On Satisfiability

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Abstract

Among the NP-hard problems is 3SAT [2, 4], which asks if there exists a satisfying assignment for the Boolean variables in a conjunctive normal form formula. This paper gives a mapping of 3SAT problems into multi-dimensional definite integrals. These integrals take on values greater than $1 - \epsilon$ if the Boolean formula is satisfiable and less than 1/2 when it is not. They thus serve as effective indicators of the satisfiability of Boolean formulas. The number of dimensions of the integral is the same as the number of variables in the Boolean formula, so deterministic methods are unlikely to be successful. However, Monte Carlo techniques look promising. Regardless, the transformation of the purely combinatorial problem of satisfiability to the evaluation of a definite integral (of simple, continuous arguments), allows us to apply the vast arsenal of modern analysis to a problem that hitherto seemed an unlikely target.

Given a Boolean formula, the problem of determining the existence of a satisfying assignment to its variables is known as *propositional satisfiability* or SAT. This is the simplest form of theorem proving and occurs in artificial intelligence, automatic verification of programs, circuit synthesis [11] and diagnosis [13] and reasoning about programs. In addition to its interest for applications, SAT was the first problem shown to be a member of the class of NP-hard tasks [2]. Thus, the complexity of solving a satisfiability problem is polynomially equivalent to a wide range of other problems including the traveling salesman problem, graph coloring etc. [4].

Existing techniques for determining satisfiability are, at their core, some form of combinatorial search [17, 16, 6, 5, 3]. Although the satisfiability problem can be converted into a linear programming problem [14, 15], the resulting problem still involves a searching phase to avoid local minimums. On the other hand, the recent results on the statistical behavior of satisfiability [12, 8]—the existence of "phase transition" type behavior—might be explainable using the analytic tools developed in this paper.

In contrast, this paper demonstrates that SAT problems can be resolved by computing the value of a multidimensional integral, with a particularly simple integrand. This result provides the first direct relationship between an NP hard combinatorial problem and a natural, continuous problem. Direct, deterministic methods for computing the multidimensional integral require an exponential number of evaluations of the integrand, although Monte Carlo methods look encouraging. Regardless, the continuous form of this result allows the tools of modern analysis to be directly applied to the satsifiability problem and may provide a deeper understanding of the complexity of the NP hard problems.

1. Introduction

To be precise, the satisfiability problems we consider are 3SAT Boolean formulas in the variables $X_1, \ldots, X_n$. These formulas have the form

$$F(X_1, \ldots, X_n) \\ = (V_1 \vee V_2 \vee V_3) \wedge \cdots \wedge (V_{3k-2} \vee V_{3k-1} \vee V_{3k}), \quad (1)$$

where each V_j is either an X_i or its negation, $\neg X_i$. Logical AND is indicated by $\wedge$ and logical OR by $\vee$. The parenthesized logical OR's are called *clauses*. Each clause consists of precisely three V_j. Distributing $\wedge$ over $\vee$ allows us to write F as

$$F(X_1, \ldots, X_n) = M_1 \vee M_2 \vee \cdots \vee M_{3^k},$$

where each of the M_j is the logical AND of k V_i, one from each parenthesized clause in (1). F is satisfiable if at least one of the M_j is satisfiable. Since the M_j are conjunctions of variables and their complements the only way an M_j could not be satisfiable is if the conjunction contains both an X_i and $\neg X_i$, for some i.

