n\times n}$ and $E : \Omega \to \mathbb{C}^{n\times n}$ are analytic, and let*

$$\Omega_\epsilon \equiv \{z \in \Omega : \|E(z)\| < \epsilon\}.$$

Then

$$\Lambda(T + E) \cap \Omega_\epsilon \quad \subset \quad \Lambda_\epsilon(T) \cap \Omega_\epsilon.$$

Furthermore, if $\mathcal{U}$ is a bounded connected component of $\Lambda_\epsilon(T)$ such that $\bar{\mathcal{U}} \subset \Omega_\epsilon$, then $\mathcal{U}$ contains exactly the same number of eigenvalues of T and $T + E$.

Proof. The inclusion result is obvious based on the characterization of the pseudospectra as unions of spectra of perturbations to T. The counting result follows from the continuity of eigenvalues: the set $\Lambda_\epsilon(T) \cap \Omega_\epsilon$ contains $\Lambda(T + sE) \cap \Omega_\epsilon$ for all $0 \le s \le 1$, so for each eigenvalue of $T + E$ in $\mathcal{U}$, there is a continuously-defined path to a corresponding eigenvalue T that remains in $\mathcal{U}$. □

5. Nonlinear perturbations of linear eigenvalue problems. A *linearization* of a matrix polynomial $P : \mathbb{C} \to \mathbb{C}^{n\times n}$ is a pair $(A, B) \in \mathbb{C}^{(nd)\times(nd)}$ such that the polynomial P and the pencil (A, B) have the same spectrum and the same Jordan structure. There are many possible linearizations, and significant effort has gone into characterizing linearizations and their structural properties [34, 35]. More recent work addresses similar linearizations for rational eigenvalue problems [47]. One way to find the spectrum of a nonlinear matrix function T is to approximate T by some rational or polynomial function $\hat{T}$, then find eigenvalues of $\hat{T}$ through a linearization. In this case, the spectrum of T can be analyzed as a nonlinear perturbation of a linearization of $\hat{T}$.

We follow a simple strategy to generalize standard perturbation theorems for linear eigenvalue problems to the case where the perturbations are nonlinear. Let $T : \Omega \to \mathbb{C}^{n\times n}$ have the form

$$T(z) = A - zB + E(z),$$

and suppose we can bound E, either in norm or in the magnitude of individual components, over a domain $\Omega_E \subset \Omega$. We then apply perturbation theorems from the

linear case that are valid for any *fixed* perturbation which is similarly controlled. This argument gives us a set that includes all eigenvalues of T inside Ω_E. By continuity of the eigenvalues, if $\mathcal{U}$ is a bounded connected component such that $\bar{\mathcal{U}} \subset \Omega_E$, then $\mathcal{U}$ contains the same number of eigenvalues of T as of the linear pencil $A - zB$.

Perhaps the simplest bound of this sort involves the pseudospectra of the generalized eigenvalue problem:

COROLLARY 5.1. *Suppose $E : \Omega \to \mathbb{C}^{n\times n}$ is analytic, and let*

$$\Omega_\epsilon \equiv \{z \in \Omega : \|E(z)\| < \epsilon\}.$$

Suppose also that (A, B) is a regular pencil. Then for $T = A - zB + E(z)$,

$$\Lambda(T) \cap \Omega_\epsilon \subset \Lambda_\epsilon(A, B),$$

where $\Lambda_\epsilon(A, B)$ denotes the ϵ-pseudospectrum for the pencil $A - zB$, i.e.

$$\Lambda_\epsilon(A, B) \equiv \{z \in \mathbb{C} : \|(A - zB)^{-1}\| > \epsilon^{-1}\}.$$

Furthermore, if $\mathcal{U}$ is a bounded connected component of $\Lambda_\epsilon(A, B)$ such that $\bar{\mathcal{U}} \subset \Omega_\epsilon$, then $\mathcal{U}$ contains exactly the same number of eigenvalues of T and of the pencil (A, B).

Proof. This is a special case of Theorem 4.4. □

The pseudospectral bound is simple, but computing the pseudospectra of a pencil may be expensive. Consequently, we may be better served by Gershgorin bounds.

COROLLARY 5.2. *Suppose*

$$T(z) = D - zI + E(z)$$

where $D \in \mathbb{C}^{n\times n}$ is diagonal and $E : \Omega \to \mathbb{C}^{n\times n}$ is analytic. Suppose also that the absolute row and column sums of E are uniformly bounded, i.e. $\forall z \in \Omega$,

$$\sum_{j=1}^{n} |e_{ij}(z)| \le r_i, \quad \sum_{i=1}^{n} |e_{ij}(z)| \le c_j.$$

Then for any $0 \le \alpha \le 1$, the eigenvalues of T lie in $\bigcup_{i=1}^{n} G_i$, where the G_i are generalized Gershgorin disks

$$G_i \equiv \{z \in \mathbb{C} : |z - d_{ii}| \le \rho_i\}, \quad \rho_i \equiv r_i^\alpha c_i^{1-\alpha}.$$

Furthermore, if $\mathcal{U}$ is a union of k disks which are disjoint from the remaining disks, and if $\mathcal{U} \subset \Omega$, then $\mathcal{U}$ contains exactly k eigenvalues.

Proof. This is a direct corollary of Theorem 3.1, noting that in this case $D(z) = D - zI$ has exactly k eigenvalues in the region $\mathcal{U}$. □

Like the ordinary Gershgorin theorem, Theorem 3.1 and Corollary 5.2 are particularly powerful in combination with an appropriate change of basis. As an example, we have the following nonlinear version of a well-known corollary of a theorem due to Bauer and Fike [6, Theorem IV]:

THEOREM 5.3. *Suppose*

$$T(z) = A - zI + E(z),$$

where $A \in \mathbb{C}^{n\times n}$ has a complete basis of eigenvectors $V \in \mathbb{C}^{n\times n}$ and $E : \Omega \to \mathbb{C}^{n\times n}$ is analytic. Suppose also that $|E(z)| \le F$ componentwise for all $z \in \Omega$. Then the eigenvalues of T in Ω lie in the union of disks

$$\bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - \lambda_i| \le \phi_i\}, \quad \phi_i \equiv n\|F\|_2 \sec(\theta_i),$$

where (λ_i, w_i, v_i) are eigentriples of A and θ_i is the angle between the left and right eigenvectors w_i and v_i. If $\mathcal{U}$ is a union of any k of these disks that are disjoint from the remaining disks, and if $\mathcal{U} \subset \Omega$, then $\mathcal{U}$ contains exactly k eigenvalues of T.

