Comput Optim Appl (2018) 70:1–32
https://doi.org/10.1007/s10589-017-9973-y

Real eigenvalues of nonsymmetric tensors

Jiawang Nie[1] · **Xinzhen Zhang**[2]

Received: 30 May 2017 / Published online: 4 December 2017

Abstract This paper discusses the computation of real Z-eigenvalues and H-eigenvalues of nonsymmetric tensors. A generic nonsymmetric tensor has finitely many Z-eigenvalues, while there may be infinitely many ones for special tensors. The number of H-eigenvalues is finite for all tensors. We propose Lasserre type semidefinite relaxation methods for computing such eigenvalues. For every tensor that has finitely many real Z-eigenvalues, we can compute all of them; each of them can be computed by solving a finite sequence of semidefinite relaxations. For every tensor, we can compute all its real H-eigenvalues; each of them can be computed by solving a finite sequence of semidefinite relaxations.

Keywords Tensor · Z-eigenvalue · H-eigenvalue · Lasserre's hierarchy · Semidefinite relaxation

Mathematics Subject Classification 15A18 · 15A69 · 90C22

1 Introduction

For positive integers m and $n_1, n_2, \ldots, n_m$, an m-order and $(n_1, n_2, \ldots, n_m)$-dimensional real tensor can be viewed as an array in the space $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$. Such a tensor $\mathcal{A}$ can be indexed as

✉ Xinzhen Zhang
xzzhang@tju.edu.cn

Jiawang Nie
njw@math.ucsd.edu

1 Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

2 School of Mathematics, Tianjin University, Tianjin 300072, China

$$\mathcal{A} = (\mathcal{A}_{i_1 \ldots i_m}), \quad 1 \le i_j \le n_j, \quad j = 1, \ldots, m. \tag{1.1}$$

When $n_1 = \cdots = n_m = n$, $\mathcal{A}$ is called an m-order n-dimensional tensor. In such case, the tensor space $\mathbb{R}^{n_1 \times \cdots \times n_m}$ is denoted as $\mathtt{T}^m(\mathbb{R}^n)$. A tensor in $\mathtt{T}^m(\mathbb{R}^n)$ is said to be *symmetric* if its entries are invariant under permutations of indices [i.e., $\mathcal{A}_{i_1 \ldots i_m} = \mathcal{A}_{j_1 \ldots j_m}$ whenever $(i_1, \ldots, i_m)$ is a permutation of $(j_1, \ldots, j_m)$]. The subspace of symmetric tensors in $\mathtt{T}^m(\mathbb{R}^n)$ is denoted as $\mathtt{S}^m(\mathbb{R}^n)$. For $\mathcal{A} \in \mathtt{T}^m(\mathbb{R}^n)$ and $x := (x_1, \ldots, x_n)$, we use the notation

$$\begin{cases} \mathcal{A}x^m := \displaystyle\sum_{1 \le i_1, \ldots, i_m \le n} \mathcal{A}_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \\ \mathcal{A}x^{m-1} := \left(\displaystyle\sum_{1 \le i_2, \ldots, i_m \le n} \mathcal{A}_{j i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} \right)_{j=1,\ldots,n}. \end{cases} \tag{1.2}$$

Note that $\mathcal{A}x^{m-1}$ is an n-dimensional vector. First, we give some definitions of tensor eigenvalues, which can be found in [21,31].

Definition 1.1 For a tensor $\mathcal{A} \in \mathtt{T}^m(\mathbb{R}^n)$, a pair $(\lambda, u) \in \mathbb{R} \times \mathbb{R}^n$ is called a real Z-eigenpair of $\mathcal{A}$ if

$$\mathcal{A}u^{m-1} = \lambda u, \quad u^T u = 1. \tag{1.3}$$

(The superscript ^{T} denotes the transpose.) Such λ is called a real Z-eigenvalue, and such u is called a real Z-eigenvector associated to λ.

Definition 1.2 For a tensor $\mathcal{A} \in \mathtt{T}^m(\mathbb{R}^n)$, a pair $(\lambda, u) \in \mathbb{R} \times \mathbb{R}^n$ is called a real H-eigenpair of $\mathcal{A}$ if

$$\mathcal{A}u^{m-1} = \lambda u^{[m-1]}, \quad u \ne 0. \tag{1.4}$$

(The symbol $u^{[m-1]}$ denotes the vector such that $(u^{[m-1]})_i = (u_i)^{m-1}$ for $i = 1, \ldots, n$). Such λ is called a real H-eigenvalue, and such u is called a real H-eigenvector associated to λ.

Tensor eigenvalues are defined in Lim [21]. For real symmetric tensors, they are also defined in Qi [31]. The Z-eigenvalus and H-eigenvalues have applications in signal processing, control, and diffusion imaging [35,37,38]. For recent work on tensor computations, we refer to [22,30,34]. In particular, Z-eigenvalues have important applications in higher order Markov chains, which are shown in Sect. 2.2. Definitions 1.1 and 1.2 are about real eigenvalues. Complex eigenvalues can be similarly defined, if the complex values are allowed for (λ, u). We refer to the work [3,32]. There also exist other types of tensor eigenvalues in [4,5]. In this paper, we focus on the computation of real Z/H-eigenvalues.

When a tensor is symmetric, the computation of real Z/H-eigenvalues is discussed in [5]. For the case $(m, n) = (3, 2)$, computing the largest Z-eigenvalues is discussed in [36]. Shifted power methods are proposed in [15,40]. In [24], a method is proposed to find best rank-1 approximations, which is equivalent to computing the largest Z-eigenvalues. As shown in [12,40], it is NP-hard to compute extreme eigenvalues of tensors. For nonnegative tensors, the largest H-eigenvalues can be computed by using Perron–Frobenius theorem [4,23].

For nonsymmetric tensors, there is no much earlier work on computing all real Z/H-eigenvalues. An elementary approach for this task is to solve the polynomial systems (1.3) and (1.4) directly, for getting all complex solutions by symbolic methods, and then choose real ones. This approach is typically very expensive to be used in computation. A numerical experiment for this is given in Sect. 6.3.

There are fundamental differences between symmetric and nonsymmetric tensor eigenvalues. For symmetric tensors, the Eqs. (1.3) and (1.4) follow from the Karush–Kuhn–Tucker (KKT) conditions of some polynomial optimization problems, which show the existence of real Z-eigenvalues and H-eigenvalues. Furthermore, every symmetric tensor has finitely many Z-eigenvalues [5]. However, the Eqs. (1.3) and (1.4) for nonsymmetric tensors may not be the KKT conditions of some optimization problems. The following examples show the differences between symmetric and nonsymmetric tensor eigenvalues.

Example 1.3 Consider the tensor $\mathcal{A} \in \mathrm{T}^4(\mathbb{R}^2)$ such that $\mathcal{A}_{ijkl} = 0$ except

$$\mathcal{A}_{1112} = \mathcal{A}_{1222} = 1, \mathcal{A}_{2111} = \mathcal{A}_{2122} = -1.$$

By the definition, (λ, x) is a Z-eigenpair if and only if

$$\begin{cases} (x_1^2 + x_2^2)x_2 = \lambda x_1, \\ -(x_1^2 + x_2^2)x_1 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

One can check that $\mathcal{A}$ has no real Z-eigenvalues and neither complex ones. This is confirmed by Theorem 3.1(i), because the semidefinite relaxation (3.4) is infeasible for the order $k = 3$. By the definition, (λ, x) is an H-eigenpair if and only if it satisfies

$$\begin{cases} (x_1^2 + x_2^2)x_2 = \lambda x_1^3, \\ -(x_1^2 + x_2^2)x_1 = \lambda x_2^3, \\ (x_1, x_2) \neq (0, 0). \end{cases}$$

The tensor $\mathcal{A}$ has no real H-eigenvalues. This is also confirmed by Theorem 4.2(i), because the semidefinite relaxation (4.5) is infeasible for the order $k = 4$.

It is possible that a tensor has infinitely many Z-eigenvalues.

Example 1.4 Consider the tensor $\mathcal{A} \in \mathrm{T}^4(\mathbb{R}^2)$ such that $\mathcal{A}_{ijkl} = 0$ except

$$\mathcal{A}_{1111} = \mathcal{A}_{2112} = 1.$$

Then, (λ, x) is a Z-eigenpair of $\mathcal{A}$ if and only if

$$\begin{cases} x_1^3 = \lambda x_1, \\ x_1^2 x_2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases} \tag{1.5}$$

One can check that every $\lambda \in [0, 1]$ is a real Z-eigenvalue, associated to Z-eigenvectors $(\pm\sqrt{\lambda}, \pm\sqrt{1-\lambda})$.

Note that the set of Z/H-eigenvalues of a nonsymmetric tensor is different with different definitions on $\mathcal{A}x^{m-1}$. To clear this fact, we consider (2)-mode $[\mathcal{A}x^{m-1}]_{(2)}$ defined by

$$[\mathcal{A}x^{m-1}]_{(2)} := \Big(\sum_{1\le i_1,i_3,\ldots,i_m\le n} \mathcal{A}_{i_1 j i_3 \ldots i_m} x_{i_1} x_{i_3} \cdots x_{i_m} \Big)_{j=1,\ldots,n}.$$

By directly computation, there is only 1 Z-eigenvalue and 1 H-eigenvalue with (2)-mode for tensor $\mathcal{A}$ in Examples 1.3 and 1.4. These are different from Examples 1.3 and 1.4.

However, we would like to remark that the above examples are not generic cases. That is, every generic tensor has finitely many Z-eigenvalues, and its number can be given by explicit formula. Here, the meaning of the word "*generic*" is in the sense of Zariski topology (i.e., there exists a polynomial ϕ in the entries of $\mathcal{A}$ such that if $\phi(\mathcal{A}) \neq 0$ then $\mathcal{A}$ has finitely many Z-eigenvalues). We refer this result to [3]. On the other hand, every tensor has finitely many H-eigenvalues, which is shown in Proposition 4.1. From these facts, it is a well-posed question to compute all real Z/H-eigenvalues, for generic tensors.

In this paper, we propose numerical methods for computing all real Z-eigenvalues (if there are finitely many ones) and all real H-eigenvalues. For symmetric tensors, the Z-eigenvalues and H-eigenvalues are critical values of some polynomial optimization problems. This property was used very much in [5]. The method in [5] is based on Jacobian SDP relaxations [26], which are used for polynomial optimization. Indeed, the same kind of method can be used to compute all local minimum values of polynomial optimization [29]. However, the method in [5] is not suitable for computing eigenvalues of nonsymmetric tensors, because their eigenvalues are no longer critical values of polynomial optimization problems.

This paper is organized as follows. Section 2 presents some preliminaries on polynomial optimization, tensor eigenvalues and an application example in higher order Markov chain. Section 3 proposes Lasserre type semidefinite relaxations for computing Z-eigenvalues. If there are finitely many real Z-eigenvalues, each of them can be computed by solving a finite sequence of semidefinite relaxations. Section 4 proposes Lasserre type semidefinite relaxations for computing all H-eigenvalues. Each of them can be computed by solving a finite sequence of semidefinite relaxations. Numerical examples are shown in Sect. 5. We make some discussions in Sect. 6.

2 Preliminaries and motivation

In this section, we recall some basics on polynomial optimization and tensor eigenvalues. After that, we present an application example arising in higher order Markov chain.

2.1 Preliminaries

In this subsection, we review some basics in polynomial optimization. We refer to [17,18] for surveys in the area. In the space $\mathbb{R}^n$, $\|u\|$ denotes the standard Euclidean norm of a vector u. For $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer not smaller than t. Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients and in variables $x := (x_1, \ldots, x_n)$. The degree of a polynomial p, denoted by $\deg(p)$, is the total degree, i.e., the highest degree of its monomials. Denote by $\mathbb{R}[x]_d$ the set of real polynomials in $\mathbb{R}[x]$ whose degrees are no greater than d. For a polynomial tuple $h = (h_1, h_2, \ldots, h_s)$, the ideal generated by h is the set

$$I(h) := h_1 \cdot \mathbb{R}[x] + h_2 \cdot \mathbb{R}[x] + \cdots + h_s \cdot \mathbb{R}[x].$$

The k-th truncation of $I(h)$ is the set

$$I_k(h) := h_1 \cdot \mathbb{R}[x]_{k-\deg(h_1)} + \cdots + h_s \cdot \mathbb{R}[x]_{k-\deg(h_s)}.$$

The complex and real algebraic varieties of h are respectively defined as

$$\mathcal{V}_{\mathbb{C}}(h) := \{x \in \mathbb{C}^n \mid h(x) = 0\}, \quad \mathcal{V}_{\mathbb{R}}(h) := \mathcal{V}_{\mathbb{C}}(h) \cap \mathbb{R}^n.$$

A polynomial p is said to be sum of squares (SOS) if there exist $p_1, p_2, \ldots p_r \in \mathbb{R}[x]$ such that $p = p_1^2 + p_2^2 + \cdots + p_r^2$. The set of all SOS polynomials is denoted as $\Sigma[x]$. For a given degree m, denote

$$\Sigma[x]_m := \Sigma[x] \cap \mathbb{R}[x]_m.$$

The *quadratic module* generated by a polynomial tuple $g = (g_1, \ldots, g_t)$ is

$$Q(g) := \Sigma[x] + g_1 \cdot \Sigma[x] + \cdots + g_t \cdot \Sigma[x].$$

The k-th truncation of the quadratic module $Q(g)$ is the set

$$Q_k(g) := \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{2k-\deg(g_1)} + \cdots + g_t \cdot \Sigma[x]_{2k-\deg(g_t)}.$$

Clearly, it holds that

$$I(h) = \bigcup_{k \geq 1} I_k(h), \quad Q(g) = \bigcup_{k \geq 1} Q_k(g).$$

If the tuple g is empty, then $Q(g) = \Sigma[x]$ and $Q_k(g) = \Sigma[x]_{2k}$.