We map Boolean formulas into polynomials according to the rules

$$X_i \rightarrow \left(\frac{1+x_i}{2}\right) \quad \text{and} \quad \neg X_i \rightarrow \left(\frac{1-x_i}{2}\right).$$

Logical OR is represented by addition and logical AND is represented by multiplication. Throughout this paper capital letters represent Boolean variables or Boolean formulas and lower case symbols represent their continuous counterparts. So, the Boolean function $F(X_1, \ldots, X_n)$ is mapped to the *Boolean polynomial* $\mathscr{P}(F) = f(x_1, \ldots, x_n)$. For example,

$$(X_1 \vee \neg X_2) \wedge (\neg X_1 \vee X_2) \rightarrow \left(\frac{1+x_1}{2} + \frac{1-x_2}{2}\right)\left(\frac{1-x_1}{2} + \frac{1+x_2}{2}\right).$$

Note the important property of the transformation that leaves the structure of the Boolean formula unchanged.

$$\begin{aligned}&\mathscr{P}((V_1 \vee V_2) \wedge (V_3 \vee V_4)) \\ &\quad = (\mathscr{P}(V_1) + \mathscr{P}(V_2)) \cdot (\mathscr{P}(V_3) + \mathscr{P}(V_4)), \\ &\quad = \mathscr{P}(V_1) \cdot \mathscr{P}(V_3) + \cdots + \mathscr{P}(V_2) \cdot \mathscr{P}(V_4), \\ &\quad = \mathscr{P}((V_1 \wedge V_3) \vee (V_1 \wedge V_4) \vee (V_2 \wedge V_3) \vee (V_2 \wedge V_4)).\end{aligned} \tag{2}$$

Thus when a Boolean formula is expressed as an OR of conjunctions (sum of products), the corresponding polynomial is a sum of products of terms of the form

$$\frac{1}{2} \pm \frac{x_i}{2}.$$

We call these products *Boolean monomials.*

The next step in our approach is to compute a definite integral of these polynomials over each of the n dimensions. The integral's kernel is chosen such that the definite integral of a satisfiable monomial dominates the value of the non-satisfiable monomials. The key to our approach is the observation that

$$\begin{aligned}&\int_{-1}^{1} \left(\frac{1 \pm x}{2}\right)^r \left(\frac{\omega}{2} \cos \frac{\omega}{2} x\right) dx \\ &\qquad = \sin \frac{\omega}{2} + \frac{r}{\omega} \cos \frac{\omega}{2} - \frac{r(r-1)}{\omega} \sin \frac{\omega}{2} + \cdots .\end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}&\int_{-1}^{1} \left(\frac{1+x}{2}\right)^r \left(\frac{1-x}{2}\right) \left(\frac{\omega}{2} \cos \frac{\omega}{2} x\right) dx \\ &= -\frac{1}{\omega} \cos \frac{\omega}{2} + \frac{2(r+1)}{\omega^2} \sin \frac{\omega}{2} + \frac{3r(r+1)}{\omega^3} \cos \frac{\omega}{2} + \cdots .\end{aligned} \tag{4}$$

As ω increases (at odd integral multiples of π), the first integral tends to 1 and the second integral tends to zero. The coefficients of the higher order terms grow slowly so we can precisely bound the omitted terms.

Given the Boolean formula $F(X_1, \ldots, X_n)$ we consider the integral

$$\begin{aligned}&\mathscr{I}(F) = \\ &\int_{-1}^{1} \cdots \int_{-1}^{1} f(x_1, \ldots, x_n) \left(\frac{\omega}{2} \cos \frac{\omega}{2} x_1\right) \times \cdots \\ &\qquad \times \left(\frac{\omega}{2} \cos \frac{\omega}{2} x_n\right) dx_1 \cdots dx_n,\end{aligned} \tag{5}$$

which is the multivariate version of the integral given in (3) and (4). We call $\mathscr{I}(F)$ the *indicator* of F. If f is a Boolean polynomial we will also use $\mathscr{I}(f)$ to represent the integral given above. The absolute value of $\mathscr{I}(F)$ is the *ω-norm* of the F, which we denote by $\|F\|_\omega = |\mathscr{I}(F)| = \|f\|_\omega$. We show that for large values of ω, the value of the ω-norm identifies whether or not the boolean formula is satisfiable.