Proof. The proof follows by applying Corollary 5.2 to $V^{-1}T(z)V$ and bounding the row sums of $|V^{-1}E(z)V|$. Without loss of generality, assume the columns of V are normalized to unit Euclidean length. For any $z \in \Omega$, note that the absolute row sum $r_i(z)$ of $V^{-1}EV$ is bounded by

$$\begin{aligned} r_i(z) &= \sum_{j=1}^{n} |e_i^*(V^{-1}E(z)V)e_j| \\ &\leq e_i^T \, |V^{-1}| \, F \, |V| \, e, \end{aligned}$$

where $e_i \in \mathbb{R}^n$ is the ith standard basis vector and $e \in \mathbb{R}^n$ is the vector of all ones. Let $w_i^* = e_i^* V^{-1}$ be the ith left eigenvector, and note that the normalization of V implies that $\| \, |V|e \, \|_2 \leq n$. Therefore,

$$|r_i(z)| \leq \|e_i^T |V^{-1}|\|_2 \, \|F\|_2 \, \||V|e\|_2 \leq \|w_i\|_2 \|F\|_2 n.$$

Note that $w_i^* v_i = 1$ and $\|v_i\| = 1$ by the normalization conditions, so

$$\|w_i\|_2 = \frac{\|w_i\|_2 \|v_i\|_2}{|w_i^* v_i|} = \sec(\theta_i).$$

Therefore, we have the uniform bound

$$|r_i(z)| \leq \phi_i = n\|F\|_2 \sec(\theta_i).$$

□

6. Applications. In general, the spectrum of a nonlinear eigenvalue problem can be more complicated than that of linear or polynomial eigenvalue problems, with infinitely many eigenvalues scattered across the complex plane. The analysis required to localize its spectrum is thus inherently more involved. In this section, we give three examples with infinitely many eigenvalues. In each case, we use our localization results to compare eigenvalues of the original problem to those of simpler problems. Because different approximating problems yield accurate eigenvalue estimates in different regions, several approximations may be necessary to get a complete picture.

6.1. Hadeler. The Hadeler problem in the NLEVP collection has the form

$$T(z) = B(\exp(z) - 1) + Az^2 - \alpha I$$

where $A, B \in \mathbb{R}^{8\times 8}$ are real and symmetric positive definite. The eigenvalues over part of the complex plane are shown in Figure 6.1. The spectrum consists of sixteen simple real eigenvalues and infinitely many complex eigenvalues arranged in groups of eight near certain curves $z_m(\theta)$ described later. We use Theorem 3.1 to compare $T(z)$ to two simpler problems in order to localize both the eigenvalues close to the real line and those that are farther away.

6.1.1. Comparison to a polynomial problem. We first consider the problem of localizing eigenvalues for the Hadeler example near the real axis. To do this, we approximate the Hadeler function

$$T(z) = B(\exp(z) - 1) + Az - \alpha I$$

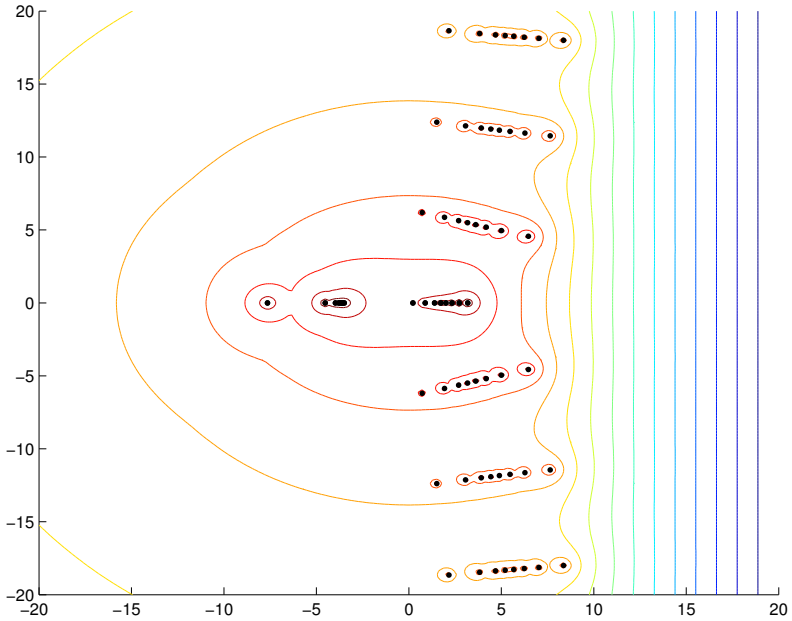

FIG. 6.1. *Spectrum and pseudospectra for the Hadeler problem. To compute the eigenvalues, we approximate eigenvalues of $T(z)$ by eigenvalues of a polynomial interpolating $T(z)$ through Chebyshev points on parts of certain curves $z_m(\theta)$ and along the real axis, then refine these estimates by a few steps of Newton iteration.*