Let $\mathbb{N}$ be the set of nonnegative integers. For $x := (x_1, \ldots, x_n)$, $\alpha := (\alpha_1, \ldots, \alpha_n)$ and a degree d, denote

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}.$$

Denote by $\mathbb{R}^{\mathbb{N}^n_d}$ the space of all vectors y that are indexed by $\alpha \in \mathbb{N}^n_d$. For $y \in \mathbb{R}^{\mathbb{N}^n_d}$, we can write it as

$$y = (y_\alpha)_{\alpha \in \mathbb{N}^n_d}.$$

For $f = \sum_{\alpha \in \mathbb{N}^n_d} f_\alpha x^\alpha \in \mathbb{R}[x]_d$ and $y \in \mathbb{R}^{\mathbb{N}^n_d}$, we define the operation

$$\langle f, y \rangle := \sum_{\alpha \in \mathbb{N}^n_d} f_\alpha y_\alpha. \tag{2.1}$$

For every fixed $f \in \mathbb{R}[x]_d$, $\langle f, y\rangle$ is a linear functional in y; every fixed $y \in \mathbb{R}^{\mathbb{N}^n_d}$, $\langle f, y\rangle$ is a linear functional in f. The operation $\langle \cdot, \cdot \rangle$ gives a dual relationship between the polynomial space $\mathbb{R}[x]_d$ and the vector space $\mathbb{R}^{\mathbb{N}^n_d}$. In $\langle f, y\rangle$, the polynomial f is in the left side, while the vector y is in the right. For an integer $t \le d$ and $y \in \mathbb{R}^{\mathbb{N}^n_d}$, denote the t-th truncation of y as

$$y|_t := (y_\alpha)_{\alpha \in \mathbb{N}^n_t}. \tag{2.2}$$

Let $q \in \mathbb{R}[x]$ with $\deg(q) \le 2k$. For each $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$, $\langle qp^2, y\rangle$ is a quadratic form in $\mathrm{vec}(p)$, the coefficient vector of the polynomial p, with $\deg(qp^2) \le 2k$. Let $L^{(k)}_q(y)$ be the symmetric matrix such that

$$\langle qp^2, y\rangle = \mathrm{vec}(p)^T \left(L^{(k)}_q(y) \right) \mathrm{vec}(p). \tag{2.3}$$

The matrix $L^{(k)}_q(y)$ is called the k-th localizing matrix of q generated by y. It is linear in y. For instance, when $n = 2$, $k = 2$ and $q = x_1x_2 - x_1^2 - x_2^2$,

$$L^{(2)}_{x_1x_2 - x_1^2 - x_2^2}(y) = \begin{pmatrix} y_{11} - y_{20} - y_{02} & y_{21} - y_{30} - y_{12} & y_{12} - y_{21} - y_{03} \\ y_{21} - y_{30} - y_{12} & y_{31} - y_{40} - y_{22} & y_{22} - y_{31} - y_{13} \\ y_{12} - y_{21} - y_{03} & y_{22} - y_{31} - y_{13} & y_{13} - y_{22} - y_{04} \end{pmatrix}.$$

If $q = (q_1, \ldots, q_r)$ is a tuple of polynomials, we then define

$$L^{(k)}_q(y) := \begin{bmatrix} L^{(k)}_{q_1}(y) & & \\ & \ddots & \\ & & L^{(k)}_{q_r}(y) \end{bmatrix}.$$

It is a block diagonal matrix. When $q = 1$ (the constant 1 polynomial), $L^{(k)}_1(y)$ is called the k-th moment matrix generated by y, and we denote

$$M_k(y) := L^{(k)}_1(y). \tag{2.4}$$

For instance, when $n = 2$ and $k = 2$,

$$M_2(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix}.$$

For a degree d, denote the monomial vector

$$[x]_d := \left[1,\ x_1,\ \ldots,\ x_n,\ x_1^2,\ x_1x_2,\ \ldots,\ x_n^2,\ \ldots,\ x_1^m,\ \ldots,\ x_n^m\right]^T. \tag{2.5}$$

As shown in Example 1.4, there may be infinitely many real $\mathbb{Z}$-eigenvalues for some tensors. But this is not the case for a generic tensor. In (1.3), a real $\mathbb{Z}$-eigenpair (λ, u) of a tensor $\mathcal{A}$ is called *isolated* if there exists $\varepsilon > 0$ such that no other real $\mathbb{Z}$-eigenpair (μ, v) satisfies

$$|\lambda - \mu| + \|u - v\| < \varepsilon.$$

Similarly, a real $\mathbb{Z}$-eigenvalue λ is called *isolated* if there exists $\varepsilon > 0$ such that no other real $\mathbb{Z}$-eigenvalue μ satisfies $|\lambda - \mu| < \varepsilon$. In practice, we need to check whether a $\mathbb{Z}$-eigenvalue is isolated or not. Denote the function

$$F(\lambda, x) := \begin{bmatrix} x^T x - 1 \\ \mathcal{A}x^{m-1} - \lambda x \end{bmatrix}. \tag{2.6}$$

Clearly, (λ, u) is a $\mathbb{Z}$-eigenpair if and only if $F(\lambda, u) = 0$. Let $J(\lambda, x)$ be the Jacobian matrix of the vector function $F(\lambda, x)$ with respect to (λ, x).

Lemma 2.1 *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and λ be a real $\mathbb{Z}$-eigenvalue of $\mathcal{A}$.*

(i) *If u is a real $\mathbb{Z}$-eigenvector associated to λ and $J(\lambda, u)$ is nonsingular, then (λ, u) is isolated.*
(ii) *If $u_1, \ldots, u_N$ are the all real $\mathbb{Z}$-eigenvectors of $\mathcal{A}$ associated to λ and each $J(\lambda, u_i)$ is nonsingular, then λ is an isolated real $\mathbb{Z}$-eigenvalue of $\mathcal{A}$.*

Lemma 2.1 follows directly from the Inverse Function Theorem. For cleanness of the paper, its proof is omitted here.

2.2 Motivation: application in higher order Markov chain

In higher order Markov chains [19,20], an mth order Markov chain fits observed data by an m-order nonsymmetric tensor $\mathcal{P} \in \mathbb{T}^m(\mathbb{R}^n)$ whose entries are given such that

$$\begin{cases} \mathcal{P}_{i_1 i_2 \ldots i_m} = \text{Prob}(X_t = i_1 |\ X_{t-1} = i_2; \cdots; X_{t-m+1} = i_m) \in [0, 1], \\ \sum\limits_{i_1=1}^{n} \mathcal{P}_{i_1 i_2 \ldots i_m} = 1, \ \ \forall\ i_2, \ldots, i_m \in \{1, \ldots, n\}. \end{cases}$$

Such $\mathcal{P}$ is called a transition probability tensor. A vector $v \in \mathbb{R}^n$ is said to be a limiting (or stationary) probability distribution for $\mathcal{P}$ if

$$\mathcal{P}v^{m-1} = v, \quad e^T v = 1, \quad v \geq 0,$$

where e is the vector of all ones. The existence and uniqueness of stationary probability distributions were discussed in [19]. Earlier work on this subject can be found in [13,19,20]. Interestingly, the stationary probability distributions can be obtained by computing the Z-eigenvalue of nonsymmetric tensor $\mathcal{P}$. This can be seen as follows.

Suppose v is a stationary probability distribution of $\mathcal{P}$. Let $u = v/\|v\|$, then

$$u^T u = 1, \quad u \geq 0, \quad 1 = e^T v = \|v\|(e^T u), \quad v = u/(e^T u),$$
$$\mathcal{P}u^{m-1} = \frac{1}{\|v\|^{m-1}}\mathcal{P}v^{m-1} = (e^T u)^{m-1}v = (e^T u)^{m-2}u.$$

That is, u is a Z-eigenvector of $\mathcal{P}$ associated to the Z-eigenvalue $(e^T u)^{m-2}$. Conversely, if λ is a Z-eigenvalue with the Z-eigenvector $u \geq 0$, then

$$\mathcal{P}u^{m-1} = \lambda u, \quad u^T u = 1,$$
$$\lambda(e^T u) = e^T(\mathcal{P}u^{m-1}) = \sum_{1\leq i_1,i_2,\ldots,i_m\leq n} \mathcal{P}_{i_1 i_2 \ldots i_m} u_{i_2} \cdots u_{i_m} =$$
$$\sum_{1\leq i_2,\ldots,i_m\leq n} u_{i_2} \cdots u_{i_m} \sum_{i_1=1}^{n} \mathcal{P}_{i_1 i_2 \ldots i_m} = \sum_{1\leq i_2,\ldots,i_m\leq n} u_{i_2} \cdots u_{i_m} = (e^T u)^{m-1}.$$

So, $\lambda = (e^T u)^{m-2}$. If we let $v = \frac{u}{e^T u}$, then

$$\mathcal{P}v^{m-1} = \frac{1}{(e^T u)^{m-1}}\mathcal{P}u^{m-1} = \frac{\lambda u}{(e^T u)^{m-1}} = v, \quad e^T v = 1, \quad v \geq 0.$$

That is, v is a stationary probability distribution vector. The above can be summarized as follows:

- If v is a stationary probability distribution vector for $\mathcal{P}$, then $u = \frac{v}{\|v\|}$ is a nonnegative Z-eigenvector.
- If u is a nonnegative Z-eigenvector for $\mathcal{P}$, then $v = \frac{u}{e^T u}$ is a stationary probability distribution vector.

Therefore, the stationary probability distribution vectors can be obtained by computing all nonnegative Z-eigenvectors.

3 Computing real Z-eigenvalues

In this section, we first reformulate Z-eigenvalue computation as polynomial optimization problem.

Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ be a tensor. A real pair (λ, u) is a Z-eigenpair of $\mathcal{A}$ if and only if $\mathcal{A}u^{m-1} = \lambda u$ and $u^T u = 1$. So,

$$\lambda = \lambda u^T u = u^T \mathcal{A}u^{m-1} = \mathcal{A}u^m.$$

Hence, u is a Z-eigenvector if and only if

$$\mathcal{A}u^{m-1} = (\mathcal{A}u^m)u, \quad u^T u = 1,$$

and the associated Z-eigenvalue is $\mathcal{A}u^m$. A generic tensor has finitely many Z-eigenvalues that are all simple, as shown in [3]. For special tensors, there might be infinitely many ones (see Example 1.4).

In this section, we aim at computing all real Z-eigenvalues when there are finitely many ones. Let h be the polynomial tuple:

$$h = \big(\mathcal{A}x^{m-1} - (\mathcal{A}x^m)x,\ x^T x - 1\big). \tag{3.1}$$

Then, u is a Z-eigenvector of $\mathcal{A}$ if and only if $h(u) = 0$. Let $Z(\mathbb{R}, \mathcal{A})$ denote the set of Z-eigenvalues of $\mathcal{A}$. If it is a finite set, we list $Z(\mathbb{R}, \mathcal{A})$ monotonically as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N.$$

We aim at computing them sequentially, from the smallest to the largest.