Using (2) we can focus on a single monomial. We show that all non-satisfiable monomials, M_{NSAT}, satisfy

$$\|M_{\mathrm{NSAT}}\|_\omega \leq \frac{B}{\omega^2},$$

where B is effectively computable. If a 3SAT Boolean formula F_{NSAT} is not satisfiable then each of the 3^k monomials in its sum of products form is also not satisfiable. Thus,

$$\|F_{\mathrm{NSAT}}\|_\omega \leq \frac{3^k B}{\omega^2}.$$

If M_{SAT} is a satisfiable monomial, we show that

$$1 - \frac{A}{\omega^2} \leq \mathscr{I}(M_{\mathrm{SAT}}) = \|M_{\mathrm{SAT}}\|_\omega,$$

where A is an explicitly computable value. A satisfiable formula has at least one satisfiable monomial. The non-satisfiable monomials can be as small as $-B\omega^{-2}$, so

$$1 - \frac{A}{\omega^2} - \frac{3^k B}{\omega^2} \leq \|F_{\mathrm{SAT}}\|_\omega,$$

If ω is chosen such that $\omega^2 > \delta(A + 3^k B)$, where δ is a real number larger than 1 then

$$\|F_{\mathrm{NSAT}}\|_\omega \leq \frac{1}{\delta} \qquad \text{and} \qquad 1 - \frac{1}{\delta} \leq \|F_{\mathrm{SAT}}\|_\omega.$$

By increasing ω moderately we can make δ as large as desired, increasing the gap between the ω norm of satisfiable and non-satisfiable Boolean formulas.

Note that even if A and B are exponentially large they will not affect the size of ω much. Although ω is at least 3^k, *i.e.*, exponential in the number of clauses in F, the number of bits required to represent ω is still polynomial and thus the number of bit operations remains polynomial in the size of the input.

In Section 2 we provide the details of the computations of A and B given above. Section 3 discusses approaches to computing $\|F\|_\omega$ for Boolean formulas and discusses the problems and promises of Monte Carlo techniques for this problem. Finally, in Section 4 we discuss the potential impact of these results.

2. Effective Bounds

In this section we prove precise bounds on the size of integrals of the form given in (5), where f is a Boolean polynomial. As noted in the introduction, initially we develop estimates for Boolean monomials, *i.e.*, expressions of the form

$$f(x_1, \ldots, x_n) = \left(\frac{1}{2} \pm \frac{x_1}{2}\right) \cdots \left(\frac{1}{2} \pm \frac{x_n}{2}\right),$$

which are the product of $k \geq n$ terms. Recall that n is the number of variables of the Boolean formula, and k is the number of terms in the conjunction.

For Boolean monomials the multidimensional integral can be written as a product of one dimensional integrals. For instance,

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} &\left(\frac{1}{2} \pm \frac{x_1}{2}\right)\left(\frac{1}{2} \pm \frac{x_2}{2}\right)\left(\frac{\omega}{2}\cos\frac{\omega}{2}x_1\right) \times \\ &\times \left(\frac{\omega}{2}\cos\frac{\omega}{2}x_2\right)\, dx_1 dx_2 \\ &= \left[\int_{-1}^{1}\left(\frac{1}{2} \pm \frac{x_1}{2}\right)\left(\frac{\omega}{2}\cos\frac{\omega}{2}x_1\right)\, dx_1\right] \\ &\quad \cdot \left[\int_{-1}^{1}\left(\frac{1}{2} \pm \frac{x_2}{2}\right)\left(\frac{\omega}{2}\cos\frac{\omega}{2}x_2\right)\, dx_2\right].\end{aligned}$$

So, we begin by computing bounds on *univariate Boolean monomials* of the form

$$\begin{aligned}N_m^{(r)}(\omega) &= \mathscr{I}(T^{m-r} \wedge (\neg T)^r) \\ &= \int_{-1}^{1}\left(\frac{1}{2} + \frac{t}{2}\right)^{m-r}\left(\frac{1}{2} - \frac{t}{2}\right)^{r}\left(\frac{\omega}{2}\cos\frac{\omega}{2}t\right)\, dt.\end{aligned}$$

In Section 2.1 we derive upper and lower bounds for satisfiable univariate boolean monomials, in Section 2.2 we give upper bounds for non-satisfiable univariate monomials—lower bounds are not needed. In Section 2.3 we combine these results to get precise bounds on the satisfiable and non-satisfiable (multivariate) monomials, and Section 2.4 we state the full theorem for Boolean formulas.