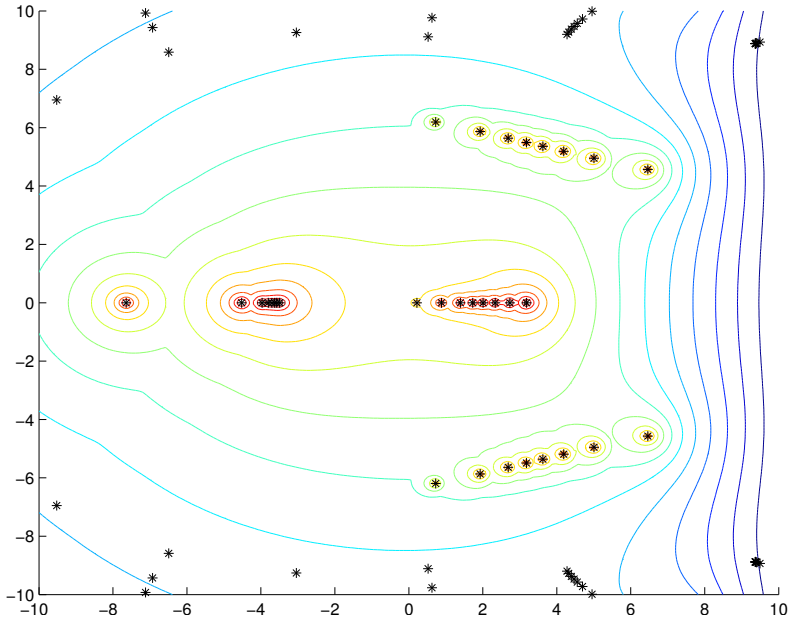

FIG. 6.2. *Spectrum for a Chebyshev approximation to the Hadeler problem (stars), together with the pseudospectrum for the Hadeler function.*

by a polynomial

$$P(z) = Bq(z) + Az - \alpha I.$$

where $q(z)$ is the polynomial interpolating $\exp(z) - 1$ through a Chebyshev grid on some interval $[z_{\min}, z_{\max}]$ guaranteed to contain all the eigenvalues, which we obtain using the Gershgorin bounds from the previous section.

Suppose we write $P(z) = Q(x)$ where $z = (1-x)z_{\min}/2 + (1+x)z_{\max}/2$; that is, Q is a rescaled version of P. If we expand Q in terms of first-kind Chebyshev polynomials T_j as

$$Q(x) = \sum_{j=0}^{n} A_j T_j(x),$$

then, assuming A_n is invertible, $\det(A_n^{-1}Q(x)) = \det(C - xI)$, where C is the *colleague matrix* linearization [18]:

$$C = \frac{1}{2}\begin{bmatrix} 0 & 2I & & & & \\ I & 0 & I & & & \\ & I & 0 & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & 0 & I \\ & & & & I & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} & & & \\ & & & \\ & & & \\ A_n^{-1}A_0 & A_n^{-1}A_1 & \dots & A_n^{-1}A_{n-1} \end{bmatrix}.$$

Note that if λ is an eigenvalue of T, then it corresponds (after an appropriate rescaling of variables) to an eigenvalue of

$$\hat{C} = \frac{1}{2}\begin{bmatrix} 0 & 2I & & & & \\ I & 0 & I & & & \\ & I & 0 & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & 0 & I \\ & & & & I & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} & & & \\ & & & \\ & & & \\ A_n^{-1}\hat{A}_0 & A_n^{-1}A_1 & \dots & A_n^{-1}A_{n-1} \end{bmatrix}.$$

where $\hat{A}_0 - A_0 = (\exp(\lambda) - 1 - q(\lambda))\, B$. Because we have expressed our polynomial in a Chebyshev basis, the colleague linearization is convenient, but other linearizations are natural for polynomials expressed in other bases [3]. One could also write the spectrum of T in terms of a nonlinear perturbation to one of these other linearizations, and this would generally lead to different bounds.

By first balancing and then computing an eigendecomposition, we find S such that

$$S^{-1}CS = D_C.$$

Furthermore, any eigenvalue λ for the fully nonlinear problem is an eigenvalue of

$$S^{-1}\hat{C}S = D_C + r(\lambda)S^{-1}E_0S,$$

where $r(\lambda) = \exp(\lambda) - 1 - q(\lambda)$ is the error in the Chebyshev approximation and E_0 is a block matrix with $A_n^{-1}B/2$ in the $(n,1)$ block and zeros elsewhere. Therefore,

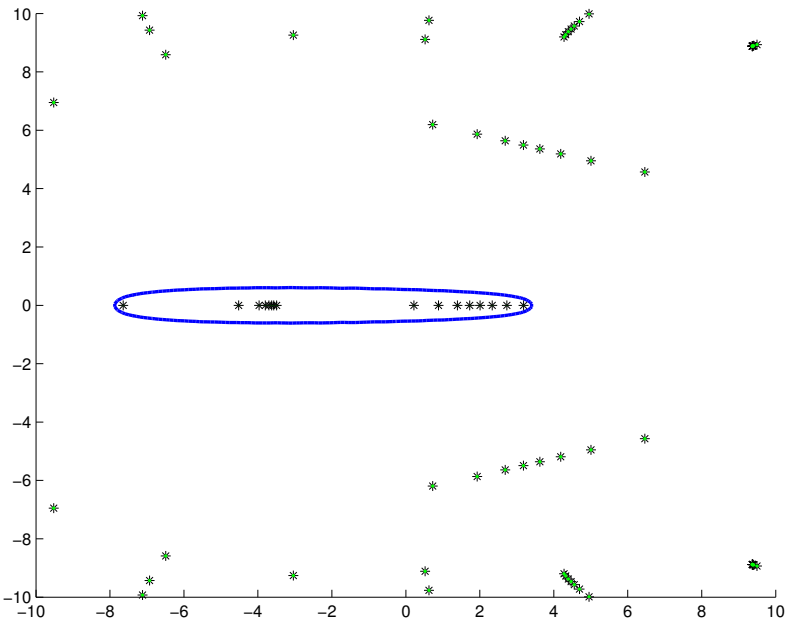

FIG. 6.3. *Region where the interpolation error $r(z)$ for the Chebyshev approximation to* $\exp(z)-1$ *is bounded by* $\epsilon = 10^{-10}$. *The Gershgorin disks in this case are all distinct, and all have radii less than* 10^{-9}.

for any $\epsilon > 0$, the eigenvalues inside the region where $|r(z)| < \epsilon$ lie in the union of Gershgorin disks of radius $\epsilon\rho_j$ about the eigenvalues of C, where ρ_j are the absolute row or column sums of $S^{-1}E_0S$. The standard theory for convergence of Chebyshev approximations tells us that for an appropriate range of ϵ values, $|r(z)| < \epsilon$ for z in a Bernstein ellipse whose radius depends on ϵ; see [53, Chapter 8].