3.1 The smallest Z-eigenvalue

To compute the smallest Z-eigenvalue λ_1, we consider the polynomial optimization problem

$$\min \quad f(x) := \mathcal{A}x^m \quad \text{s.t.} \quad h(x) = 0, \tag{3.2}$$

where h is as in (3.1). Note that u is a Z-eigenvector if and only if $h(u) = 0$, with the Z-eigenvalue $f(u)$. The optimal value of (3.2) is λ_1, if $\mathcal{V}_{\mathbb{R}}(h) \neq \emptyset$. Let

$$k_0 = \lceil (m+1)/2 \rceil. \tag{3.3}$$

Lasserre's hierarchy [16] of semidefinite relaxations for solving (3.2) is

$$\begin{cases} f_1^{1,k} := \min \langle f, y \rangle \\ \qquad \text{s.t. } \langle 1, y \rangle = 1,\ L_h^{(k)}(y) = 0, \\ \qquad\quad M_k(y) \succeq 0,\ y \in \mathbb{R}^{\mathbb{N}^n_{2k}}, \end{cases} \tag{3.4}$$

for the orders $k = k_0, k_0 + 1, \ldots$. See (2.3)–(2.4) for the notation $L_h^{(k)}(y)$ and $M_k(y)$. In the above, $X \succeq 0$ means that the symmetric matrix X is positive semidefinite. The dual optimization problem of (3.4) is

$$\begin{cases} f_1^{2,k} := \max \gamma \\ \quad \text{s.t. } f - \gamma \in I_{2k}(h) + \Sigma[x]_{2k}. \end{cases} \tag{3.5}$$

As in [16], it can be shown that for all k

$$f_1^{2,k} \le f_1^{1,k} \le \lambda_1$$

and the sequences $\{f_1^{1,k}\}$ and $\{f_1^{2,k}\}$ are monotonically increasing.

Theorem 3.1 *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and let $Z(\mathbb{R}, \mathcal{A})$ be the set of its real* Z*-eigenvalues. Then, we have the properties:*

(i) *The set $Z(\mathbb{R}, \mathcal{A}) = \emptyset$ if and only if the semidefinite relaxation (3.4) is infeasible for some order k.*
(ii) *If $Z(\mathbb{R}, \mathcal{A}) \neq \emptyset$ and λ_1 is the smallest real* Z*-eigenvalue, then*

$$\lim_{k\to\infty} f_1^{2,k} = \lim_{k\to\infty} f_1^{1,k} = \lambda_1. \tag{3.6}$$

If, in addition, $Z(\mathbb{R}, \mathcal{A})$ is a finite set, then for all k sufficiently big

$$f_1^{2,k} = f_1^{1,k} = \lambda_1. \tag{3.7}$$

(iii) *Let k_0 be as in (3.3). Suppose y^* is a minimizer of (3.4). If there exists $t \le k$ such that*

$$\text{rank } M_{t-k_0}(y^*) = \text{rank } M_t(y^*), \tag{3.8}$$

then $\lambda_1 = f_1^{1,k}$ and there are $r := \text{rank } M_t(y^*)$ distinct real Z*-eigenvectors $u_1, \ldots, u_r$ associated to λ_1.*
(iv) *Suppose $Z(\mathbb{R}, \mathcal{A})$ is a finite set. If there are finitely many real* Z*-eigenvectors associated to λ_1, then, for all k big enough and for every minimizer y^* of (3.4), the condition (3.8) is satisfied for some $t \le k$.*

Proof (i) We prove the equivalence in two directions.

Sufficiency: Assume (3.4) is infeasible for some k. Then $\mathcal{A}$ has no real Z-eigenpairs. Suppose otherwise (λ, u) is such a one. Then $[u]_{2k}$ [see (2.5) for the notation] is feasible for (3.4) for all values of k, which is a contradiction. So $Z(\mathbb{R}, \mathcal{A}) = \emptyset$.

Necessity: Assume $Z(\mathbb{R}, \mathcal{A}) = \emptyset$. Then the equation $h(x) = 0$ has no real solutions. By the Positivstellensatz [2], $-1 \in I(h) + \Sigma[x]$. So, when k is big enough, $-1 \in I_{2k}(h) + \Sigma[x]_{2k}$, and then the optimization (3.5) is unbounded from above. By duality theory, (3.4) must be infeasible, for all k big enough.

(ii) Note that $x^T x - 1$ is a polynomial in the tuple h. So, $-(x^T x - 1)^2 \in I(h)$ and the set $-(x^T x - 1)^2 \ge 0$ is compact. The ideal $I(h)$ is archimedean [16]. The asymptotic convergence (3.6) is proved in Theorem 4.2 of [16].

Next, we prove the finite convergence (3.7) when $Z(\mathbb{R}, \mathcal{A}) \neq \emptyset$ is a finite set. Write $Z(\mathbb{R}, \mathcal{A}) = \{\lambda_1, \ldots, \lambda_N\}$, with $\lambda_1 < \cdots < \lambda_N$. Let $b_1, \ldots, b_N \in \mathbb{R}[t]$ be the

univariate real polynomials in t such that $b_i(\lambda_j) = 0$ when $i \neq j$ and $b_i(\lambda_j) = 1$ when $i = j$. For $i = 1, \ldots, N$, let

$$s_i := (\lambda_i - \lambda_1)\Big(b_i(f(x))\Big)^2.$$

Let $s := s_1 + \cdots + s_N$. Then, $s \in \Sigma[x]_{2k_1}$ for some $k_1 > 0$. The polynomial

$$\hat{f} := f - \lambda_1 - s$$

vanishes identically on $\mathcal{V}_{\mathbb{R}}(h)$. By the Real Nullstellensatz [2, Corollary 4.1.8], there exist an integer $\ell > 0$ and $q \in \Sigma[x]$ such that

$$\hat{f}^{2\ell} + q \in I(h).$$

For all $\varepsilon > 0$ and $c > 0$, we can write $\hat{f} + \varepsilon = \phi_\varepsilon + \theta_\varepsilon$, where

$$\begin{aligned}\phi_\varepsilon &= -c\varepsilon^{1-2\ell}(\hat{f}^{2\ell} + q),\\ \theta_\varepsilon &= \varepsilon\Big(1 + \hat{f}/\varepsilon + c(\hat{f}/\varepsilon)^{2\ell}\Big) + c\varepsilon^{1-2\ell}q.\end{aligned}$$

By Lemma 2.1 of [25], when $c \geq \frac{1}{2\ell}$, there exists $k_2 \geq k_1$ such that, for all $\varepsilon > 0$,

$$\phi_\varepsilon \in I_{2k_2}(h), \quad \theta_\varepsilon \in \Sigma[x]_{2k_2}.$$

Hence, we can get

$$f - (\lambda_1 - \varepsilon) = \phi_\varepsilon + \sigma_\varepsilon,$$

where $\sigma_\varepsilon = \theta_\varepsilon + s \in \Sigma[x]_{2k_2}$ for all $\varepsilon > 0$. This implies that, for all $\varepsilon > 0$, $\gamma = \lambda_1 - \varepsilon$ is feasible in (3.5) for the order k_2. Thus, we get $f_1^{2,k_2} \geq \lambda_1$. Note that $f_1^{2,k} \leq f_1^{1,k} \leq \lambda_1$ for all k and the sequence $\{f_1^{2,k}\}$ is monotonically increasing. So, (3.7) must be true when $k \geq k_2$.

(iii) Note that $M_t(y^*) \succeq 0$ and $L_h^{(t)}(y^*) = 0$, because $t \leq k$. When (3.8) is satisfied, by Theorem 1.1 of [6], there exist $r := \operatorname{rank} M_t(y^*)$ vectors $u_1, \ldots, u_r \in \mathcal{V}_{\mathbb{R}}(h)$ such that (we refer to (2.5) for the notation $[u_i]_{2t}$)

$$y^*|_{2t} = c_1[u_1]_{2t} + \cdots + c_r[u_r]_{2t},$$

with numbers $c_1, \ldots, c_r > 0$. The condition $\langle 1, y^* \rangle = 1$ implies that

$$c_1 + \cdots + c_r = 1.$$

By the notation $\langle \cdot, \cdot \rangle$ as in (2.1), we can see that $\langle f, [u_i]_{2k} \rangle = f(u_i)$, so

$$f_1^{1,k} = \langle f, y^* \rangle = c_1 f(u_1) + \cdots + c_r f(u_r),$$

$$f_1^{1,k} \le f(u_i) \quad i = 1, \ldots, r,$$

because each $[u_i]_{2k}$ is a feasible point for (3.4). Thus, we must have

$$f_1^{1,k} = f(u_1) = \cdots = f(u_r).$$

Also note that $f_1^{1,k} \le \lambda_1$ and each u_i is a Z-eigenvector. So,

$$f_1^{1,k} = f(u_1) = \cdots = f(u_r) = \lambda_1.$$

(iv) When $Z(\mathbb{R}, \mathcal{A})$ is a finite set, both $\{f_1^{1,k}\}$ and $\{f_1^{2,k}\}$ have finite convergence to λ_1, by item (iii). If (3.2) has finitely many minimizers, i.e., λ_1 has finitely many Z-eigenvectors, the rank condition (3.8) must be satisfied when k is sufficiently big. The conclusion follows from Theorem 2.6 of [27]. □

In computation, we start to solve the semidefinite relaxation (3.4) from the smallest value of k, which is k_0 as in (3.3). If (3.4) is infeasible for $k = k_0$, then $\mathcal{A}$ has no real Z-eigenvalues; if (3.4) is feasible for $k = k_0$, then solve it for an optimizer y^*. If y^* satisfies (3.8), then we get $\lambda_1 = f_1^{1,k}$; otherwise, increase the value of k by one, and then repeat the above process. See Algorithm 3.6 for the implementation details. The tensor $\mathcal{A}$ in Example 1.3 has no real Z-eigenvalues. This is confirmed by that the relaxation (3.4) is infeasible for $k = 3$.

Remark 3.2 (1) The rank condition (3.8) can be used as a criterion to check whether $f_1^{1,k} = \lambda_1$ or not. If it is satisfied, then we can get r distinct minimizers $u_1, \ldots, u_r$ of (3.2), i.e., each u_i is Z-eigenvector of $\mathcal{A}$. The vectors u_i can be computed by the method in [10], which is implemented in `GloptiPoly 3` [11].

(2) Suppose (3.8) holds. If rank $M_k(y^*)$ is maximum among the set of all optimizers of (3.4), then we can get all mimizers of (3.2) [18, §6.6], i.e., we can get all real Z-eigenvectors associated to λ_1. When (3.4)–(3.5) are solved by primal-dual interior point methods, typically we can get all real Z-eigenvectors associated to λ_1, provided there are finitely many.

(3) The condition (3.8) requires to know the ranks of $M_{t-k_0}(y^*)$ and $M_t(y^*)$. It would be hard to evaluate them when the matrices are singular or close to be singular. The matrix rank is equal to the number of positive singular values. In common practice, people often determine the rank by counting the number of singular values bigger than a tolerance (say, 10^{-6}). This is a classical problem in numerical linear algebra. We refer to Demmel's book [8] for this question.

(4) For generic polynomial optimization problems, Lasserre's hierarchies of semidefinite relaxations have finite convergence, as shown in [28]. However, the problem (3.2) is not generic, because h depends on f. So, the finite convergence of (3.4) cannot be proved by using the results of [28].

3.2 Larger z-eigenvalues

Suppose the ith smallest Z-eigenvalue λ_i is known. We want to determine whether the next larger one λ_{i+1} exists or not. If it exists, we show how to compute it; if it does not, we give a certificate for the nonexistence. Let $\delta > 0$ be a small number. Consider the problem

$$\min \quad f(x) \quad \text{s.t.} \quad h(x) = 0, \quad f(x) \geq \lambda_i + \delta. \tag{3.9}$$

Clearly, the optimal value of (3.9) is the smallest Z-eigenvalue that is greater than or equal to $\lambda_i + \delta$. Let

$$\omega(\lambda_i + \delta) := \min\{\lambda \in Z(\mathbb{R}, \mathcal{A}) : \lambda \geq \lambda_i + \delta\}. \tag{3.10}$$

Lasserre's hierarchy of semidefinite relaxations for solving (3.9) is

$$\begin{cases} f_{i+1}^{1,k} := \min \langle f, y\rangle \\ \qquad \text{s.t. } \langle 1, y\rangle = 1,\ L_h^{(k)}(y) = 0, \\ \qquad\quad M_k(y) \succeq 0,\ L_{f-\lambda_i-\delta}^{(k)}(y) \succeq 0,\ y \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases} \tag{3.11}$$

for the orders $k = k_0, k_0 + 1, \ldots$. The dual problem of (3.11) is

$$\begin{cases} f_{i+1}^{2,k} := \max \gamma \\ \qquad \text{s.t. } f - \gamma \in I_{2k}(h) + Q_k(f - \lambda_i - \delta). \end{cases} \tag{3.12}$$

Similar to Theorem 3.1, we have the following convergence result.