2.1. Satisfiable Univariate Monomials

Define S_m to be the indicator of a satisfiable univariate monomial:

$$\begin{aligned}S_m = N_m^{(0)} &= \mathscr{I}(T^m) = \mathscr{I}((\neg T)^m) \\ &= \int_{-1}^{1}\left(\frac{1}{2} \pm \frac{t}{2}\right)^m\left(\frac{\omega}{2}\cos\frac{\omega}{2}t\right)\, dt.\end{aligned}$$

First, observe that the value of S_m is independent of the sign chosen for the $\pm$ in the integral since

$$\int_{-1}^{1} t^r \left(\frac{\omega}{2}\cos\frac{\omega}{2}t\right)\, dt = 0$$

whenever r is odd.

By simple integration we have

$$S_0 = 2\sin\frac{\omega}{2}, \qquad \text{and} \qquad S_1 = \sin\frac{\omega}{2}.$$

Integrating by parts twice gives the following recurrence, which permits computation of higher values of S_m:

$$\begin{aligned}\int &\left(\frac{1}{2} + \frac{t}{2}\right)^m\left(\frac{\omega}{2}\cos\frac{\omega}{2}t\right)\, dt \\ &= \left(\frac{1}{2} + \frac{t}{2}\right)^m \sin\frac{\omega}{2}t + \frac{m}{\omega}\left(\frac{1}{2} + \frac{t}{2}\right)^{m-1}\cos\frac{\omega}{2}t \\ &\quad - \frac{m(m-1)}{\omega^2}\int\left(\frac{1}{2} + \frac{t}{2}\right)^{n-2}\left(\frac{\omega}{2}\cos\frac{\omega}{2}t\right)\, dt.\end{aligned}$$

Thus,

$$S_m = \sin\frac{\omega}{2} + \frac{m}{\omega}\cos\frac{\omega}{2} - \frac{m(m-1)}{\omega^2}S_{m-2}. \tag{6}$$

This recurrence can be unrolled into

$$\begin{aligned}S_m = \sin\frac{\omega}{2}&\left(\sum_{i=0}^{m/2}\frac{(-1)^i}{\omega^{2i}}\frac{m!}{(m-2i)!} + \frac{(-1)^{m/2}m!}{\omega^m}\right) \\ &+ \cos\frac{\omega}{2}\sum_{i=0}^{m/2}\frac{(-1)^i}{\omega^{2i+1}}\frac{m!}{(m-2i-1)!},\end{aligned} \tag{7}$$

when m is even and

$$\begin{aligned}S_m = \sin\frac{\omega}{2}&\sum_{i=0}^{(m-1)/2}\frac{(-1)^i}{\omega^{2i}}\frac{m!}{(m-2i)!} \\ &+ \cos\frac{\omega}{2}\sum_{i=0}^{(m-1)/2}\frac{(-1)^i}{\omega^{2i+1}}\frac{m!}{(m-2i-1)!},\end{aligned} \tag{8}$$

when m is odd. Notice that the only difference between these two formulas is the doubling of the last term in the even case. This technical complication arises from the fact that S_0 is twice S_1. From (7) and (8) we note that the coefficients of ω^{-2i} are bounded by m^{2i} and the heighest power of ω^{-1} that occurs is no greater than m.