If we apply the above procedure with a degree 20 interpolant on the interval from $z_{\min} = -7.7650$ to $z_{\max} = 3.3149$, we obtain a polynomial eigenvalue problem whose eigenvalues are shown in Figure 6.2. The polynomial provides good estimates for the real eigenvalues, and reasonable estimates for the first clusters of complex eigenvalues near the real axis. The other eigenvalues of the polynomial interpolant do not approximate any eigenvalues of T.

For this problem, the largest Gershgorin radius $\epsilon\rho_j$ is less than 7ϵ. Figure 6.3 shows the region where $|r(z)| < \epsilon = 10^{-10}$; the corresponding Gershgorin disks in this case are so tight that they are not visible in the figure. Thus, we can trust these approximations to the real eigenvalues to an absolute error of less than 10^{-9}.

A more interesting bound involves the eigenvalues farther from the real axis. Without the comparison to the previously computed spectrum of T, it would initially be unclear whether the cluster of eigenvalues with imaginary part near 6 is spurious or not. If we set $\epsilon = 0.1$ and $\epsilon = 1.6$, we get the Gershgorin disks shown in Figure 6.4; these are sufficient to show that the polynomial eigenvalue clusters closest to the real line also approximate eigenvalues of T, and to bound the approximation error.

6.1.2. Comparison to a simplified function. The polynomial approximation in the previous section resolves eigenvalues near the real axis, but tells us nothing about eigenvalues deeper in the complex plane. However, for $|z| \gg 1$, $T(z)$ is dominated by either $B\exp(z)$ or Az^2, and the remaining constant term becomes rel-

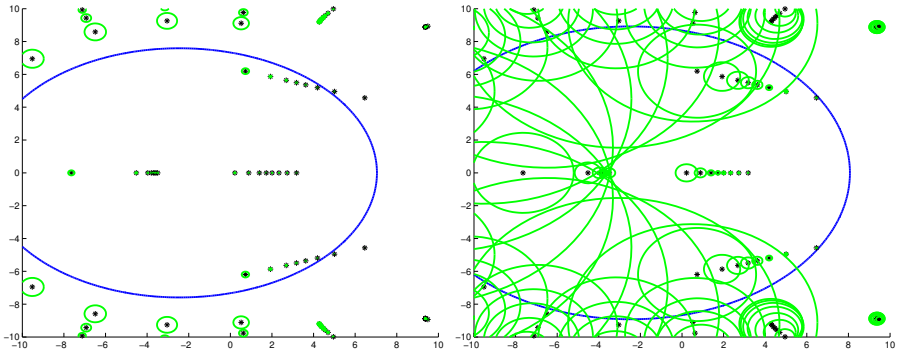

FIG. 6.4. *Region where the interpolation error $r(z)$ for the Chebyshev approximation to* $\exp(z)-1$ *is bounded by* $\epsilon = 0.1$ *(left) and* $\epsilon = 1.6$ *(right). The Gershgorin disks of radii* $\epsilon\rho_j$ *are shown in green in each case.*

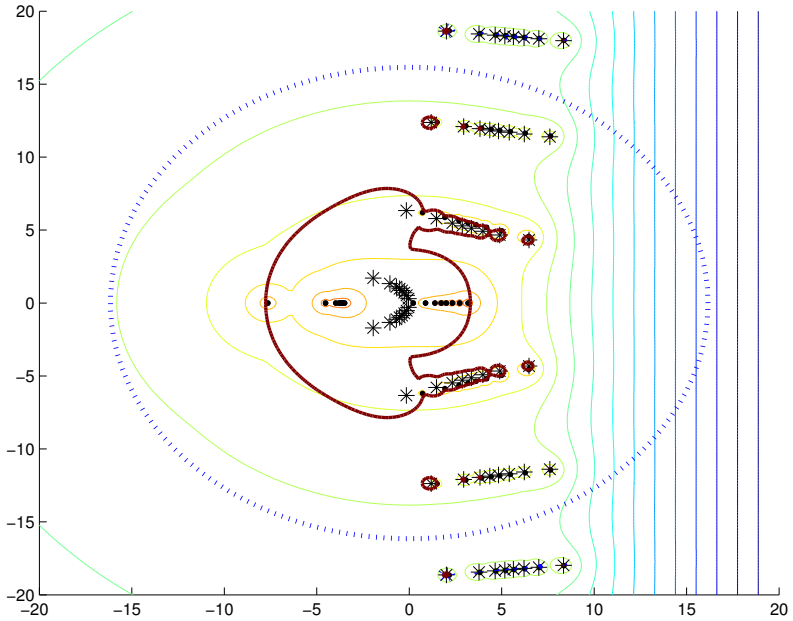

FIG. 6.5. *Gershgorin region (solid line) containing eigenvalues of the Hadeler function T (dots) and the simplified problem $\hat{T}$ (stars). Each connected component contains the same number of eigenvalues for both problems. The Gersgorin region is somewhat complicated, but it can be shown that the components of the Gershgorin regions containing eigenvalues $\hat{\lambda} \in \Lambda(\hat{T})$ outside a disk of radius about* 16.3 *(dashed line) is contained in a union of disks of radius $O(|\hat{\lambda}|^{-2})$.*

atively insignificant. Therefore, we can localize eigenvalues far from the origin using a simplified problem without the constant term.

Let U be a matrix of A-orthonormal eigenvectors for the pencil (B, A), and define

$$\tilde{T}(z) = U^T T(z) U = D_B \exp(z) + Iz^2 + E$$

where $D_B = \operatorname{diag}(\beta_1, \ldots, \beta_8)$, $\beta_j > 0$, and $E = -U^T(\alpha I + B)U$ is a constant matrix.