Theorem 3.3 *Let $\mathcal{A} \in \mathrm{T}^m(\mathbb{R}^n)$ and $Z(\mathbb{R}, \mathcal{A})$ be the set of real Z-eigenvalues of $\mathcal{A}$. Suppose $\lambda_i \in Z(\mathbb{R}, \mathcal{A})$. Then, we have the properties:*

(i) *The intersection $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \emptyset$ if and only if the semidefinite relaxation (3.11) is infeasible for some order k.*

(ii) *If $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) \neq \emptyset$, then*

$$\lim_{k\to\infty} f_{i+1}^{2,k} = \lim_{k\to\infty} f_{i+1}^{1,k} = \omega(\lambda_i + \delta). \tag{3.13}$$

If, in addition, $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty)$ is a finite set, then

$$f_{i+1}^{2,k} = f_{i+1}^{1,k} = \omega(\lambda_i + \delta) \tag{3.14}$$

for all k sufficiently big.

(iii) *Suppose y^* is a minimizer of (3.11). If (3.8) is satisfied for some $t \leq k$, then $\omega(\lambda_i + \delta) = f_{i+1}^{1,k}$ and there are $r := \operatorname{rank} M_t(y^*)$ distinct real Z-eigenvectors $u_1, \ldots, u_r$, associated to $\omega(\lambda_i + \delta)$.*

(iv) *Suppose $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty)$ is a finite set and $\omega(\lambda_i + \delta)$ has finitely many real Z-eigenvectors. Then, for all k big enough, and for every minimizer y^* of (3.11), there exists $t \leq k$ satisfying (3.8).*

Proof The proof is mostly the same as for Theorem 3.1. In the following, we only list the major differences.

(i) Necessity: If $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \emptyset$, then (3.9) is infeasible. By the Positivstellensatz,

$$-1 \in I(h) + Q(f - \lambda_i - \delta).$$

The rest of the proof is the same as for Theorem 3.1(i).

(ii) The ideal $I(h)$ is archimedean, and so is $I(h) + Q(f - \lambda_i - \delta)$. The asymptotic convergence (3.13) can be implied by Theorem 4.2 of [16]. To prove the finite convergence (3.14), we follow the same proof as for Theorem 3.1(ii). Suppose $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \{v_1, \ldots, v_L\}$. Construct the polynomial s same as there and let

$$\hat{f} = f - \omega(\lambda_i + \delta) - s.$$

Then $\hat{f}$ vanishes identically on the set

$$\{x \in \mathbb{R}^n : h(x) = 0, f(x) - \lambda_i - \delta \geq 0\}.$$

By the Real Nullstellensatz [2, Corollary 4.1.8], there exist an integer $\ell > 0$ and $q \in Q(f - \lambda_i - \delta)$ such that $\hat{f}^{2\ell} + q \in I(h)$. The rest of the proof is the same, except replacing $\Sigma[x]$ by $Q(f - \lambda_i - \delta)$, and $\Sigma[x]_{2k}$ by $Q_k(f - \lambda_i - \delta)$.

(iii)–(iv) The proof is the same as for Theorem 3.1(iii)–(iv). □

The convergence of semidefinite relaxations (3.11)–(3.12) can be checked by the condition (3.8). When it is satisfied, the real Z-eigenvectors $u_1, \ldots, u_r$ can be computed by the method in [10]. Typically, we can get all real Z-eigenvectors if primal-dual interior-point methods are used to solve the semidefinite relaxations. We refer to Remark 3.2.

Next, we show how to use $\omega(\lambda_i + \delta)$ to determine λ_{i+1}. Assume λ_i is isolated (otherwise, there are infinitely many Z-eigenvalues). If λ_i is the largest real Z-eigenvalue, then $\lambda_1, \ldots, \lambda_i$ are the all real Z-eigenvalues and we can stop; otherwise, the next larger one λ_{i+1} exists. For such case, if $\delta > 0$ in (3.9) is small enough, then $\omega(\lambda_i + \delta)$ as in (3.10) equals λ_{i+1}. Consider the optimization problem

$$\begin{cases} v^+(\lambda_i, \delta) := \max\ f(x) \\ \qquad\qquad \text{s.t. } h(x) = 0,\ f(x) \leq \lambda_i + \delta. \end{cases} \tag{3.15}$$

The optimal value of (3.15) is the largest Z-eigenvalue of $\mathcal{A}$ that is smaller than or equal to $\lambda_i + \delta$, i.e.,

$$v^+(\lambda_i, \delta) = \max\{\lambda \in \mathrm{Z}(\mathbb{R}, \mathcal{A}) : \lambda \leq \lambda_i + \delta\}.$$

The next larger Z-eigenvalue λ_{i+1} can be determined as follows.

Theorem 3.4 *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $\delta > 0$. Assume λ_i is an isolated Z-eigenvalue of $\mathcal{A}$ and λ_{max} is the largest one. Then, we have the properties:*

(i) *For all $\delta > 0$ sufficiently small, $\nu^+(\lambda_i, \delta) = \lambda_i$.*
(ii) *If $\nu^+(\lambda_i, \delta) = \lambda_i$ and (3.11) is infeasible for some k, then $\lambda_i = \lambda_{max}$ and the next larger Z-eigenvalue λ_{i+1} does not exist.*
(iii) *If $\nu^+(\lambda_i, \delta) = \lambda_i$ and the condition (3.8) is satisfied for some k, then the next larger Z-eigenvalue $\lambda_{i+1} = f_{i+1}^{1,k}$.*

Proof (i) Note that $\nu^+(\lambda_i, \delta)$ is the smallest Z-eigenvalue greater than or equal to $\lambda_i + \delta$. When λ_i is isolated, for $\delta > 0$ sufficiently small, we must have $\nu^+(\lambda_i, \delta) = \lambda_i$.

(ii) When (3.11) is infeasible for some k, by Theorem 3.3(i), all the Z-eigenvalues are smaller than $\lambda_i + \delta$. Note that $\nu^+(\lambda_i, \delta)$ is the largest Z-eigenvalue that is smaller than or equal to $\lambda_i + \delta$. If $\lambda_i = \nu^+(\lambda_i, \delta)$, then λ_i must be the largest Z-eigenvalue, i.e., $\lambda_i = \lambda_{max}$.

(iii) When (3.8) is satisfied for some k, by Theorem 3.3(iii), we know

$$\omega(\lambda_i + \delta) = f_{i+1}^{1,k}.$$

Note that $\nu^+(\lambda_i, \delta)$ is the largest Z-eigenvalue that is smaller than or equal to $\lambda_i + \delta$, while $\omega(\lambda_i + \delta)$ is the smallest Z-eigenvalue that is bigger than or equal to $\lambda_i + \delta$. Since $\lambda_i = \nu^+(\lambda_i, \delta)$ and λ_i is isolated, we must have

$$\lambda_{i+1} = \omega(\lambda_i + \delta),$$

which is the smallest Z-eigenvalue bigger than λ_i. □

In the next, we show how to check if a real Z-eigenvalue λ_i is isolated or not. Lemma 2.1 can be used to verify the isolatedness. It only gives a sufficient condition, which may not be necessary sometimes. Here, we give a sufficient and necessary condition. Consider the optimization problem

$$\begin{cases} \nu^-(\lambda_i, \delta) := \min\ f(x) \\ \qquad\qquad \text{s.t. } h(x) = 0,\ f(x) \geq \lambda_i - \delta. \end{cases} \tag{3.16}$$

Clearly, for all $\delta > 0$, it holds that

$$\nu^-(\lambda_i, \delta) \leq \lambda_i \leq \nu^+(\lambda_i, \delta).$$

Lemma 3.5 *Let $\nu^+(\lambda_i, \delta)$, $\nu^-(\lambda_i, \delta)$ be the optimal values as in (3.15) and (3.16). Then, λ_i is an isolated real Z-eigenvalue of $\mathcal{A}$ if and only if for some $\delta > 0$*

$$\nu^+(\lambda_i, \delta) = \nu^-(\lambda_i, \delta). \tag{3.17}$$

When the above holds, λ_i is the unique Z-eigenvalue of $\mathcal{A}$ in $[\lambda_i - \delta, \lambda_i + \delta]$.

Proof By the construction of h as in (3.1), u is a Z-eigenvector if and only if $h(u) = 0$. So, $\nu^+(\lambda_i, \delta)$ is the largest Z-eigenvalue that is smaller than or equal to $\lambda_i + \delta$, while $\nu^-(\lambda_i, \delta)$ is the smallest one that is greater than or equal to $\lambda_i - \delta$. We prove the equivalence in two directions.

Necessity: If λ_i is isolated, then for $\delta > 0$ small enough, λ_i is the unique Z-eigenvalue of $\mathcal{A}$ in the interval $[\lambda_i - \delta, \lambda_i + \delta]$. So, (3.17) holds.

Sufficiency: Assume (3.17) holds for some $\delta > 0$, then $\nu^-(\lambda_i, \delta) = \lambda_i = \nu^+(\lambda_i, \delta)$. So, λ_i is the unique Z-eigenvalue in $[\lambda_i - \delta, \lambda_i + \delta]$, and it must be isolated. □

The problems (3.15) and (3.16) are polynomial optimization. Their optimal values $\nu^+(\lambda_i, \delta), \nu^-(\lambda_i, \delta)$ can be computed by solving Lasserre type semidefinite relaxations that are similar to (3.11)–(3.12).

3.3 An algorithm for computing real Z-eigenvalues

For a given tensor $\mathcal{A}$, we compute its real Z-eigenvalues if they exist, from the smallest to the largest. First, we compute λ_1 if it exists, by solving (3.4)–(3.5). After getting λ_1, we solve (3.11)–(3.12) and then determine λ_2. If λ_2 does not exist, we stop; otherwise, we then determine λ_3. Repeating this procedure, we can get all the real Z-eigenvalues, when there are finitely many ones. This results in the following algorithm.

Algorithm 3.6 *For a given tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$, compute its real Z-eigenvalues as follows:*

Step 0: *Let $k := k_0$, with k_0 as in (3.3).*

Step 1: *Solve the relaxation (3.4). If it is infeasible, then $\mathcal{A}$ has no real Z-eigenvalues and stop; if it is feasible, compute an optimizer y^*.*

Step 2: *If (3.8) is satisfied, then $\lambda_1 = f_1^{1,k}$ and go to Step 3 with $i = 1$. Otherwise, let $k := k + 1$ and go to Step 1.*

Step 3: *Let $\delta = 0.05$. Solve (3.15), (3.16) for the optimal values $\nu^+(\lambda_i, \delta)$, $\nu^-(\lambda_i, \delta)$. If $\nu^+(\lambda_i, \delta) = \nu^-(\lambda_i, \delta)$, then λ_i is an isolated Z-eigenvalue and go to Step 4. Otherwise, let $\delta := \delta/5$ and compute $\nu^+(\lambda_i, \delta)$, $\nu^-(\lambda_i, \delta)$ again. Repeat this process until we get $\nu^+(\lambda_i, \delta) = \nu^-(\lambda_i, \delta)$.*

Step 4: *Let $k := k_0$, with k_0 as in (3.3).*

Step 5: *Solve the relaxation (3.11). If it is infeasible, the largest Z-eigenvalue is λ_i and stop. Otherwise, compute an optimizer y^* for it.*

Step 6: *If (3.8) is satisfied, then $\lambda_{i+1} = f_{i+1}^{1,k}$ and go to Step 3 with $i := i + 1$. Otherwise, let $k := k + 1$ and go to Step 5.*

The correctness of Algorithm 3.6 and its convergence properties are proved in Theorems 3.1, 3.3 and 3.4.

4 Computing real H-eigenvalues

In this section, we compute real H-eigenvalues. For every tensor $\mathcal{A}$, the number of H-eigenvalues is always finite. In Definition 1.2, if λ, u are allowed to achieve com-

plex values, then we call such λ a complex H-eigenvalue and such u a complex H-eigenvector.

