Asymmetric LSH (ALSH) for Sublinear Time Maximum Inner Product Search (MIPS)

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The Maximum Inner Product Search (MIPS) Problem

Given a query $q \in \mathbb{R}^D$ and a giant collection $\mathcal{C}$ of N vectors in $\mathbb{R}^D$, search for $p \in \mathcal{C}$ s.t.,

$$p = \arg\max_{x \in \mathcal{C}} \quad q^T x$$

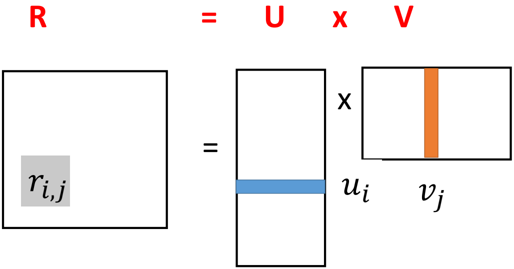
- Worst case O(N) for any query. N is huge.
- O(N) quite expensive for frequent queries.

Our goal is to solve this efficiently (something sub-linear)

Not same as the classical near-neighbor search problem.

$$\arg\min_{x \in \mathcal{C}} ||q - x||_2^2 = \arg\min_{x \in \mathcal{C}} (||x||_2^2 - 2q^T x) \neq \arg\max_{x \in \mathcal{C}} \quad q^T x$$

Scenario 1: User-Item Recommendations



- Matrix factorizations for collaborative filtering.
- Given a user u_i, the best item to recommend is a MIPS instance

$$Item = \arg\max_j \; r_{i,j} = \arg\max_j \; u_i^T v_j$$

Vectors u_i and v_j are learned. No control over norms.

Scenario 2: Multi-Class Prediction

Standard multi-class SVM in practice learns a weight vector w_i for each of the class label $i \in \mathcal{L}$.

Predicting a new x_{test}, is a MIPS instance:

$$y_{test} = \arg\max_{i \in \mathcal{L}} \; x_{test}^T \, w_i$$

w_is are learned and usually have varying norms.

Note: Fine grain ImageNet classification has 100,000 classes, in practice this number can be much higher.

Scenario 3: Web-Scale Deep Architectures

Activation of hidden node i monotonic in $x^T w_i$.

MAXOUT (Goodfellow et. al. 13) only requires updating hidden nodes having max activations.

$$h(x) = \max_j \quad x^T w_j;$$

Max-Product Networks (Gens & Domingos 12)
Adaptive Dropouts (Ba & Frey 13)

Problems:

- Each iteration, find max activation for every data point in every layer.
- **Networks with millions (or billions) of hidden nodes ?**

Efficient MIPS $\Longrightarrow$ Fast Training and Testing of Giant Networks

Note: No control over norms.

Many More !!

- Max of affine function approximations. (Prof. Nesterov's Talk)
- Active Learning
- Deformable parts model in vision
- Cutting plane methods and Greedy coordinate ascent.
- Greedy matching pursuit.
- ...

Outline of the talk

- Brief Introduction to Locality Sensitive Hashing (LSH).
- A Negative Result, Symmetry Cannot Solve MIPS.
- Construction of Asymmetric Hashing (ALSH) for MIPS
- Experiments.
- Extensions and More Connections.

Locality Sensitive Hashing (LSH)

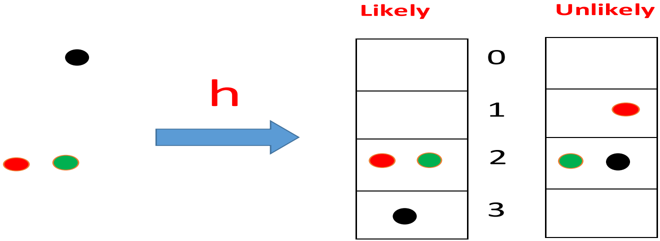
Hashing: Function (randomized) h that maps a given data vector $x \in \mathbb{R}^D$ to an integer key $h : \mathbb{R}^D \Rightarrow [0, 1, 2, ..., N]$

Locality Sensitive: Additional property

$$Pr_h\Big[h(x) = h(y)\Big] = f(sim(x, y)),$$

where f is monotonically increasing.

Similar points are more likely to have the same hash value (hash collision) compared to dissimilar points.