When ω is restricted to odd multiples of π, (6) reduces to

$$S_m = 1 - \frac{m(m-1)}{\omega^2} S_{m-2}, \tag{9}$$

and

$$S_0 = 2, \qquad S_1 = 1, \qquad \text{and} \qquad S_2 = 1 - \frac{4}{\omega^2}.$$

A short table of the S_m is included in Appendix A. When $\omega \geq 2m$, S_1 and S_2 are positive and less than or equal to 1. Applying induction to (9), we see that this is also true for all m.

To compute a lower bound for $|S_m|$, the triangle inequality is applied to (9),

$$|S_m| \geq |1| - \left| \frac{m(m-1)}{\omega^2} S_{m-2} \right| \geq 1 - \frac{m(m-1)}{\omega^2}.$$

last inequality follows from the upper bound on S_m.

These results are summarized in the following proposition.

Proposition 1 *If m is positive and ω is an odd multiple of π greater than $2m$, then*

$$1 - \frac{m(m-1)}{\omega^2} \leq \|T^m\|_\omega = |S_m| \leq 1.$$

If $m = 0$ then $S_0 = 2$.

2.2. Unsatisfiable Univariate Monomials

For the unsatisfiable univariate monomials, we only need to compute upper bounds. That is, we need to compute upper bounds for $N_m^{(r)}$ where,

$$\begin{aligned} N_m^{(r)} &= \mathscr{I}(T^{m-r} \wedge (\neg T)^r) \\ &= \int_{-1}^{1} \left(\frac{1}{2} + \frac{t}{2}\right)^{m-2r} \left(\frac{1}{4} - \frac{t^2}{4}\right)^r \left(\frac{\omega}{2} \cos \frac{\omega}{2} t\right) dt. \end{aligned}$$

The "degree" of the Boolean monomial is m.

Observe that if $f(t)$ is a function of t and $\tilde{f}$ is its indicator,

$$\tilde{f} = \int_{-1}^{1} f(t) \left(\frac{\omega}{2} \cos \frac{\omega}{2} t\right) dt,$$

then the indicator of $f(t)(\frac{1}{4} - \frac{t^2}{4})$ is

$$\frac{\tilde{f}}{4} + \omega \frac{\partial^2}{\partial \omega^2} \left(\frac{\tilde{f}}{\omega}\right) = \int_{-1}^{1} f(t) \left(\frac{1}{4} - \frac{t^2}{4}\right) \left(\frac{\omega}{2} \cos \frac{\omega}{2} t\right) dt. \tag{10}$$

That is,

$$f(t) \times \left(\frac{1}{4} - \frac{t^2}{4}\right) \Longleftrightarrow \mathscr{N}(\tilde{f}) = \frac{\tilde{f}(\omega)}{4} + \omega \frac{\partial^2}{\partial \omega^2} \tilde{f}(\omega).$$

So, $N_m^{(0)} = S_m$ and

$$N_m^{(r)} = \frac{N_{m-2}^{(r-1)}}{4} + \omega \frac{\partial^2}{\partial \omega^2} \left(\frac{N_{m-2}^{(r-1)}}{\omega} \right) = \mathscr{N}^r(S_{m-2r}).$$

From the previous section we know that S_m is a sum of terms of the form

$$\frac{a_i}{\omega_i} \sin \frac{\omega}{2} \quad \text{and} \quad \frac{b_i}{\omega_i} \cos \frac{\omega}{2} \tag{11}$$

where the a_i and b_i are monic polynomials in m of degree less than i. Applying $\mathscr{N}$ to these expressions we have

$$\begin{aligned} &\mathscr{N}\left(\frac{a_j}{\omega^j} \sin \frac{\omega}{2}\right) \to \\ &\quad -\frac{a_j \cdot (j+1)}{\omega^{j+1}} \cos \frac{\omega}{2} + \frac{a_j \cdot (j+1)(j+2)}{\omega^{j+2}} \sin \frac{\omega}{2}, \\ &\mathscr{N}\left(\frac{b_j}{\omega^j} \cos \frac{\omega}{2}\right) \to \\ &\quad \frac{b_j \cdot (j+1)}{\omega^{j+1}} \sin \frac{\omega}{2} + \frac{b_j \cdot (j+1)(j+2)}{\omega^{j+2}} \cos \frac{\omega}{2}. \end{aligned}$$