Proposition 4.1 *Every tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{C}^n)$ has $n(m-1)^{n-1}$ complex H-eigenvalues, counting their multiplicities.*

Proof Let $\mathcal{I} \in \mathbb{T}^m(\mathbb{C}^n)$ be the identity tensor whose only non-zero entries are $\mathcal{I}_{ii\ldots i} = 1$ for $i = 1, 2, \ldots, n$. Note that λ is a complex H-eigenvalue if and only if there exists $0 \neq u \in \mathbb{C}^n$ such that $\mathcal{A}u^{m-1} = \lambda u^{[m-1]}$, that is, $(\mathcal{A} - \lambda\mathcal{I})u^{m-1} = 0$. By the definition of resultant [39], which we denote by Res, λ is an H-eigenvalue if and only if

$$\mathrm{Res}\left((\mathcal{A} - \lambda\mathcal{I})x^{m-1}\right) = 0. \tag{4.1}$$

The resultant $\mathrm{Res}\left((\mathcal{A} - \lambda\mathcal{I})x^{m-1}\right)$ is homogeneous in the entries of $\mathcal{A}$ and λ. It has degree $D := n(m-1)^{n-1}$. We can expand it as

$$\mathrm{Res}\left((\mathcal{A} - \lambda\mathcal{I})x^{m-1}\right) = p_0(\mathcal{A}) + p_1(\mathcal{A})\lambda + \cdots + p_D(\mathcal{A})\lambda^D.$$

By the homogeneity of Res, we know

$$p_D(\mathcal{A}) = \mathrm{Res}\left(-\mathcal{I}x^{m-1}\right) \neq 0,$$

because the homogeneous polynomial system $-\mathcal{I}x^{m-1} = 0$ has no nonzero complex solutions. Hence, the leading coefficient of the polynomial $\mathrm{Res}\left((\mathcal{A} - \lambda\mathcal{I})x^{m-1}\right)$ in λ is not zero, and the degree is D. This implies that (4.1) has D complex solutions, counting multiplicities, and the lemma is proved. □

Recall that (λ, u) is a real H-eigenpair if and only if

$$\mathcal{A}u^{m-1} = \lambda u^{[m-1]}, \quad 0 \neq u \in \mathbb{R}^n.$$

Let m_0 be the largest even number less than or equal to m, i.e.,

$$m_0 = 2\lceil (m-1)/2 \rceil.$$

Note that $m - 1 \leq m_0 \leq m$. We can normalize u as

$$(u_1)^{m_0} + \cdots + (u_n)^{m_0} = 1. \tag{4.2}$$

Under this normalization, the H-eigenvalue λ can be given as

$$\lambda = \lambda(u^{[m_0-m+1]})^T u^{[m-1]} = (u^{[m_0-m+1]})^T \mathcal{A}u^{m-1}.$$

Let h be the polynomial tuple

$$h := \left(\mathcal{A}x^{m-1} - \left((x^{[m_0-m+1]})^T \mathcal{A}x^{m-1}\right)x^{[m-1]}, \sum_{i=1}^{n}(x_i)^{m_0} - 1\right). \tag{4.3}$$

Then, u is an H-eigenvector normalized as in (4.2) if and only if $h(u) = 0$. Since $\mathcal{A}$ has finitely many H-eigenvalues, we can order the real ones monotonically as

$$\mu_1 < \mu_2 < \cdots < \mu_N,$$

if at least one of them exists. We call μ_i the ith smallest H-eigenvalue.

4.1 The smallest H-eigenvalue

In this subsection, we show how to determine μ_1. Let h be as in (4.3), then μ_1 equals the optimal value of the optimization problem

$$\begin{cases} \min\ f(x) := (x^{[m_0-m+1]})^T \mathcal{A} x^{m-1} \\ \text{s.t. } h(x) = 0. \end{cases} \tag{4.4}$$

Lasserre's hierarchy [16] of semidefinite relaxations for solving (4.4) is

$$\begin{cases} \rho_1^{1,k} := \min\ \langle f, z \rangle \\ \qquad\quad \text{s.t. } \langle 1, z \rangle = 1,\ L_h^{(k)}(z) = 0, \\ \qquad\qquad M_k(z) \succeq 0,\ z \in \mathbb{R}^{\mathbb{N}^n_{2k}}, \end{cases} \tag{4.5}$$

for the orders $k = k_0, k_0 + 1, \ldots,$ where

$$k_0 := \lceil (m_0 + m - 1)/2 \rceil. \tag{4.6}$$

The dual optimization problem of (4.5) is

$$\begin{cases} \rho_1^{2,k} := \max\ \gamma \\ \qquad\quad \text{s.t. } f - \gamma \in I_{2k}(h) + \Sigma[x]_{2k}. \end{cases} \tag{4.7}$$

As can be shown in [16], $\rho_1^{2,k} \le \rho_1^{1,k} \le \mu_1$ for all k, and the sequences $\{\rho_1^{1,k}\}$ and $\{\rho_1^{2,k}\}$ are monotonically increasing.

Theorem 4.2 *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $H(\mathbb{R}, \mathcal{A})$ be the set of its real H-eigenvalues. Then, we have the properties:*

(i) *The set $H(\mathbb{R}, \mathcal{A}) = \emptyset$ if and only if the semidefinite relaxation (4.5) is infeasible for some order k.*

(ii) *If $H(\mathbb{R}, \mathcal{A}) \neq \emptyset$, then for all k sufficiently big*

$$\rho_1^{1,k} = \rho_1^{2,k} = \mu_1. \tag{4.8}$$

(iii) *Let k_0 be as in (4.6). Suppose z^* is a minimizer of (4.5). If there exists an integer $t \le k$ such that*

$$\text{rank } M_{t-k_0}(z^*) = \text{rank } M_t(z^*), \tag{4.9}$$

then $\mu_1 = \rho_1^{1,k}$ *and there are* $r := \operatorname{rank} M_t(z^*)$ *distinct real* H*-eigenvectors* $v_1, \ldots, v_r$ *associated to* μ_1 *and normalized as in* (4.2).

(iv) *Suppose* $\mathcal{A}$ *has finitely many real* H*-eigenvectors associated to* μ_1. *Then, for all* k *big enough and for every minimizer* z^* *of* (4.5), *there exists an integer* $t \le k$ *satisfying* (4.9).

Proof It can be proved in the same way as for Theorem 3.1. The only difference is that $H(\mathbb{R}, \mathcal{A})$ is always a finite set, by Proposition 4.1. To avoid being repetitive, the proof is omitted here. □

The tensor $\mathcal{A}$ in Example 1.3 has no real H-eigenvalues. This can be confirmed by Theorem 4.2(i), because the semidefinite relaxation (4.5) is infeasible for the order $k = 4$. The rank condition (4.9) is a criterion for checking the finite convergence in (4.8). When it is satisfied, the real H-eigenvectors $u_1, \ldots, u_r$ can be computed by the method in [10]. Typically, we can get all H-eigenvectors if primal-dual interior-point methods are used to solve (4.5). We refer to Remark 3.2.

4.2 Larger H-eigenvalues

Suppose the ith smallest H-eigenvalue μ_i is known. We want to determine the next larger one μ_{i+1}. If it exists, we show how to compute it; if not, we get a certificate for the nonexistence. Let $\delta > 0$ be a small number. Consider the optimization problem

$$\min \quad f(x) \quad \text{s.t.} \quad h(x) = 0, \quad f(x) \ge \mu_i + \delta, \tag{4.10}$$

where f, h are same as in (4.4). The optimal value of (4.10) is the smallest H-eigenvalue of $\mathcal{A}$ that is greater than or equal to $\mu_i + \delta$. Denote

$$\varpi(\mu_i + \delta) := \min\{\mu \in H(\mathbb{R}, \mathcal{A}) : \mu \ge \mu_i + \delta\}. \tag{4.11}$$

Lasserre's hierarchy of semidefinite relaxations for solving (4.10) is

$$\begin{cases} \rho_{i+1}^{1,k} := \min \langle f, z \rangle \\ \qquad\quad \text{s.t. } \langle 1, z \rangle = 1, L_h^{(k)}(z) = 0, \\ \qquad\quad\quad M_k(z) \succeq 0, \ L_{f-\mu_i-\delta}^{(k)}(z) \succeq 0, \ z \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases} \tag{4.12}$$

for the orders $k = k_0, k_0 + 1, \ldots$. The dual problem of (4.12) is

$$\begin{cases} \rho_{i+1}^{2,k} := \max \gamma \\ \qquad\quad \text{s.t. } f - \gamma \in I_{2k}(h) + Q_k(f - \mu_i - \delta). \end{cases} \tag{4.13}$$

The properties of relaxations (4.12)–(4.13) are as follows.

Theorem 4.3 *Let* $\mathcal{A} \in \mathrm{T}^m(\mathbb{R}^n)$ *and* $H(\mathbb{R}, \mathcal{A})$ *be the set of its real* H*-eigenvalues. Assume that* $\mu_i \in H(\mathbb{R}, \mathcal{A})$. *Then, we have the properties:*

(i) *The intersection* $H(\mathbb{R}, \mathcal{A}) \cap [\mu_i + \delta, +\infty) = \emptyset$ *if and only if the semidefinite relaxation* (4.12) *is infeasible for some order* k.
(ii) *If* $H(\mathbb{R}, \mathcal{A}) \cap [\mu_i + \delta, +\infty) \neq \emptyset$, *then for all* k *sufficiently big*

$$\rho_{i+1}^{2,k} = \rho_{i+1}^{1,k} = \varpi(\mu_i + \delta). \tag{4.14}$$

(iii) *Let* z^* *be a minimizer of* (4.12). *If* (4.9) *is satisfied for some* $t \leq k$, *then there exists* $r := \operatorname{rank} M_t(z^*)$ *real* H*-eigenvectors* $v_1, \ldots, v_r$ *that are associated to* $\varpi(\mu_i + \delta)$ *and that are normalized as in* (4.2).
(iv) *Suppose* $\mathcal{A}$ *has finitely many real* H*-eigenvectors that are associated to* $\varpi(\mu_i + \delta)$ *and that are normalized as in* (4.2). *Then, for all* k *big enough and for all minimizer* z^* *of* (4.12), *there exists* $t \leq k$ *satisfying* (4.9).

Proof It can be proved in the same way as for Theorem 3.3. Note that $H(\mathbb{R}, \mathcal{A})$ is always a finite set, by Proposition 4.1. □

In the following, we show how to use $\varpi(\mu_i + \delta)$ to determine μ_{i+1}. Consider the maximization problem

$$\begin{cases} \upsilon_i := \max\ f(x) \\ \quad \text{s.t. } h(x) = 0,\ f(x) \leq \mu_i + \delta. \end{cases} \tag{4.15}$$

The optimal value of (4.15) is the largest H-eigenvalue of $\mathcal{A}$ that is smaller than or equal to $\mu_i + \delta$, i.e.,

$$\upsilon_i = \max\{\mu \in H(\mathbb{R}, \mathcal{A}) :\ \mu \leq \mu_i + \delta\}.$$

The next larger H-eigenvalue μ_{i+1} can be determined as follows.

Theorem 4.4 *Let* $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ *and* $\delta > 0$. *Suppose* $\mu_i \in H(\mathbb{R}, \mathcal{A})$ *and* μ_{max} *is the maximum real* H*-eigenvalue. Let* $\varpi(\lambda_i + \delta)$ *be as in* (4.11). *Then, we have:*

(i) *For all* $\delta > 0$ *small enough,* $\upsilon_i = \mu_i$.
(ii) *If* $\upsilon_i = \mu_i$ *and* (4.12) *is infeasible for some* k, *then* $\mu_i = \mu_{max}$.
(iii) *If* $\upsilon_i = \mu_i$ *and* (4.9) *is satisfied for some* k, *then* $\mu_{i+1} = \rho_{i+1}^{1,k}$.

Proof The proof is the same as for Theorem 3.4. Note that $\mathcal{A}$ has finitely many H-eigenvalues, and μ_i is always isolated, as in Proposition 4.1. □

Since (4.15) is a polynomial optimization, the optimal value υ_i can also be computed by solving Lasserre type semidefinite relaxations that are similar to (4.12)–(4.13).

4.3 An algorithm for all H-eigenvalues

We can compute all real H-eigenvalues of a tensor $\mathcal{A}$ sequentially, from the smallest one to the largest one, if they exist. A similar version of Algorithm 3.6 can be applied.

Algorithm 4.5 *For a given tenosr $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$, compute its real* H*-eigenvalues as follows:*

Step 0: *Let $k = k_0$, with k_0 as in (4.6).*
Step 1: *Solve the relaxation (4.5). If it is infeasible, then $\mathcal{A}$ has no real* H*-eigenvalues and stop. Otherwise, compute an optimizer z^*.*
Step 2: *If (4.9) is satisfied, then $\mu_1 = \rho_1^{1,k}$ and go to Step 3 with $i := 1$. Otherwise, let $k := k + 1$ and go to Step 1.*
Step 3: *Let $\delta = 0.05$. Solve (4.15) for its optimal value υ_i. If $\upsilon_i = \mu_i$, go to Step 4. Otherwise, let $\delta := \delta/5$ and compute υ_i again. Repeat this process until we get $\upsilon_i = \mu_i$.*
Step 4: *Let $k = k_0$, with k_0 as in (4.6).*
Step 5: *Solve the relaxation (4.12). If it is infeasible, the largest* H*-eigenvalue is μ_i and stop. If it is feasible, compute an optimizer z^*.*
Step 6: *If (4.9) is satisfied, then $\mu_{i+1} = \rho_{i+1}^{1,k}$ and go to Step 3 with $i := i + 1$. Otherwise, let $i := i + 1$ and go to Step 5.*

The correctness of Algorithm 4.5 and its convergence properties are proved in Theorems 4.2, 4.3 and 4.4.