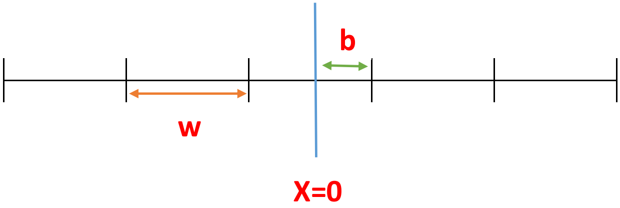
Hashing for L_2 Distance

L2-LSH

$$h_w(x) = \left\lfloor \frac{r^T x + b}{w} \right\rfloor$$

where $r \in R^D$ drawn independently from $N(0, \mathcal{I})$, b is drawn uniformly from $[0, w]$. w is a parameter. $\lfloor\rfloor$ is the floor operation.

It can be shown that $Pr(h_w(x) = h_w(y))$ **is monotonic in** $||x - y||_2$



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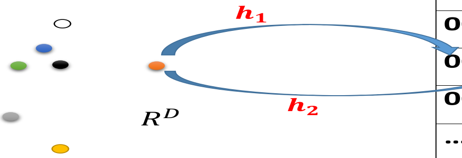
b **hurts**. 1-bit (Sign) or 2-bit hashing is preferred (Li et. al. ICML 14)

Sub-linear Near Neighbor Search: Idea



$h_1, h_2 : R^D \rightarrow \{0, 1, 2, 3\}$

h_1	h_2	Buckets (pointers only)
00	00	...
00	01	...
00	10	Empty
...	...	...
11	11	...

- Given query q, if $h_1(q) = 11$ **and** $h_2(q) = 01$, then probe bucket with index **1101**. It is a good bucket !!
- (LSH Property) $h_i(q) = h_i(x)$ is an indicator of high similarity between q and x for all i.

The Classical LSH Algorithm

Table 1

h_1^1	...	h_K^1	Buckets
00	...	00	...
00	...	01	...
00	...	10	Empty
...	...	...	...
11	...	11	...

...

Table L

h_1^L	...	h_K^L	Buckets
00	...	00	...
00	...	01	...
00	...	10	
...	...	...	...
11	...	11	Empty

Querying: Report union of L buckets.

- We can use K concatenation. To improve recall we can repeat the process L times and take union.
- K and L are two knobs.

Theory says we have a sweet spot. Provable sub-linear algorithm.

Negative Result: LSH Cannot Solve MIPS !!

- For inner products, we can have x and y, s.t $x^T y > x^T x$. Self similarity is not the highest similarity.

- Under any hash function $Pr(h(x) = h(x)) = 1$. But we need

$$Pr(h(x) = h(y)) > Pr(h(x) = h(x)) = 1$$

We cannot have Locality Sensitive Hashing for inner products !

Extend Framework: Asymmetric LSH (ALSH)

Main concern: $Pr(h(x) = h(x)) = 1$, an obvious identity.

How about asymmetry ?

- We can use $P(.)$ for creating buckets.
- While querying probe buckets using $Q(.)$ with $P \neq Q$.

$$Pr(Q(x) = P(x)) \neq 1$$

All we need is $Pr(Q(q) = P(x))$ to be monotonic in $q^T x$.

- Same proofs work !!
- Symmetry in hashing is unnecessary part of LSH definition.

Fine ! How do I construct P and Q ?

Reduce the problem to known domain we are comfortable with.
(Tea-kettle principle in mathematics)

Construction

Known: $Pr\big(h(q) = h(x)\big) = f(||q||_2^2 + ||x||_2^2 - 2q^Tx)$ **(L2 LSH)**

Idea: Construct P and Q s.t. $||Q(q)||_2^2 + ||P(x)||_2^2 - 2Q(q)^TP(x)$ is monotonic (or approximately) in q^Tx. This is also sufficient !!
(We can expand dimensions as part of P and Q.)