We compare $\tilde{T}$ to the simplified function

$$\hat{T}(z) = D_B \exp(z) + Iz^2 = \operatorname{diag}\left(f_j(z)\right)_{j=1}^{8},$$
$$f_j(z) \equiv \beta_j \exp(z) + z^2 = 4e^z \left(\frac{\beta_j}{4} + \left[-\frac{z}{2}\exp\left(-\frac{z}{2}\right)\right]^2\right).$$

The eigenvalues of $\hat{T}$ lie along the curves $z_m(\theta) = (2\theta + (2m-1)\pi)(\cot(\theta) + i)$, which are the preimage of $i\mathbb{R}$ under the mapping $z \mapsto (-z/2)\exp(-z/2)$. More precisely, the zeros of f_j can be written as

$$\hat{\lambda}_{kj}^{\pm} = -2W_k\left(\pm\frac{i}{2}\sqrt{\beta_j}\right), \quad \text{for } k \in \mathbb{Z},$$

where W_k denotes the kth branch of via the Lambert W function [15], the multi-valued solution to the equation $W(z)\exp W(z) = z$.

Using Theorem 3.1, we know that

$$\Lambda(\tilde{T}) \subset \bigcup_{j=1}^{n} G_j^1 \equiv \bigcup_{j=1}^{n} \{z : |f_j(z)| \le \rho_j\}, \quad \rho_j \equiv \sum_{k=1}^{n} |e_{jk}|.$$

Furthermore, any connected component of this region contains the same number of eigenvalues of $\tilde{T}$ and $\hat{T}$. In Figure 6.5, we show a plot of the Gershgorin regions in the complex plane, with the location of the eigenvalues of $\hat{T}$ marked by asterisks. These regions are somewhat complicated, but we can bound some of them in simpler sets. If $\hat{\lambda}$ is a zero of f_j, then Taylor expansion about $\hat{\lambda}$ yields

$$f_j(\hat{\lambda} + w) = bw + R(w), \quad b \equiv \hat{\lambda}(2 - \hat{\lambda}), \quad |R(w)| \le \left(1 + (|\hat{\lambda}|^2/2)\exp|w|\right)|w|^2.$$

If $1 + (|\hat{\lambda}|^2/2)\exp(2\rho_j/|b|) < |b|^2/(4\rho_j)$, then $|f_j(\hat{\lambda} + w)| > \rho_j$ for $|w| = 2\rho_j/|b|$. The condition always holds for $|\hat{\lambda}| > R \approx 16.3$, and so the component of G_j^1 containing $\hat{\lambda} \in \Lambda(\hat{T})$ outside this disk must lie in a disk of radius $2\rho_j/|b| = O(|\hat{\lambda}|^{-2})$. Thus, outside the disk of radius R, every eigenvalue of T is approximated by an eigenvalue of $\hat{T}$ with less than 2% relative error, with better accuracy farther from the origin.

6.2. Time delay. Another example from the NLEVP toolbox is the `time_delay` example, which comes from applying transform methods to a delay differential equation. The function is

$$T(z) = -zI + A_0 + A_1\exp(-z),$$

where A_0 is a companion matrix and A_1 is rank one. The spectrum and pseudospectra for this problem over part of the complex plane are shown in Figure 6.6.

As with the Hadeler example, we can get good estimates of the eigenvalues far from the origin by dropping the constant term in the problem. In order to analyze this case, let us transform T into a convenient basis. We choose a basis of eigenvectors V for A_1 so that $V^{-1}A_1V = D_1 = \operatorname{diag}(\mu_1, 0, 0)$ and so that the trailing 2-by-2 submatrix of $E = V^{-1}A_0V$ is diagonal. The eigenvalues of T are thus also eigenvalues of

$$\tilde{T}(z) = -zI + D_1\exp(-z) + E;$$

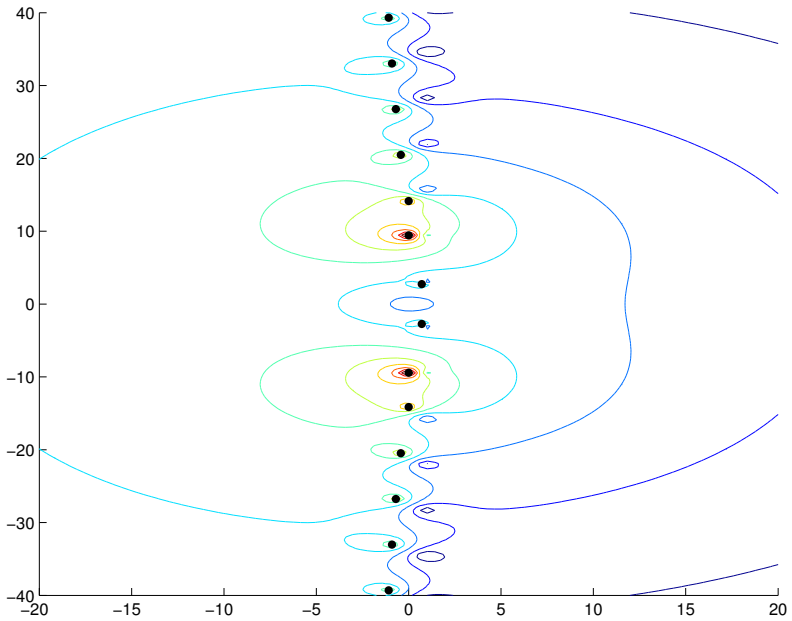

FIG. 6.6. *Spectrum (in dots) and pseudospectra for the time delay example. The spectrum closest to the real axis was computed using a degree 40 Chebyshev interpolant of T on the interval $[-12i, 12i]$; farther out in the complex plane, we get an initial guess from a simplified problem, then refine using Newton iteration.*