5 Numerical examples

In this section, we present numerical experiments for computing real Z-eigenvalues and H-eigenvalues. The Algorithms 3.6 and 4.5 are implemented in MATLAB 7.10 on a Dell Desktop with Linux as OS with 8GB memory and Intel(R) CPU 2.8GHz. The software `Gloptipoly 3` [11] is used to solve the semidefinite relaxations in the algorithms. The computational results are displayed with four decimal digits, for cleanness of the presentation. The isolatedness of Z-eigenvalues are checked by Lemma 2.1 or Lemma 3.5. In checking conditions (3.8) and (4.9), the rank of a matrix is evaluated as the number of its singular values that are greater than 10^{-6}. For odd order tensors, the Z-eigenvalues always appear in $\pm$ pairs, so only nonnegative Z-eigenvalues are displayed for them. In each table of this section, the `time` denotes the consuming time in seconds of the computation.

5.1 Numerical examples of Z(H)-eigenvalues

Example 5.1 [9, Example 3] Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^2)$ with entries $\mathcal{A}_{i_1 i_2 i_3 i_4} = 0$ except

$$\mathcal{A}_{1111} = 25.1, \quad \mathcal{A}_{1212} = 25.6, \quad \mathcal{A}_{2121} = 24.8, \quad \mathcal{A}_{2222} = 23.$$

Applying Algorithms 3.6 and 4.5, we get all the real Z/H-eigenvalues. They are shown in Table 1. It took about 1 s to compute Z-eigenpairs, and about 3 s to compute H-eigenpairs. The convergence $f_i^{1,k} \to \lambda_i$ and $\rho_i^{1,k} \to \mu_i$ occurs as follows:

$$f_1^{1,3} = \lambda_1, \ f_2^{1,3} = \lambda_2, \quad \rho_1^{1,4} = \mu_1, \ \rho_2^{1,5} = \mu_2, \ \rho_3^{1,4} = \mu_3.$$

Table 1 Z/H-eigenpairs of the tensor in Example 5.1

Z-eigenvalue	23.000	25.1000	
Z-eigenvector	$\pm(0, 1)$	$\pm(1, 0)$	
H-eigenvalue	23.0000	25.1000	49.2687
H-eigenvector	$\pm(0, 1)$	$\pm(1, 0)$	$\pm(0.8527, \pm 0.8285)$

The Z-eigenvalues λ_1, λ_2 are isolated. This can be verified by Lemma 3.5, because

$$\nu^+(\lambda_i, 0.05) = \nu^-(\lambda_i, 0.05)$$

for all λ_i. Their isolatedness can also be verified by Lemma 2.1. For the function $F(\lambda, u)$ as in (2.6), its Jacobian matrices at (λ_1, u_1), (λ_2, u_2) are respectively

$$\begin{bmatrix} 0 & 2 & 0 \\ 2.6 & 0 & 0 \\ 0 & 46 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 50.2 & 0 & -1 \\ 0 & -0.3 & 0 \end{bmatrix}.$$

They are both nonsingular. By a direct calculation, one can show that the real Z-eigenvalues are $\lambda_1 = 23$, $\lambda_2 = 25.1$, while the real H-eigenvalues are $\mu_1 = 23$, $\mu_2 = 25.1$, $\mu_3 = (\sqrt{254393}+481)/20$. The computed Z/H-eigenpairs are all correct, up to round-off errors. Actually, we have $\|\mathcal{A}u_i^3 - \lambda_i u_i\| = \|\mathcal{A}v_i^3 - \mu_i v_i^{[3]}\| = 0$ for $i = 1, 2$ and $\|\mathcal{A}v_3^3 - \mu_3 v_3^{[3]}\| \approx 10^{-14}$.

Example 5.2 [33, Example 1] Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^3)$ with the entries $\mathcal{A}_{i_1 i_2 i_3} = 0$ except

$$\begin{aligned}
&\mathcal{A}_{111} = 0.4333,\ \mathcal{A}_{121} = 0.4278,\ \mathcal{A}_{131} = 0.4140,\ \mathcal{A}_{211} = 0.8154,\ \mathcal{A}_{221} = 0.0199,\\
&\mathcal{A}_{231} = 0.5598,\ \mathcal{A}_{311} = 0.0643,\ \mathcal{A}_{321} = 0.3815,\ \mathcal{A}_{331} = 0.8834,\ \mathcal{A}_{112} = 0.4866,\\
&\mathcal{A}_{122} = 0.8087,\ \mathcal{A}_{132} = 0.2073,\ \mathcal{A}_{212} = 0.7641,\ \mathcal{A}_{222} = 0.9924,\ \mathcal{A}_{232} = 0.8752,\\
&\mathcal{A}_{312} = 0.6708,\ \mathcal{A}_{322} = 0.8296,\ \mathcal{A}_{332} = 0.125,\ \mathcal{A}_{113} = 0.3871,\ \mathcal{A}_{123} = 0.0769,\\
&\mathcal{A}_{133} = 0.3151,\ \mathcal{A}_{213} = 0.1355,\ \mathcal{A}_{223} = 0.7727,\ \mathcal{A}_{233} = 0.4089,\ \mathcal{A}_{313} = 0.9715,\\
&\mathcal{A}_{323} = 0.7726,\ \mathcal{A}_{333} = 0.5526.
\end{aligned}$$

By Algorithms 3.6 and 4.5, we get all the real Z-eigenvalues (only nonnegative ones are computed because the order is odd), and get all the real H-eigenvalues. Three Z-eigenpairs (λ_i, u_i) and four H-eigenpairs (μ_i, v_i) are obtained. The results are shown in Table 2. It took about 4 s to compute the Z-eigenvalues, and about 5 s to get the H-eigenvalues. The three Z-eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are all isolated. This can be verified by Lemma 3.5, because

$$\nu^-(\lambda_i, 0, 05) = \nu^+(\lambda_i, 0.05)$$

Table 2 Z/H-eigenpairs of the tensor in Example 5.2

Z-eig. λ_i	Z-eigvec. u_i	$\|\mathcal{A}u_i^2 - \lambda_i u_i\|$
0.2331	(0.3736, 0.1703, −0.9118)	3.8×10^{-9}
0.4869	(0.6666, 0.4324, −0.6072)	1.2×10^{-7}
2.7418	(0.4086, 0.6588, 0.6317)	9.1×10^{-8}
H-eig. μ_i	H-eigvec. v_i	$\|\mathcal{A}v_i^2 - \mu_i v_i^{[2]}\|$
1.3586	(0.5795, −0.4276, 0.6938)	1.3×10^{-7}
1.4985	(0.6782, 0.6327, 0.6938)	8.6×10^{-7}
1.5526	(−0.4367, 0.8090, 0.3935)	7.7×10^{-7}
4.7303	(0.4956, 0.6174, 0.6109)	4.6×10^{-8}

for all λ_i. The isolatedness is also confirmed by Lemma 2.1. The smallest singular values of the Jacobian matrix of $F(\lambda, x)$ at (λ_i, u_i) are respectively 0.6015, 0.7061, 0.5614. They are all nonsingular. The computed eignpairs are all correct, up to tiny numerical errors. The residual errors $\|\mathcal{A}u_i^2 - \lambda_i u_i\|$, $\|\mathcal{A}v_i^2 - \mu_i v_i^{[2]}\|$ are shown in Table 2.

Example 5.3 [14, §4.1] Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^3)$ with $\mathcal{A}_{i_1 i_2 i_3} = 0$ except

$$\begin{aligned}
&\mathcal{A}_{111} = 0.0072, \quad \mathcal{A}_{121} = -0.4413, \quad \mathcal{A}_{131} = 0.1941, \quad \mathcal{A}_{211} = -0.4413, \quad \mathcal{A}_{221} = 0.0940,\\
&\mathcal{A}_{231} = 0.5901, \quad \mathcal{A}_{311} = 0.1941, \quad \mathcal{A}_{321} = -0.4099, \quad \mathcal{A}_{331} = -0.1012, \quad \mathcal{A}_{112} = -0.4413,\\
&\mathcal{A}_{122} = 0.0940, \quad \mathcal{A}_{132} = -0.4099, \quad \mathcal{A}_{212} = 0.0940, \quad \mathcal{A}_{222} = 0.2183, \quad \mathcal{A}_{232} = 0.2950,\\
&\mathcal{A}_{312} = 0.5901, \quad \mathcal{A}_{322} = 0.2950, \quad \mathcal{A}_{332} = 0.2229, \quad \mathcal{A}_{113} = 0.1941, \quad \mathcal{A}_{123} = 0.5901,\\
&\mathcal{A}_{133} = -0.1012, \quad \mathcal{A}_{213} = -0.4099, \quad \mathcal{A}_{223} = 0.2950, \quad \mathcal{A}_{233} = 0.2229, \quad \mathcal{A}_{313} = -0.1012,\\
&\mathcal{A}_{323} = 0.2229, \quad \mathcal{A}_{333} = -0.4891.
\end{aligned}$$

By Algorithms 3.6 and 4.5, we get all the real Z-eigenvalues (only nonnegative ones are computed because the order is odd), and get all the real H-eigenvalues. They are shown in Table 3. It took about 4 s to compute the Z-eigenvalues, and about 3 s to get the H-eigenvalues. It can be verified that

$$\nu^-(\lambda_i, 0.05) = \nu^+(\lambda_i, 0.05)$$

for all λ_i. By Lemma 3.5, we know λ_1, λ_2 are isolated Z-eigenvalues. This is also confirmed by Lemma 2.1. The smallest singular value of the Jacobian matrix of $F(\lambda, x)$ as in (2.6) at (λ_1, u_1), (λ_2, u_2) are respectively 0.9999, 0.9501. (For λ_2, there are four Z-eigenvectors u_2. At each of them, the smallest singular value is the same.) They are all nonsingular. The computed eigenpairs (λ_i, u_i) and (μ_i, v_i) are correct, up to small numerical errors. Their residual errors are shown in Table 3.

Table 3 Z/H-eigenvalues of the tensor in Example 5.3

Z-eig. λ_i	Z-eigvec. u_i	$\|\mathcal{A}u_i^2 - \lambda_i u_i\|$
0.0000	$(-0.6059, 0.3195, 0.7285)$	2.8×10^{-8}
0.5774	$(-0.8062, -0.4872, 0.3354)$	1.3×10^{-7}
	$(-0.1121, 0.1995, -0.9734)$	2.1×10^{-7}
	$(0.8120, -0.5685, 0.1323)$	1.9×10^{-7}
	$(0.1065, 0.8562, 0.5054)$	3.2×10^{-7}
H-eig. μ_i	H-eigvec. v_i	$\|\mathcal{A}v_i^2 - \mu_i v_i^{[2]}\|$
0.0000	$(0.7954, 0.2492, 0.5524)$	7.1×10^{-7}
0.7875	$(0.1760, 0.8067, 0.5642)$	3.6×10^{-8}

Table 4 Z/H-eigenvalues of the tensor in Example 5.4

n	Z-eig. λ_i (≥ 0)	time	H-eig. μ_i	time
2	10.5518	0.6	None	0.1
3	0.2336, 1.6614, 10.5063	3.3	-2.5615, 0.3456;	2.1
4	3.3651, 8.8507, 10.4981	5.2	$-6.2888, -0.7048$, 2.8947, 5.9245	7.9
5	1.7701, 1.9260, 4.0140, 4.1174, 4.3543, 8.8414, 14.4904	38.8	$-8.8357, -6.8068, -6.5504$, 3.3380, 6.5247, 7.1458, 8.4572, 11.0901	66.3

Example 5.4 [24, Example 3.19] Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3} = \tan\left(i_1 - \frac{i_2}{2} + \frac{i_3}{3}\right).$$

Applying Algorithms 3.6 and 4.5, we get all the real Z-eigenvalues (only nonnegative ones are computed because the order is odd), and get all the real H-eigenvalues. The results are shown in Table 4, for the dimensions $n = 2, 3, 4, 5$. There are no real H-eigenvalues when $n = 2$. The Z-eigenvalues are all isolated. This is verified by Lemma 3.5, because $\nu^-(\lambda_i, 0.05) = \nu^+(\lambda_i, 0, 05)$ for all λ_i. Since there are 28 eigenvalues, the eigenvectors are not shown, for neatness of the paper. They are all correct, up to small numerical errors.

