How Bad is Selfish Routing?

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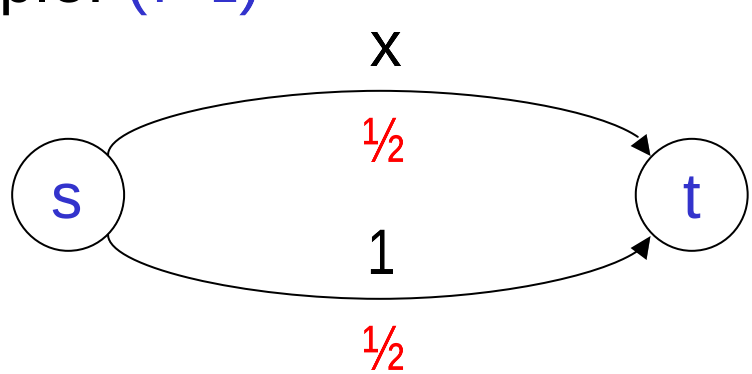
Traffic in Congested Networks

Given:

- A directed graph $G = (V,E)$
- A source s and a sink t
- A rate r of traffic from s to t
- For each edge e, a latency function $l_e(\bullet)$

Example: $(r=1)$



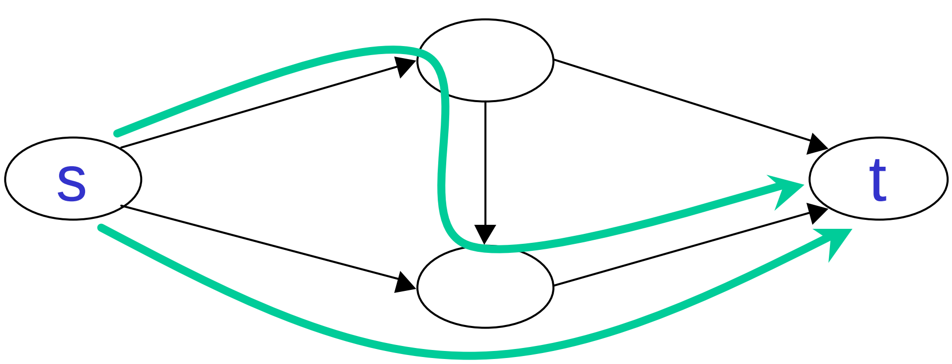
Flows and their Cost

Traffic and Flows:

- f_P = amount of traffic routed on s-t path P
- flow vector f ⇔ traffic pattern at steady-state

The Cost of a Flow:

- $l_P(f)$ = sum of latencies of edges on P (w.r.t. the flow f)
- $C(f)$ = cost or total latency of flow f: $S_P \; f_P \bullet l_P(f)$

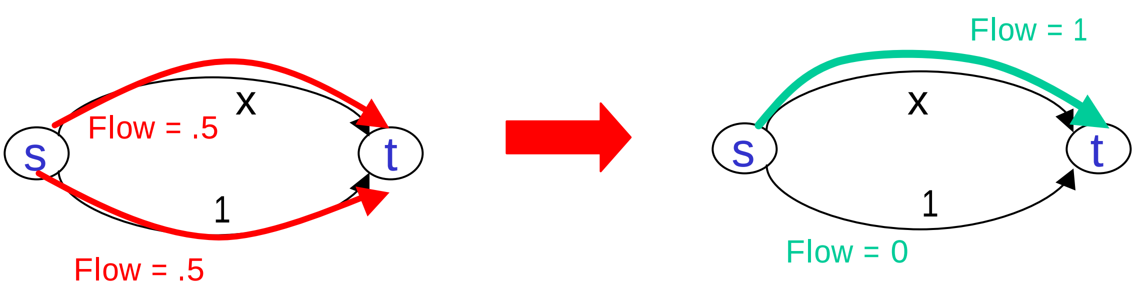


Flows and Game Theory

- flow = routes of many noncooperative agents
- Examples:
  - cars in a highway system
  - packets in a network
    - [at steady-state]
- cost (total latency) of a flow as a measure of social welfare
- agents are selfish
  - do not care about social welfare
  - want to minimize personal latency

Flows at Nash Equilibrium

Def: A flow is at Nash equilibrium (is a Nash flow) if no agent can improve its latency by changing its path

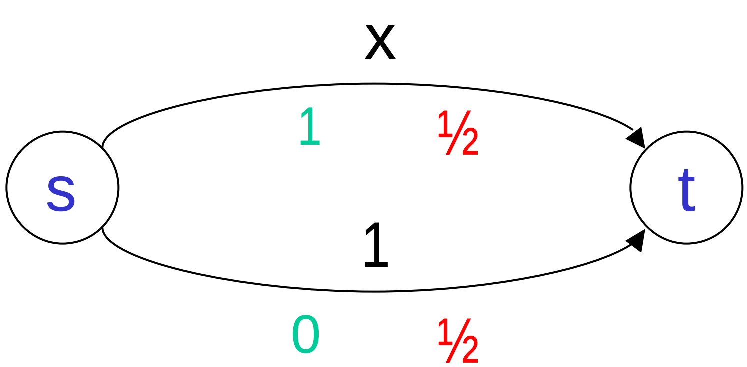


Assumption: edge latency functions are continuous, nondecreasing

Lemma: f is a Nash flow if and only if all flow travels along minimum-latency paths (w.r.t. f)

Nash Flows and Social Welfare

Central Question: To what extent does a Nash flow optimize social welfare? What is the cost of the lack of coordination in a Nash flow?



