# Tensor eigenvectors and stochastic processes

*Austin R. Benson and David F. Gleich · SIAM ALA2018 · Hong Kong*

*Topics covered.*

- Tensor eigenvectors
- Z-eigenvectors
- Irreducible tensors
- Higher-order Markov chains
- Spacey random walks
- Vertex reinforced random walks
- Dynamical systems as algorithms
- Fitting spacey walks to data
- Multilinear PageRank models
- Clustering tensors
- *And lots of open problems in this area!*

*Outline.*

Act 1. 10:45-10:55am Overview
Act 2. 10:55-11:05am Motivating applications
Act 3. 11:05-11:30am Stochastic processes & Markov chains
Act 4. 11:40-12:10pm Spacey random walk stochastic process
Act 5. 12:10-12:30pm Theory of spacey random walks
Act 6. 12:30-12:45pm Applications of spacey random walks

*Basic objects.*

| | |
|---|---|
| scalars | $\alpha$, $x_i$, $A_{ij}$, $\underline{\boldsymbol{P}}(i, j, k)$, $\underline{\boldsymbol{P}}_{i,j,k}$, $[\boldsymbol{A}^2]_{i,j}$ |
| vectors | $\mathbf{x}$, $\mathbf{x}_k$ |
| matrices | $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{Q}$, $\boldsymbol{P}$ |
| tensors | $\underline{\boldsymbol{A}}$, $\underline{\boldsymbol{P}}$, $\underline{\boldsymbol{T}}$ |

*Special objects.*

| | | |
|---|---|---|
| 3-mode transition probability tensor | $\underline{\boldsymbol{P}}$ | $\sum_{i=1}^{n} \underline{\boldsymbol{P}}_{i,j,k} = 1$ |
| $m$-mode transition probability tensor | $\underline{\boldsymbol{P}}$ | $\sum_{i=1}^{n} \underline{\boldsymbol{P}}_{i,j,k,\ldots,l} = 1$ |

*Operations.*

| | | |
|---|---|---|
| matrix-vector | $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ | $y_i = \sum_{j=1}^{n} \boldsymbol{A}_{i,j} x_j$ |
| tensor-vector | $\mathbf{y} = \underline{\boldsymbol{T}}\mathbf{x}^2$ | $y_i = \sum_{j,k} \underline{\boldsymbol{T}}_{i,j,k} x_j x_k$ |
| tensor-vector | $\mathbf{y} = \underline{\boldsymbol{T}}\mathbf{x}^{m-1}$ | $y_i = \sum_{j,k,\ldots,l} \underline{\boldsymbol{T}}_{i,j,k,\ldots,l} x_j x_k \cdots x_l$ |
| tensor-collapse | $\mathbf{y} = \underline{\boldsymbol{T}}[\mathbf{x}]$ | $Y_{i,j} = \sum_{k} \underline{\boldsymbol{T}}_{i,j,k} x_k$ |
| tensor-collapse | $\mathbf{y} = \underline{\boldsymbol{T}}[\mathbf{x}]^{m-1}$ | $Y_{i,j} = \sum_{k,\ldots,l} \underline{\boldsymbol{T}}_{i,j,k,\ldots,l} x_k \cdots x_l$ |

*Tensor eigenvector equations.*

| | | |
|---|---|---|
| $Z$-eigenvector | $\underline{\boldsymbol{T}}\mathbf{x}^2 = \lambda\mathbf{x}$ | $\sum_{j,k} \underline{\boldsymbol{T}}_{i,j,k} x_j x_k = \lambda x_i$ |
| $Z$-eigenvector | $\underline{\boldsymbol{T}}\mathbf{x}^{m-1} = \lambda\mathbf{x}$ | $\sum_{j,k,\ldots,l} \underline{\boldsymbol{T}}_{i,j,k,\ldots,l} x_j x_k \cdots x_l = \lambda x_i$ |
| $H$-eigenvector | $\underline{\boldsymbol{T}}\mathbf{x}^2 = \lambda\mathbf{x}^2$ | $\sum_{j,k} \underline{\boldsymbol{T}}_{i,j,k} x_j x_k = \lambda x_i^2$ |
| $H$-eigenvector | $\underline{\boldsymbol{T}}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{m-1}$ | $\sum_{j,k,\ldots,l} \underline{\boldsymbol{T}}_{i,j,k,\ldots,l} x_j x_k \cdots x_l = \lambda x_i^{m-1}$ |

*Stochastic processes.*

| | |
|---|---|
| sequence of random variables | $Z_0, Z_1, \ldots, Z_t, Z_{t+1}, \ldots$ |
| first-order Markov chain | $\Pr(Z_{t+1} = i \mid Z_t = j) = P_{i,j}$ |
| second-order Markov chain | $\Pr(Z_{t+1} = k \mid Z_t = j, Z_{t-1} = i) = \underline{P}_{i,j,k}$ |
| limiting distribution | $x_k = \lim_{t\to\infty} \frac{1}{t} \sum_{j=1}^{t} \operatorname{Ind}[Z_j = k]$ |

*Spacey random walks.*

| | |
|---|---|
| occupancy vector | $[\mathbf{w}_t]_k = \frac{1}{t+n}(1 + \sum_{j=1}^{t} \operatorname{Ind}[Z_j = k])$ |
| GVRRW map | $\boldsymbol{M}(\mathbf{w}_t) = \underline{\boldsymbol{P}}[\mathbf{w}_t]^{m-1}$, $\boldsymbol{M}(\mathbf{w}_t)$ is column stochastic matrix |
| GVRRW dynamical system | $d\mathbf{x}/dt = \pi[\boldsymbol{M}(\mathbf{x})] - \mathbf{x}$, $\pi$ maps $\boldsymbol{M}(\mathbf{x})$ to its Perron vector |
| SRW dynamical system | $d\mathbf{x}/dt = \pi[\underline{\boldsymbol{P}}[\mathbf{x}]^{m-2}] - \mathbf{x}$, $\pi$ maps $\underline{\boldsymbol{P}}[\mathbf{x}]^{m-2}$ to its Perron vector |