Example 5.5 Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}_{i_1 \ldots i_4} = \arctan(i_1 i_2^2 i_3^3 i_4^4).$$

By Algorithms 3.6 and 4.5, we get all the real Z/H-eigenvalues. It took about 3 s to get the Z-eigenvalues, and about 5 s to get the H-eigenvalues. The results are shown in Table 5. All the Z-eigenvalues λ_i are isolated, which is verified by Lemma 3.5 because

Table 5 Z/H-eigenvalues of the tensor in Example 5.5

Z-eig. λ_i	Z-eigvec. u_i	$\|\mathcal{A}u_i^3 - \lambda_i u_i\|$
-0.2700	$(0.9306,\ 0.0372,\ -0.3641)$	6.7×10^{-8}
0.0003	$(0.2402,\ -0.7552,\ 0.6099)$	1.2×10^{-7}
13.8286	$(0.5701,\ 0.5792,\ 0.5827)$	3.3×10^{-7}
H-eig. μ_i	H-eigvec. v_i	$\|\mathcal{A}u_i^3 - \lambda_i u_i^{[3]}\|$
-0.3662	$(0.9413,\ 0.3559,\ -0.6677)$	4.2×10^{-9}
0.0005	$(0.2924,\ -0.9139,\ 0.7365)$	4.4×10^{-7}
41.4705	$(0.7566,\ 0.7607,\ 0.7622)$	8.8×10^{-7}

Table 6 Z/H-eigenvalues of the tensor in Example 5.6

Z-eig. λ_i	Z-eigvec. u_i	$\|\mathcal{A}u_i^3 - \lambda_i u_i\|$
0.0000	$(0.2131, -0.7720, 0.5989)$	5.5×10^{-11}
0.0002	$(0.6244, -0.1938, -0.7567)$	4.9×10^{-9}
0.4572	$(0.6083, 0.5756, 0.5464)$	7.7×10^{-9}
H-eig. μ_i	H-eigvec. v_i	$\|\mathcal{A}v_i^3 - \mu_i v_i^{[3]}\|$
0.0000	$(0.2552, -0.9246, 0.7173)$	8.2×10^{-9}
0.0005	$(-0.8465, 0.4326, 0.8197)$	8.3×10^{-9}
1.3581	$(0.7732, 0.7593, 0.7463)$	5.0×10^{-8}

$\nu^-(\lambda_i, 0.05) = \nu^+(\lambda_i, 0.05)$ for all λ_i. The computed Z/H-eigenpairs $(\lambda_i, u_i), (\mu_i, v_i)$ are all correct, up to small numerical errors. The residual errors are shown in Table 5.

Example 5.6 Consider the tensor $\mathcal{A} \in \mathrm{T}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}_{i_1 \ldots i_4} = (1 + i_1 + 2i_2 + 3i_3 + 4i_4)^{-1}.$$

By Algorithms 3.6 and 4.5, we get all the real Z/H-eigenvalues. It took about 4 s to compute Z-eigenvalues, and about 5 s for H-eigenvalues. The results are shown in Table 6. The Z-eigenvalues λ_i are all isolated. This is verified by Lemma 3.5, because

$$\nu^-(\lambda_i, 0.0001) = \nu^+(\lambda_i, 0.0001)$$

for all λ_i. The computed eigenpairs $(\lambda_i, u_i), (\mu_i, v_i)$ are all correct, up to small numerical errors. The residual errors are shown in Table 6.

Example 5.7 Consider the tensor $\mathcal{A} \in \mathrm{T}^5(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 \ldots i_5} = \Big(\sum_{j=1}^{5} (-1)^{j-1} \exp(i_j) \Big)^{-1}.$$

Table 7 Z/H-eigenvalues of the tensor in Example 5.7

n	Z-eig. λ_i (≥ 0)	Time	H-eig. μ_i	Time
2	0.4721	1.0	0.5138, 1.2654	2.2
3	0.6158	2.9	0.5196, 2.0800, 2.2995, 2.4335	13.6
4	0.7682	4.6	0.5199, 2.0964, 2.2980, 2.3991, 2.9454, 4.4609, 4.9588, 5.4419	205.7
5	0.8384	31.8	0.5199, 2.0978, 2.2997, 2.3860, 2.4010, 2.9658, 4.4713, 4.4902, 4.6880, 4.7008, 5.0136, 5.7891, 6.0668, 7.3250, 7.3469, 8.8555	18,243

Table 8 Z/H-eigenvalues of the tensor in Example 5.8

n	Z-eig. λ_i (≥ 0)	time	H-eig. μ_i	time
2	0.0024, 0.0038, 1.4928	5.1	0.0060, 2.0960	1.5
3	0.0067, 0.0161, 3.6417	6.2	$-0.0401, -0.0243$, 0.0086, 0.0235, 0.1568, 0.6635, 1.4958, 6.2378	15.9
4	0.0000, 0.0107, 0.0396, 6.9922	14.3	$-0.0240, -0.0087, -0.0001$, 0.0000, 0.0102, 0.0258, 0.0437, 1.5761, 2.6824, 4.1089, 5.8270, 13.7960	59.6

For the dimensions $n = 2, 3, 4, 5$, all the real Z/H-eigenvalues are found by Algorithms 3.6 and 4.5. Because the order is odd, only nonnegative Z-eigenvalues are computed. The results are shown in Table 7.

The Z-eigenvalues are all isolated, verified by Lemma 3.5. This is because

$$\nu^-(\lambda_i, 0.05) = \nu^+(\lambda_i, 0.05)$$

for all λ_i. The computed eigenpairs are all correct, up to round-off errors. Since there are 34 computed eigenpairs, for neatness of the paper, the eigenvectors and the residual errors are not displayed.

Example 5.8 Consider the tensor $\mathcal{A} \in \mathrm{T}^3(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3} = \frac{1}{10}\left(i_1 + 2i_2 + 3i_3 - \sqrt{i_1^2 + 2i_2^2 + 3i_3^2}\right).$$

For the dimensions $n = 2, 3, 4$, we get all the real Z-eigenvalues (only nonnegative ones are computed because the order is odd), and get all the real H-eigenvalues. The results are shown in Table 8.

The Z-eigenvalues are all isolated, verified by Lemma 3.5. This is because

$$\nu^-(\lambda_i, 0.001) = \nu^+(\lambda_i, 0.001)$$

for all λ_i. The computed eigenpairs are all correct, up to round-off errors. Since there are 32 computed eigenpairs, for neatness of the paper, the eigenvectors and the residual errors are not displayed.

5.2 Numerical examples of stationary probability distribution vectors

In this subsection, we report numerical examples on computing all stationary probability distribution vectors.

Example 5.9 Consider the 3rd order 3-dimensional transition probability tensor $\mathcal{P} \in \mathrm{T}^3(\mathbb{R}^3)$ whose nonzero entries are

$$\begin{array}{l} \mathcal{P}_{112} = 1,\ \mathcal{P}_{121} = 1,\ \mathcal{P}_{113} = 1,\ \mathcal{P}_{131} = 1,\ \mathcal{P}_{211} = 1, \\ \mathcal{P}_{222} = 1,\ \mathcal{P}_{223} = 1,\ \mathcal{P}_{232} = 1,\ \mathcal{P}_{333} = 1. \end{array}$$

Its stationary probability distribution vectors are exactly

$$v_1 = (0, 1, 0), \quad v_2 = (0, 0, 1), \quad v_3 = (\frac{1}{2}, \frac{1}{2}, 0).$$

By Algorithm 3.3, we get all the stationary probability distribution vectors, with the relaxation orders 2, 2, 3, respectively. It took about 4 s. One can verify that $\|\mathcal{P}v_i^2 - v_i\| = 0$ for $i = 1, 2, 3$.

In the following examples, for $\mathcal{P} \in \mathrm{T}^m(\mathbb{R}^n)$, $\mathcal{P}_{ij\ldots}$ $(i, j, \ldots \in \{1, \ldots, n\})$ denotes the matrix whose entries are given as

$$(\mathcal{P}_{ij\ldots})_{t_1t_2} = \mathcal{P}_{t_1t_2ij\ldots} \quad (1 \le t_1, t_2 \le n).$$

Example 5.10 [1] Consider the transition probability tensor $\mathcal{P} \in \mathrm{T}^3(\mathbb{R}^4)$ that is given as:

$$\mathcal{P}_1 = \begin{pmatrix} 0.2091 & 0.2834 & 0.2194 & 0.1830 \\ 0.3371 & 0.3997 & 0.3219 & 0.3377 \\ 0.3265 & 0.0560 & 0.3119 & 0.2961 \\ 0.1273 & 0.2608 & 0.1468 & 0.1832 \end{pmatrix}; \mathcal{P}_2 = \begin{pmatrix} 0.1952 & 0.2695 & 0.2055 & 0.1690 \\ 0.3336 & 0.3962 & 0.3184 & 0.3342 \\ 0.2954 & 0.0249 & 0.2808 & 0.2650 \\ 0.1758 & 0.3094 & 0.1953 & 0.2318 \end{pmatrix};$$
$$\mathcal{P}_3 = \begin{pmatrix} 0.3145 & 0.3887 & 0.3248 & 0.2883 \\ 0.0603 & 0.1230 & 0.0451 & 0.0609 \\ 0.3960 & 0.1255 & 0.3814 & 0.3656 \\ 0.2293 & 0.3628 & 0.2487 & 0.2852 \end{pmatrix}; \mathcal{P}_4 = \begin{pmatrix} 0.1686 & 0.2429 & 0.1789 & 0.1425 \\ 0.3553 & 0.4180 & 0.3402 & 0.3559 \\ 0.3189 & 0.0484 & 0.3043 & 0.2885 \\ 0.1571 & 0.2907 & 0.1766 & 0.2131 \end{pmatrix}.$$

It has only one nonnegative Z-eigenvalue. By Algorithm 3.6, we get the Z-eigenvector

$$u = (0.4771,\ 0.5716,\ 0.4875,\ 0.4560).$$

It took about 2 s. By a scaling, we get its unique stationary probability distribution vector

$$v = (0.2395,\ 0.2869,\ 0.2447,\ 0.2288).$$

The residual error $\|\mathcal{P}v^2 - v\| \approx 9.2 \times 10^{-9}$.

Example 5.11 [19] Consider the transition probability tensor $\mathcal{P} \in \mathrm{T}^4(\mathbb{R}^3)$ that is given as:

$$\mathcal{P}_{11} = \begin{pmatrix} 0.3721 & 0.2600 & 0.4157 \\ 0.4477 & 0.5000 & 0.4270 \\ 0.1802 & 0.2400 & 0.1573 \end{pmatrix}; \mathcal{P}_{21} = \begin{pmatrix} 0.3692 & 0.2673 & 0.3175 \\ 0.4667 & 0.5594 & 0.5079 \\ 0.1641 & 0.1733 & 0.1746 \end{pmatrix};$$
$$\mathcal{P}_{31} = \begin{pmatrix} 0.4227 & 0.2958 & 0.2353 \\ 0.4124 & 0.5563 & 0.5588 \\ 0.1649 & 0.1479 & 0.2059 \end{pmatrix}; \mathcal{P}_{12} = \begin{pmatrix} 0.3178 & 0.2632 & 0.3194 \\ 0.5212 & 0.6228 & 0.5833 \\ 0.1610 & 0.1140 & 0.0972 \end{pmatrix};$$
$$\mathcal{P}_{22} = \begin{pmatrix} 0.2836 & 0.2636 & 0.3042 \\ 0.5012 & 0.6000 & 0.5250 \\ 0.2152 & 0.1364 & 0.1708 \end{pmatrix}; \mathcal{P}_{32} = \begin{pmatrix} 0.3382 & 0.2396 & 0.3766 \\ 0.5147 & 0.6406 & 0.4935 \\ 0.1471 & 0.1198 & 0.1299 \end{pmatrix};$$
$$\mathcal{P}_{13} = \begin{pmatrix} 0.3204 & 0.2985 & 0.3500 \\ 0.4854 & 0.5000 & 0.5000 \\ 0.1942 & 0.2015 & 0.1500 \end{pmatrix}; \mathcal{P}_{23} = \begin{pmatrix} 0.4068 & 0.2816 & 0.3594 \\ 0.3898 & 0.5143 & 0.4219 \\ 0.2034 & 0.2041 & 0.2188 \end{pmatrix};$$
$$\mathcal{P}_{33} = \begin{pmatrix} 0.3721 & 0.3529 & 0.3000 \\ 0.5349 & 0.3971 & 0.5500 \\ 0.0930 & 0.2500 & 0.1500 \end{pmatrix}.$$

By Algorithm 3.6, we get the unique Z-eigenvector

$$u = (0.4719,\ 0.8420,\ 0.2614).$$

It took about 2 s. After a scaling, we get the unique stationary probability distribution vector

$$v = (0.2995,\ 0.5345,\ 0.1659).$$

The residual error $\|\mathcal{P}v^3 - v\| \approx 7.5 \times 10^{-7}$.

6 Discussion

This paper proposed semidefinite relaxation methods for computing real Z/H-eigenvalues for nonsymmetric tensors. We can compute all real Z-eigenvalues when there are finitely many ones. Moreover, all real H-eigenvalues can be computed, since its number is always finite. In this section, we discuss some related issues.