Cost of Nash flow = 1•1 + 0•1 = 1

Cost of optimal (min-cost) flow

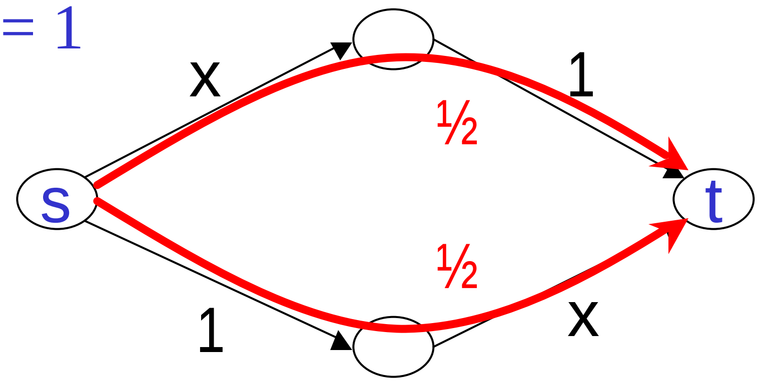
= ½•½ +½•1 = ¾

Previous Work

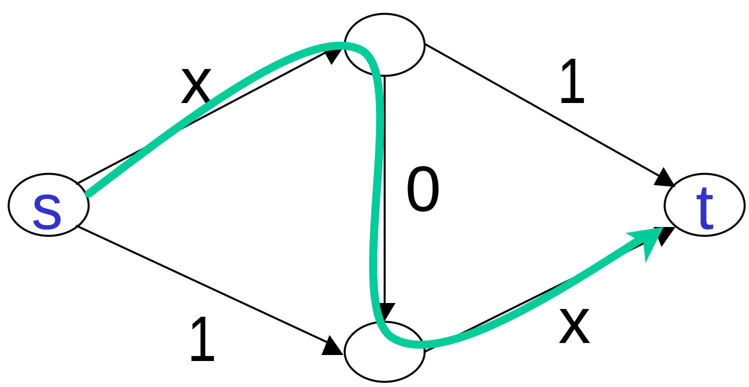
- [Beckmann et al. 56], ...
  - Existence, uniqueness of flows at Nash equilibrium
- [Dafermos/Sparrow 69], ...
  - Efficiently computing Nash and optimal flows
- [Braess 68], ...
  - Network design
- [Koutsoupias/Papadimitriou 99]
  - Quantifying the cost of a lack of coordination

Braess's Paradox

Rate: r = 1



Cost of Nash flow = 1.5



Cost of Nash flow = 2

All flow experiences more latency!

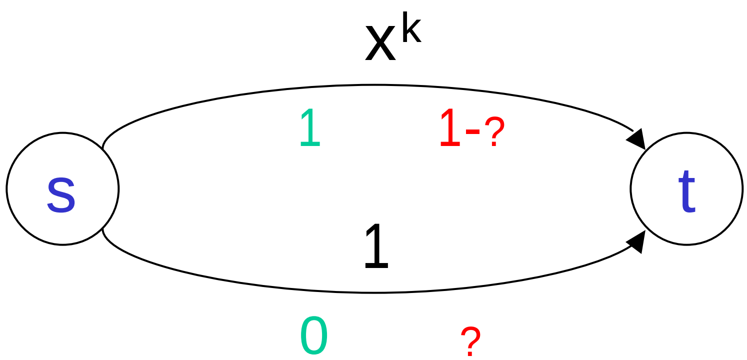
Our Results for Linear Latency

Def: a linear latency function is of the form $l_e(x)=a_e x+b_e$

Theorem 1: In a network with linear latency functions, the cost of a Nash flow is at most 4/3 times that of the minimum-latency flow.

General Latency Functions?

Bad Example: (r = 1, k large)



Nash flow has cost 1, min cost ≈ 0

Þ Nash flow can cost arbitrarily more than the optimal (min-cost) flow

– even if latency functions are polynomials

Our Results for General Latency

All is not lost: the previous example does not preclude interesting bicriteria results.

Theorem 2: In any network with continuous, nondecreasing latency functions:

The cost of a Nash flow with rate r is at most the cost of an optimal flow with rate $2r$.

Characterizing the Optimal Flow

Cost $f_e \cdot l_e(f_e)$ Þ marginal cost of increasing flow on edge e is

$$l_e(f_e) + f_e \cdot l_e'(f_e)$$

latency of new flow

Added latency of flow already on edge

Key Lemma: a flow f is optimal if and only if all flow travels along paths with minimum marginal cost (w.r.t. f).

The Optimal Flow as a Socially Aware Nash

A flow f is optimal if and only if all flow travels along paths with minimum marginal cost

Marginal cost: $l_e(f_e) + f_e \cdot l_e'(f_e)$

A flow f is at Nash equilibrium if and only if all flow travels along minimum latency paths

Latency: $l_e(f_e)$

Consequences for Linear Latency Fns

Observation: if $l_e(f_e) = a_e f_e + b_e$ (latency functions are linear) Þ marginal cost of P w.r.t. f is:

$$S_{e \in P} \; 2a_e f_e + b_e$$

Corollary: f a Nash flow with rate r in a network with linear latency fns Þ $f/2$ is optimal with rate $r/2$

Conclusions

- Multicommodity analogues of both results (can specify rate of traffic between each pair of nodes)
- Approximate versions assuming imprecise evaluation of path latency
- Open: extension to a model in which agents may control the amount of traffic (in addition to the routes)
  - Problem: how to avoid the "tragedy of the commons"?