Pre-Processing
Scale data x, such that

$$||x_i|| < 1 \quad \forall x_i \in \mathcal{C}$$

$$P(x_i) = [x_i; ||x_i||_2^2; ||x_i||_2^4; ....; ||x_i||_2^{2^m}]$$

Querying q

$$Q(q) = [q; 1/2; 1/2; ....; 1/2]$$

$$||Q(q) - P(x_i)||_2^2 = (||q||^2 + m/4) - 2q^Tx_i + ||x_i||_2^{2^{m+1}}$$

$||x_i||^{2^{m+1}} \to 0$, and m is constant. We therefore have

$$\arg\max_{x \in \mathcal{S}} q^Tx \simeq \arg\min_{x \in \mathcal{S}} ||Q(q) - P(x)||_2$$

Main Result: First Provable Hashing Algorithm for MIPS

Theorem

*(**Approximate MIPS is Efficient**) For the problem of c-approximate MIPS, one can construct a data structure having $O(n^{\rho^*}\log n)$ query time and space $O(n^{1+\rho^*})$, where $\rho^* < 1$.*

$$\rho_u^* = \min_{0<U<1, m\in N, r} \frac{\log F_r\left(\sqrt{m/2 - 2S_0\left(\frac{U^2}{M^2}\right) + 2U^{2^{m+1}}}\right)}{\log F_r\left(\sqrt{m/2 - 2cS_0\left(\frac{U^2}{M^2}\right)}\right)}$$

$$s.t. \quad \frac{U^{(2^{m+1}-2)}M^2}{S_0} < 1 - c,$$

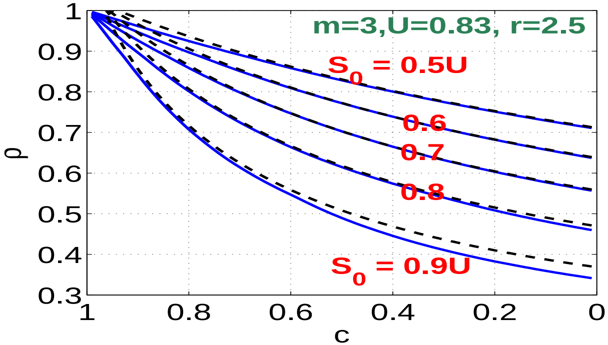
The only assumption needed is bounded norms, i.e. M is finite.

Why parameters do not bother us ?

Theory

- Even classical LSH requires computing K and L for a given c-approximate instance, and there is no fixed universal choice.
- Not hyper-parameters, can be exactly computed.
- In theory, **we do not lose any properties.**

In practice, there is a good choice: $m = 3, \quad U = 0.83, \quad r = 2.5$



The Final Algorithm

Preprocessing: Scale $\implies$ Append 3 Numbers $\implies$ Usual L2 - LSH

- Scale $x \in \mathcal{C}$ to have norm ≤ 0.83
- Append $||x_i||^2$, $||x_i||^4$, and $||x_i||^8$ to vector x_i. (just 3 scalars)
- Use standard L2 hash to create hash tables.

Querying: Append 3 Numbers $\implies$ Usual L2 - LSH

- Append 0.5 three times to the query q. (just three 0.5s)
- Use standard L2 hash on the transformed query to probe buckets.

- **A surprisingly simple algorithm.**
- Trivial to implement.

How much benefit compared to standard hash functions ?

Datasets and Settings

Datasets

- Movielens (10M)
- Netflix

Latent User Item Features:

- Matrix factorization to generate user and item latent features.
- Dimension: 150 for Movielens and 300 for Netflix.

Given a user as query, finding the best item is a MIPS instance.
Aim: Evaluate and compare computational savings.

Competing Hash Functions

- ALSH (proposed)
- L2-LSH (LSH for L2 distance)
- Signed Random Projections (SRP)