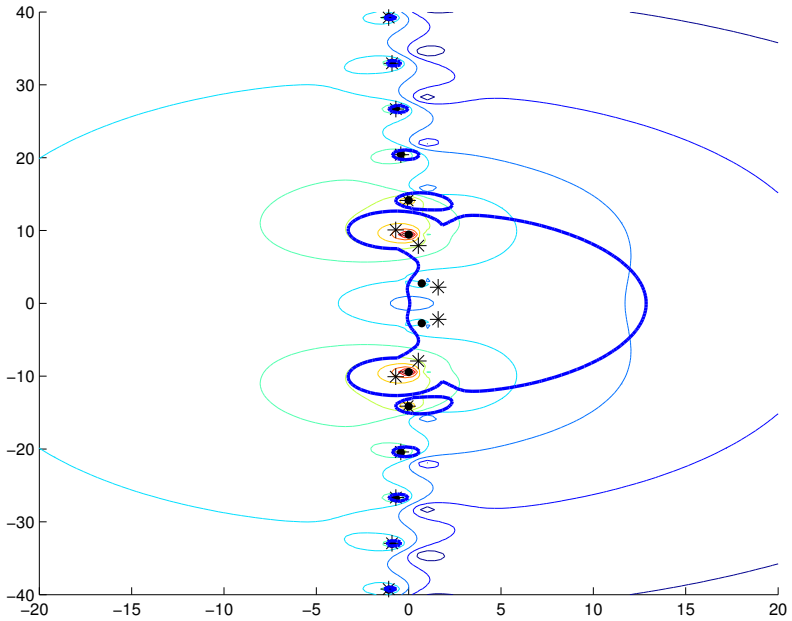

FIG. 6.7. *Spectrum of the approximation $\hat{T}$ (stars) to the time delay problem T, Gershgorin regions (thick line), and pseudospectra for T.*

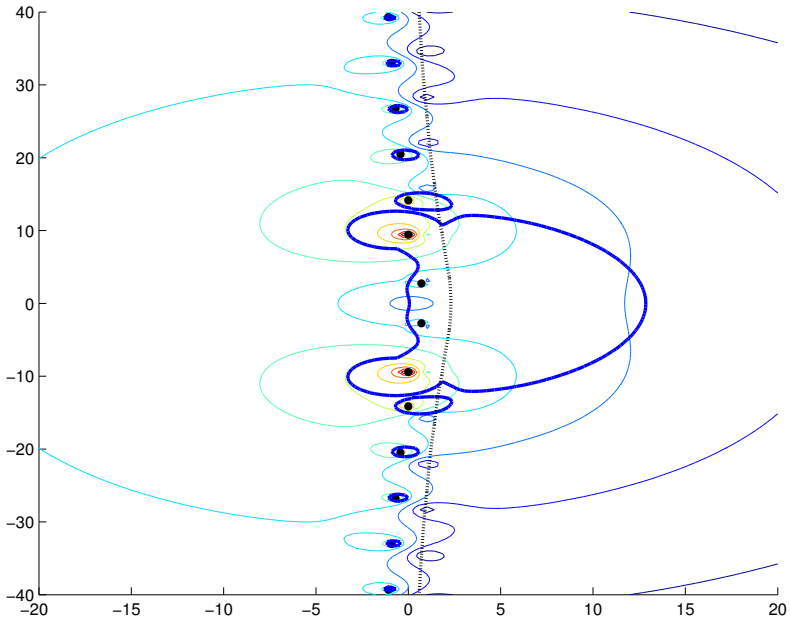

FIG. 6.8. *Gershgorin region for a simplified problem with the exponential term dropped (left of the dashed line). This is superimposed on the Gershgorin regions from Figure 6.7.*

and, as in the case of the Hadeler example, we can easily compute the eigenvalues of the related problem

$$\hat{T}(z) = -zI + D_1 \exp(-z).$$

The function $\hat{T}$ has a double eigenvalue at the origin corresponding to the zero eigenvalues of A_1; the remaining eigenvalues are solutions of the equation

$$z \exp(z) = \mu_1 \approx -13.3519,$$

which can be written as $W_k(\mu_1)$ for $k \in \mathbb{Z}$, where W_k is again the kth branch of the Lambert W function. Eigenvalues of $\tilde{T}$ must lie in the (column) Gershgorin region

$$\bigcup_{j=1}^{3} \{z : | - z + \mu_j \exp(-z)| \leq \rho_j\},$$

where ρ_j are the absolute column sums of E. In Figure 6.6, we plot this region in the complex plane, with the location of the eigenvalues of $\hat{T}$ marked by asterisks. Theorem 3.1 gives us that each component contains the same number of eigenvalues of $\tilde{T}$ and $\hat{T}$. Note in particular that this means that the central "blob" in the pseudospectrum must contain exactly six eigenvalues of T, as indeed it does – two eigenvalues closer to the origin, and a pair of degenerate double eigenvalues at $\pm 3\pi i$; see [30].

The Gershgorin regions shown in Figure 6.7 obtained by comparing T to $\hat{T}$ extend far into the right half plane. We can tighten our inclusion region somewhat by also comparing T to $A_0 - zI$ and letting $A_1 \exp(-z)$ be the error term. Define

$$\breve{T}(z) = V^{-1}T(z)V = D_0 - zI + \breve{E} \exp(-z)$$

where $D_0 = V^{-1}A_0V$ and $\breve{E} = V^{-1}A_1V$. Applying Theorem 3.1, any eigenvalue of T must live in the union of the regions $|d_i - z| \le \gamma_i |\exp(-z)|$, where γ_i is an absolute row sum of $\breve{E}$. This region is bounded from the right by the contour shown in Figure 6.8. Intersecting these bounds with the previous Gershgorin bounds give very tight control on the spectrum.

REMARK 2. The determinant of $T(z)$ above is exactly the type of scalar function studied in [7], and there are similarites between the analysis done there and in this section, i.e., dropping the constant term.

6.3. Resonances. Our final example is a problem associated with resonances of a Schrödinger operator on the positive real line [59, 11]. We seek the values of λ such that the following two-point boundary value problem has nontrivial solutions:

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + V - \lambda\right)\psi = 0 \qquad & \text{on } (0,b), \\ \psi(0) = 0 \qquad \text{and} \qquad & \psi'(b) = i\sqrt{\lambda}\psi(b), \end{aligned} \tag{6.1}$$

where V equals $V_0 > 0$ on (a,b) and is zero elsewhere. In our computations, we used $(a,b) = (2,3)$ and $V_0 = 5$. We formulate (6.1) as a finite-dimensional nonlinear eigenvalue problem by shooting from 0 to a and from a to b [5, Chapter 7], described as follows.