6.1 Eigenvalues of symmetric tensors

Algorithms 3.6 and 4.5 are designed to compute Z/H-eigenvalues for nonsymmetric tensors. As a special case, they can also be applied to symmetric tensors. For computing symmetric tensor eigenvalues, a hierarchy of semidefinite relaxations was proposed in [5, §3]. It is based on polynomial optimization and Jacobian representations. The method in this paper does not use Jacobian representations. When the tensors are symmetric, these two approaches are mathematically equivalent. An interesting question is whether the relaxation orders required for convergence are the same or not. Here, we give an example for comparison. Consider the symmetric tensor $\mathcal{A} \in \mathbb{S}^4(\mathbb{R}^2)$ [5, Example 4.4] such that $\mathcal{A}_{ijkl} = 0$ except $\mathcal{A}_{1111} = 3$, $\mathcal{A}_{2222} = 1$, and

$$\mathcal{A}_{1122} = \mathcal{A}_{1212} = \mathcal{A}_{1221} = \mathcal{A}_{2112} = \mathcal{A}_{2121} = \mathcal{A}_{2211} = \frac{1}{2}.$$

This tensor has two Z-eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ and three H-eigenvalues $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 \approx 3.8028$. Both the methods in [5] and in this paper compute the eigenvalues correctly. We list the least values of relaxation orders k, for which the convergence occurs, for the methods in [5] and in this paper (for our method, they are the smallest k such that $f_i^{1,k} = \lambda_i$ and $\rho_i^{1,k} = \mu_i$). The values of such orders k are listed as follows:

Z/H-eigenvalues	λ_1	λ_2	μ_1	μ_2	μ_3
The method in [5]	2	3	3	5	5
The method in this paper	3	3	4	5	5

By directly computation, the initial relaxation orders k_0 is larger than that of [5] in general. However, except for λ_1 and μ_1, the relaxation orders required for convergence are the same. For λ_1 (resp., μ_1), 3 (resp., 4) is the lowest order of the semidefinite relaxation (3.4), while 2 (resp., 3) is the lowest one in [5]. For the above tensor, there is no much difference for the required relaxation orders. For more general symmetric tensors, we do not know whether the same conclusion is true or not.

6.2 Tensors with infinitely many Z-eigenvalues

Consider the tensor $\mathcal{A}$ in Example 1.4. It has infinitely many real Z-eigenvalues. Indeed, every $\lambda \in [0, 1]$ is a Z-eigenvalue. By Algorithm 3.6, we can get the smallest real Z-eigenvalue $\lambda_1 = 0$, together with the eigenvector $u_1 = \pm(0, 1)$. However, λ_1 is not isolated. In Lemma 3.5, the equality $\nu^-(\lambda_1, \delta) = \nu^+(\lambda_1, \delta)$ is not satisfied for any $\delta > 0$. In Lemma 2.1, the Jacobian matrix

$$J(\lambda, x) = \begin{bmatrix} 0 & 2x_1 & 2x_2 \\ -x_1 & 3x_1^2 - \lambda & 0 \\ -x_2 & 2x_1x_2 & x_1^2 - \lambda \end{bmatrix}$$

is singular at (λ_1, u_1). The second smallest real Z-eigenvalue λ_2 does not exist.

Similarly, if we apply Algorithm 3.6 to $-\mathcal{A}$, then we can get the largest real Z-eigenvalue $\lambda_{max} = 1$, with the Z-eigenvector $\pm(1, 0)$.

6.3 Comparison with other methods

A straightforward approach for computing Z/H-eigenvalues is to solve the polynomial Eqs. (1.3) and (1.4) directly. Symbolic methods can be naturally used for the computation. As mentioned in the introduction, such methods are usually very expensive to be used. It usually takes much longer time. We compare with the symbolic solver `solve` provided by MATLAB, which calls the software MAPLE. We use it to compute the Z/H-eigenvalues of the tensor in Example 5.4, for the dimensions $n = 2, 3, 4, 5$. The consumed time by `solve` is listed as follows:

Time	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Z-eig.	1.4	5.2	41.1	4297.4
H-eig.	0.7	9.8	13646.0	$(*)$ 2574.0

The time consumed by Algorithms 3.6 and 4.5 is shown in Table 4. As one can see, the symbolic solver takes much more time, especially for $n = 4, 5$. For the case $n = 5$, `solve` can only get one H-eigenvalue but not all, which already took about 2574 s. We put the mark $(*)$ in the above table for this fact. In contrast, Algorithm 4.5 takes about 38 s to compute all the real Z-eigenvalues (resp., 66 s to compute all the real H-eigenvalues). Our semidefinite relaxation methods are faster for computing real tensor eigenvalues.

There also exist other type methods for solving polynomial systems. For instance, the Homotopy continuation method [7] can be applied. This method typically can only get one eigenvalue, instead of all, at each running. Its performance depends on the choice of the starting point. To see this, we apply it to the tensor in Example 5.1. The Z-eigenvalue 25.1 is obtained from the starting point (0.5, 0.5, 0.5); the Z-eigenvalue 23 is obtained from the starting point (8, 4, 0.5); no Z-eigenvalue is obtained from the starting point (0, 0, 0). If we use different starting points (say, randomly generated points), we might get different eigenvalues. However, this does not guarantee that all the eigenvalues can be found. Moreover, even if all the eigenvalues are obtained, the method cannot detect that no eigenvalues are missing in the computation.

6.4 Isolatedness of z-eigenvalues

The isolatedness of Z-eigenvlaues can be verified by Lemma 2.1. It requires to check whether the Jacobian matrix $J(\lambda, u)$ of the function $F(\lambda, x)$ in (2.6) is nonsingular or not. A common practice is to compute the smallest singular value σ_{min} of $J(\lambda, u)$. If σ_{min} is clearly greater than zero (e.g., $\sigma_{min} > 10^{-3}$), then it is usually safe to claim the nonsingularity of $J(\lambda, u)$. However, if σ_{min} is close to zero, then it may be hard to tell.

This issue is related to the problem of evaluating the rank of a matrix numerically. This is a classical question in numerical linear algebra. We refer to the book [8]. Moreover, Lemma 3.5 can also be applied to check isolatedness. It does not require to determine whether or not the Jacobian matrix $J(\lambda, u)$ is nonsingular.

Acknowledgements Jiawang Nie was partially supported by the NSF Grants DMS-1417985 and DMS-1619973. Xinzhen Zhang was partially supported by the National Natural Science Foundation of China (Grant No. 11471242).

References

1. Berchtold, A., Raftery, A.: The mixture transition distribution model for high-order Markov chains and non-Gaussian time series. Stat. Sci. **17**, 328–356 (2002)
2. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Springer, New York (1998)
3. Cartwright, D., Stumfels, B.: The number of eigenvalues of a tensor. Linear Algebra Appl. **438**, 942–952 (2013)
4. Chang, K., Pearson, K., Zhang, T.: Perron–Frobenius theorem for nonnegative tensors. Commun. Math. Sci. **6**, 507–520 (2008)
5. Cui, C., Dai, Y., Nie, J.: All real eigenvalues of symmetric tensors. SIAM J. Matrix Anal. Appl. **35**, 1582–1601 (2014)
6. Curto, R., Fialkow, L.: Truncated K-moment problems in several variables. J. Oper. Theory **54**, 189–226 (2005)
7. Decarolis, F., Mayer, R., Santamaria, M.: Homotopy continuation methods: an algorithm for the fixed point and Newton homotopy methods with some examples. http://home.chicago.edu/~fdc/H-topy.pdf
8. Demmel, J.: Applied Numerical Linear Algebra. Society for Industrial and Applied Mathematics, Philadelphia (1997)
9. De Lathauwer, L., De Moor, B., Vandewalle, J.: On the best rank-1 and rank-(R1, R2,· · · , RN) approximation of higher-order tensors. SIAM J. Matrix Anal. Appl. **21**, 1324–1342 (2000)
10. Henrion, D., Lasserre, J.B.: Detecting global optimality and extracting solutions in GloptiPoly. In: Positive Polynomials in Control. Lecture Notes in Control and Information Science, vol. 312, pp. 293–310. Springer, Berlin (2005)
11. Henrion, D., Lasserre, J.B., Lofberg, Y.: GloptiPoly 3: moments, optimization and semidfinite programming. Optim. Methods Softw. **24**, 761–779 (2009)
12. Hillar, C., Lim, L.H.: Most tensor problems are NP-hard. J. ACM **60**(6), 45 (2013)
13. Hu, S., Qi, L.: Convergence of a second order Markov chain. Appl. Math. Comput. **241**, 183–192 (2014)
14. Ishteva, M., Absil, P.-A., Van Dooren, P.: Jacobi algorithm for the best low multilinear rank approximation of symmetric tensors. SIAM J. Matrix Anal. Appl. **34**, 651–672 (2013)
15. Kolda, T.G., Mayo, J.R.: Shifted power method for computing tensor eigenpairs. SIAM J. Matrix Anal. Appl. **32**, 1095–1124 (2011)
16. Lasserre, J.B.: Global optimization with polynomials and the problem of moments. SIAM J. Optim. **11**, 796–817 (2001)
17. Lasserre, J.B.: Moments, Positive Polynomials and Their Applications. Imperial College Press, London (2009)
18. Laurent, M.: Sums of squares, moment matrices and optimization over polynomials. In: Putinar, M., Sullivant, S. (eds.) Emerging Applications of Algebraic Geometry, IMA Volumes in Mathematics and its Applications, vol. 149, pp. 157–270. Springer, New York (2009)
19. Li, W., Ng, M.: On the limiting probability distribution of a transition probability tensor. Linear Multilinear Algebra **62**, 362–385 (2014)
20. Li, X., Ng, M., Ye, Y.: Finding stationary probability vector of a transition probability tensor arising from a higher-order Markov chain. Technical Report, Department of Mathematics, Hong Kong Baptist University (2011). http://www.math.hkbu.edu.hk/~mng/tensor-research/report1.pdf
21. Lim, L.H.: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'05), vol. 1, pp. 129–132 (2005)

22. Lim, L.-H.: Tensors and hypermatrices. In: Hogben, L. (ed.) Handbook of Linear Algebra, 2nd edn. CRC Press, Boca Raton (2013)
23. Ng, M., Qi, L., Zhou, G.: Finding the largest eigenvalue of a non-negative tensor. SIAM J. Matrix Anal. Appl. **31**, 1090–1099 (2009)
24. Nie, J., Wang, L.: Semidefinite relaxations for best rank-1 tensor approximations. SIAM J. Matrix Anal. Appl. **35**, 1155–1179 (2014)
25. Nie, J.: Polynomial optimization with real varieties. SIAM J. Optim. **23**, 1634–1646 (2013)
26. Nie, J.: An exact Jacobian SDP relaxation for polynomial optimization. Math. Program. **137**, 225–255 (2013)
27. Nie, J.: Certifying convergence of Lasserre's hierarchy via flat truncation. Math. Program. **142**, 485–510 (2013)
28. Nie, J.: Optimality conditions and finite convergence of Lasserre's hierarchy. Math. Program. **146**, 97–121 (2014)
29. Nie, J.: The hierarchy of local minimums in polynomial optimizaion. Math. Program. **151**, 555–583 (2015)
30. Nie, J.: Generating polynomials and symmetric tensor decomposition. Found. Comput. Math. **17**, 423–465 (2017)
31. Qi, L.: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**, 1302–1324 (2005)
32. Qi, L.: Eigenvalues and invariants of tensors. J. Math. Anal. Appl. **325**, 1363–1377 (2007)
33. Qi, L.: The best rank-one approximation ratio of a tensor space. SIAM J. Matrix Anal. Appl. **32**, 430–442 (2011)
34. Qi, L., Sun, W., Wang, Y.: Numerical multilinear algebra and its applications. Front. Math. China **2**, 501–526 (2007)
35. Qi, L., Teo, K.L.: Multivariate polynomial minimization and its application in signal processing. J. Glob. Optim. **26**, 419–433 (2003)
36. Qi, L., Wang, F., Wang, Y.: Z-eigenvalue methods for a global polynomial optimization problem. Math. Program. **118**, 301–316 (2009)
37. Qi, L., Wang, Y., Wu, E.X.: D-eigenvalues of diffusion kurtosis tensor. J. Comput. Appl. Math. **221**, 150–157 (2008)
38. Qi, L., Yu, G., Wu, E.X.: Higher order positive semidefinite diffusion tensor imaging. SIAM J. Imaging Sci. **3**, 416–433 (2010)
39. Sturmfels, B.: Solving Systems of Polynomial Equations, CBMS Regional Conference Series in Mathematics, 97. American Mathematical Society, Providence (2002)
40. Zhang, X., Qi, L., Ye, Y.: The cubic spherical optimization problems. Math. Comput. **81**, 1513–1526 (2012)