Ranking Quality Based on Hash Collisions

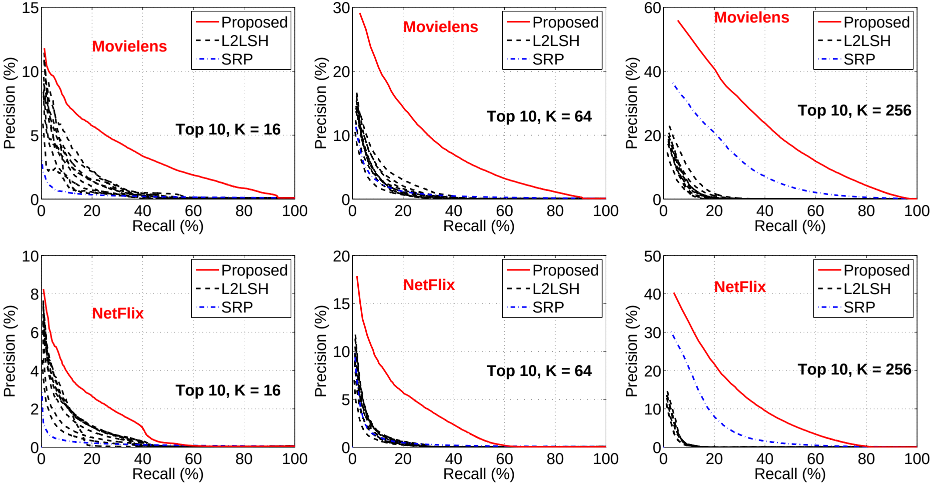


Figure: Precision-Recall curves (**higher is better**), for top-10 items.

- Previous evaluations are not true indicators of computational savings.
- **Bucketing Experiments:** We need to ensure comparison is fair.
- Find best $K \in [1 - \mathbf{40}]$ and $L \in [1 - \mathbf{400}]$ for each recall.
- Summary of **16000** experiments. Averaged over 2000 queries.

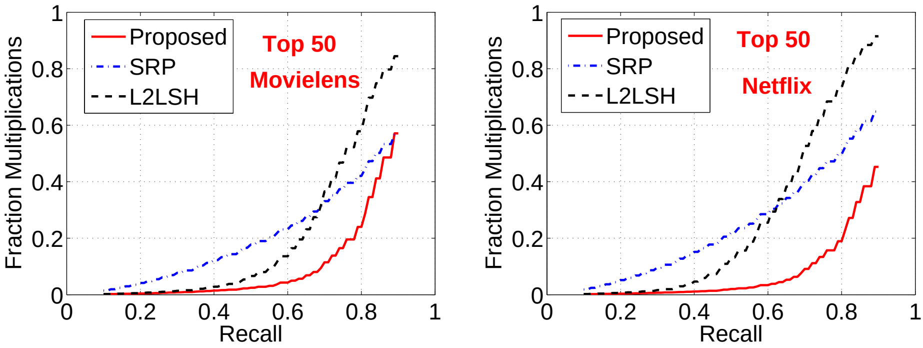


Figure: Mean inner products computed, relative to a linear scan. **Lower is better**.

A Generic Recipe for Constructing New ALSHs

In this work

- LSH for $\mathcal{S}'(q,x) = ||q - x||_2$. $\implies$ ALSH for $S(q,x) = q^Tx$.

Can we do better ? (Yes !)

- Signed Random Projections (SRP) leads to more informative hashes than L2-LSH (Li et. al. ICML 2014). **SRP as Base LSH** ?
- Use $\mathcal{S}'(q,x) = \frac{q^Tx}{||q||_2||x||_2}$, we can construct a different set of P and Q.
- **For binary data:** Minwise hashing and $\mathcal{S}'(q,x) = \frac{|q \cap x|}{|q \cup x|}$. (Jaccard)

Asymmetric Minwise Hashing for Set Intersection

(Shrivastava & Li 2014) "Asymmetric Minwise Hashing"

For sparse data (not necessarily binary):

- Minwise hashing is better than SRP (Shrivastava & Li AISTATS 14)
- $\mathcal{S}'(q,x) = \frac{|q \cap x|}{|q \cup x|}$ (Jaccard Similarity).
- Another P and Q with minwise hashing, **significant improvements**.

$$P(x) = [x; M - f_x; 0] \qquad Q(x) = [x; 0; M - f_x]$$

f_x is number of non-zeros in x, M is maximum sparsity.

Excercise in Construction of P and Q: **Expand dimensionality** and cancel out effect of terms we do not want in $\mathcal{S}'(q,x)$.

Conclusions

- We provide the first provable and practical hashing solution to MIPS.
- MIPS occurs as a subroutine in many machine learning application. All those applications will directly benefit from this work.
- Constructing asymmetric transformations to reduce MIPS to near-neighbor search was the key realization.
- In the task of item recommendation, we obtain significant savings compared to well known heuristics.
- Idea behind the ALSH construction is general and it connects MIPS to many known similarity search problems.

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Thanks for Your Attention !!