Rewriting (6.1) in first-order form, we have

$$\frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V-\lambda & 0 \end{bmatrix} u, \text{ where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}. \tag{6.2}$$

Then the matrices

$$R_{0a}(\lambda) = \exp\left(a\begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}\right), \quad R_{ab}(\lambda) = \exp\left((b-a)\begin{bmatrix} 0 & 1 \\ V_0-\lambda & 0 \end{bmatrix}\right)$$

respectively map $u(0) \mapsto u(a)$ and $u(a) \mapsto u(b)$. Thus, (6.1) is equivalent to the six-dimensional nonlinear eigenvalue problem

$$T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 0 \\ -i\sqrt{\lambda} & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0. \tag{6.3}$$

In the next section, we derive a rational approximation $\hat{T} \approx T$ whose linearization $K - \lambda M$ corresponds to a discretization of (6.1). We then use the eigenvalues of $\hat{T}$ as starting points to compute eigenvalues of T, and establish by Theorem 4.4 that this procedure finds all eigenvalues of T in a region of interest.

6.3.1. Rational approximation of the resonance problem. We construct $K - \lambda M$ by introducing auxiliary variables y whose elimination produces a rational approximation to $T(\lambda)$. That is, we write $A(\lambda) = K - \lambda M$ so that

$$\begin{bmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{bmatrix} \begin{bmatrix} u_{\text{all}} \\ y \end{bmatrix} \approx \begin{bmatrix} T(\lambda)u_{\text{all}} \\ 0 \end{bmatrix}. \tag{6.4}$$

If we eliminate the auxiliary variables and the equations that define them, we are left with a rational approximation to $T(\lambda)$ given by the leading 6-by-6 Schur complement in A:

$$T(\lambda) \approx \hat{T}(\lambda) = A_{11}(\lambda) - A_{12}(\lambda)A_{22}(\lambda)^{-1}A_{21}(\lambda).$$

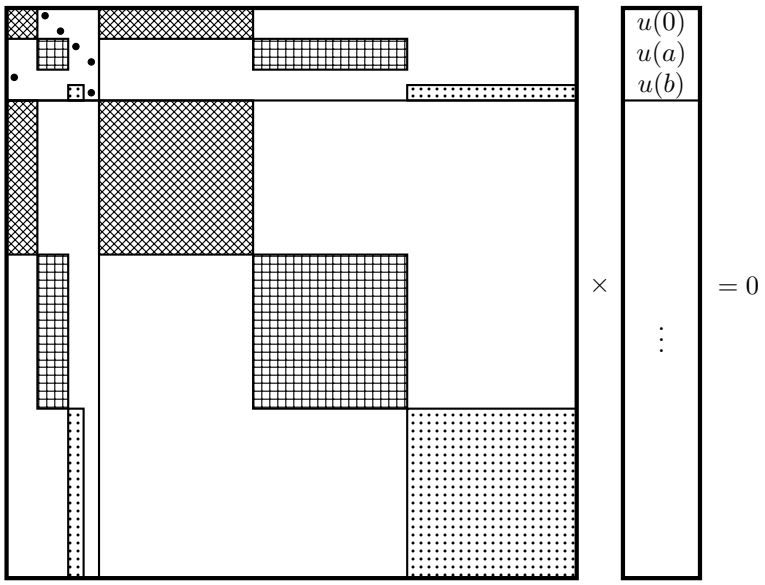

FIG. 6.9. *Linearized rational approximation to* (6.3). *The rational eigenvalue problem is a Schur complement in a linear eigenvalue problem obtained by eliminating all but the first six variables and equations. The overall matrix is assembled from linear matrix-valued functions* $A^{0a}(\lambda)$ *(cross-hatched),* $A^{ab}(\lambda)$ *(plaid), and* $A^{Z}(\lambda)$ *(dots) that generate rational approximations to* $R_{0a}(\lambda)$, $R_{ab}(\lambda)$, *and* $-i\sqrt{\lambda}$, *respectively.*

More precisely, we will define $A_{11}(\lambda)$ to be the constant part of $T(\lambda)$, i.e.

$$A_{11}(\lambda) = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix},$$

then add three submatrices $A^{0a}(\lambda)$, $A^{ab}(\lambda)$, and $A^{Z}(\lambda)$ (to be defined in a moment) that generate rational approximations to the nonlinear terms $R_{0a}(\lambda)$, $R_{ab}(\lambda)$, and $-i\sqrt{\lambda}$ when the Schur complement in A is taken. The structure of the matrix A in terms of these submatrices is shown schematically in Figure 6.9.

To define the rational approximation to $R_{0a}(\lambda)$, we start by writing the exact function $R_{0a}(\lambda)u(0)$ via the equations

$$\begin{bmatrix} 0 & B(a) \\ -I & B(0) \\ 0 & -\frac{d^2}{dx^2} - \lambda \end{bmatrix} \begin{bmatrix} u(0) \\ \psi \end{bmatrix} = \begin{bmatrix} R_{0a}(\lambda)u(0) \\ 0 \\ 0 \end{bmatrix}, \tag{6.5}$$

where

$$B(x)\psi \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix} = u(x).$$

If we discretize (6.5) by replacing ψ with a vector $\hat{\psi}$ of function values at sample points, and correspondingly replace the operators in the second column in (6.5) with

discrete approximations, we are left with a matrix equation

$$A^{(0a)}(\lambda)\begin{bmatrix} u(0) \\ \hat{\psi} \end{bmatrix} \equiv \begin{bmatrix} 0 & \hat{B}(a) \\ -I & \hat{B}(0) \\ 0 & K_H - \lambda M_H \end{bmatrix}\begin{bmatrix} u(0) \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} \hat{R}_{0a}(\lambda)u(0) \\ 0 \\ 0 \end{bmatrix}, \tag{6.6}$$

where K_H and M_H are some fixed matrices of dimension $(N-2)\times N$. For our problem, we set $\hat{\psi}$ to be function values at a Chebyshev mesh of $N = 40$ points on $[0, a]$, and $(K_H - \lambda M_H)$ represents a pseudospectral collocation discretization of $-d^2/dx^2 - \lambda$ [52]. The matrix $\hat{A}^{ab}(\lambda)$ is defined similarly.

To define A^Z, we begin with the best max-norm rational approximation to $z^{-1/2}$ on an interval $[m, M]$, which was first discovered in 1877 by Zolotarev [24, §5.9]. The approximation is

$$z^{-1/2} \approx r(z) = \sum_{j=1}^{N_Z} \frac{\gamma_j}{z - \xi_j}$$

where the poles ξ_j and the weights γ_j are defined in terms of elliptic integrals; for details, we refer to Method 3 of [23]. We approximate $-i\sqrt{\lambda}$ by $-i/r(\lambda)$, which we encode as the leading 1-by-1 Schur complement in

$$A^Z(\lambda) = \begin{bmatrix} 0 & i & & & & \\ 1 & 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{N_Z} \\ & 1 & \xi_1 - \lambda & & & \\ & 1 & & \xi_2 - \lambda & & \\ & \vdots & & & \ddots & \\ & 1 & & & & \xi_{N_Z} - \lambda \end{bmatrix}.$$

For our problem, we use the Zolotarev approximation with $N_Z = 20$ poles, chosen for optimality on the interval $[m, M] = [0.1, 500]$.

6.3.2. Analysis of the rational approximation. Our goal in this section will be to find all eigenvalues in the region D bounded by the ellipse Γ shown in Figure 6.10. D is clearly contained in Ω_ϵ, where $\epsilon = 10^{-8}$. Moreover, Γ was chosen so that $\|T(z)^{-1}\| < \epsilon^{-1}$ for all $z \in \Gamma$. This means that the contour Γ does not intersect the ϵ-pseudospectrum of T, and hence any connected component of $\Lambda_\epsilon(T)$ in D contains the same number of eigenvalues of T and $\hat{T}$ (by Theorem 4.4). It follows that the same number of eigenvalues of T and $\hat{T}$ lie in D.

Since the norm of the perturbation is small there, we expect that the eigenvalues of $\hat{T}$ in D are very good approximations to those of T. We refine each these eigenvalue estimates by Newton iteration on a bordered system [21, Chapter 3]. The absolute difference between each eigenvalue of $\hat{T}$ and the corresponding eigenvalue of T is shown in Table 6.1

7. Conclusion. In this paper, we have described several localization theorems for the spectrum of a regular analytic function $T : \Omega \to \mathbb{C}^{n\times n}$. These pseudospectral and Gershgorin inclusion results generalize well-known perturbation theory for the standard eigenvalue problem. We have also shown through several examples how these results are practical tools to localize the spectrum, count eigenvalues in parts of the complex plane, and judge which eigenvalues from an approximating eigenvalue problem are accurate approximations of true eigenvalues and which are spurious.

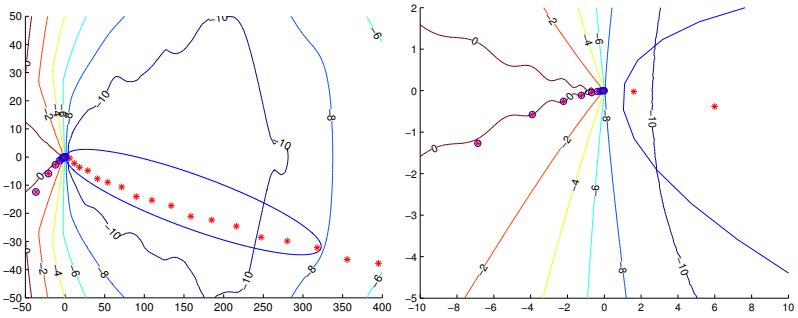

FIG. 6.10. *Computed eigenvalues for $V_0 = 5$, $(a, b) = (2, 3)$, using 20 poles in a Zolotarev square root approximation optimal on $[0.1, 500]$, and Chebyshev meshes of size 40 on $(0, a)$ and (a, b). Circled eigenvalues of $\hat{T}$ satisfy $\sigma_{\min}(T(\lambda)) > 10^{-8}$. Contour plots of $\log_{10}(\|T(z) - \hat{T}(z)\|)$ and an ellipse on which the smallest singular value of $T(z)$ is greater than 10^{-8} (left). A closer view (right).*

Eigenvalue	Error	Eigenvalue	Error
$483.76 - 44.65i$	1.34×10^{-5}	$439.47 - 40.27i$	2.39×10^{-6}
$395.11 - 37.76i$	4.56×10^{-7}	$355.60 - 36.42i$	5.67×10^{-8}
$317.83 - 32.22i$	7.19×10^{-9}	$280.15 - 29.85i$	8.45×10^{-10}
$247.21 - 28.52i$	1.23×10^{-10}	$215.94 - 24.54i$	7.10×10^{-11}
$184.96 - 22.34i$	9.90×10^{-11}	$158.59 - 21.04i$	8.45×10^{-11}
$133.80 - 17.31i$	4.61×10^{-11}	$109.55 - 15.33i$	2.20×10^{-11}
$89.76 - 14.06i$	5.61×10^{-12}	$71.41 - 10.65i$	1.60×10^{-11}
$53.94 - 8.99i$	2.08×10^{-11}	$40.77 - 7.72i$	1.97×10^{-11}
$28.79 - 4.80i$	5.84×10^{-12}	$18.24 - 3.63i$	2.04×10^{-12}
$11.78 - 2.23i$	3.22×10^{-12}	$5.99 - 0.38i$	4.75×10^{-13}
$1.60 - 0.02i$	1.46×10^{-13}		

TABLE 6.1
Error bounds for computed resonances

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