$N_m^{(r)}$ is also the sum of terms of the form (11). In particular, we can write

$$\begin{aligned} N_m^{(r)} &= a_0^{(r)} \sin \frac{\omega}{2} + b_0^{(r)} \cos \frac{\omega}{2} \\ &\quad + \frac{a_1^{(r)}}{\omega} \sin \frac{\omega}{2} + \frac{b_0^{(r)}}{\omega} \cos \frac{\omega}{2} + \cdots, \end{aligned} \tag{12}$$

where

$$\begin{aligned} a_j^{(r)} &= j b_{j-1}^{(r-1)} + j(j-1) a_{j-2}^{(r-1)}, \\ b_j^{(r)} &= -j a_{j-1}^{(r-1)} + j(j-1) b_{j-2}^{(r-1)}, \end{aligned} \tag{13}$$

and $a_i^{(r-j)}$ and $b_i^{(r-j)}$ are the coefficients of $N_{m-2j}^{(r-j)}$.

Recall from (7) and (8) that the largest k for which S_{m-2r} has non-zero terms is $m - 2r$, so $N_m^{(r)}$ will have terms of order no larger than m (in ω^{-1}). Thus the j that appear in (13) is never larger than m. By the previous section, $a_i^{(0)}$ and $b_i^{(0)}$ are bounded by $(m -$

$2r)^i \leq m^i$. So, at worst, the difference formula above doubles $a_k^{(r)}$ with each increase in r. That is,

$$|a_i^{(r)}|, |b_i^{(r)}| \leq 2^r m^i.$$

This estimate allows us to compute an upper bound on $N_m^{(r)}$, which is given in the following proposition.

Proposition 2 *Let δ be a real number larger than 1 and assume ω is an odd integral multiple of π such that*

$$\omega > \frac{\delta m}{\delta - 1}. \tag{14}$$

then

$$\|N_m^{(r)}\|_\omega \leq \delta \left(\frac{2m}{\omega}\right)^r. \tag{15}$$

Proof: As usual, we can ignore the coefficients of the cosine terms in (12). Since we are looking for upper bounds we can use the triangle inequality and take the absolute values of the coefficients of the sine terms. Furthermore, notice that the lowest order term non-zero coefficient is either the ω^{-r} term or ω^{-r-1} term depending on whether r is even or odd.

$$\|N_m^{(r)}\|_\omega \leq \frac{(2m)^r}{\omega^r}\left[1 + \frac{m}{\omega} + \cdots \frac{m^m}{\omega^m}\right]$$

Since all the terms in the series are positive, the series is bounded above by the infinite version of the series:

$$\begin{aligned}\|N_m^{(r)}\|_\omega &\leq \frac{(2m)^r}{\omega^r}\left[1 + \frac{m}{\omega} + \frac{m^2}{\omega^2} + \cdots\right] \\ &\leq \frac{(2m)^r}{\omega^r} \cdot \frac{1}{1 - \frac{m}{\omega}} \leq \delta \frac{(2m)^r}{\omega^r}.\end{aligned}$$

For the last inequality to hold we must have

$$\frac{1}{\delta} < 1 - \frac{m}{\omega},$$

which is equivalent to (14). □

Notice that the quantity inside parentheses in (15) is less than 1 so that $|N_m^{(r)}|$ is largest when $r = 1$. Furthermore, due to the fact that leading term of S_m does not include a cosine term, the leading term of $N_m^{(r)}$, for odd r does not include a sine term. Thus $\|N_m^{(1)}\|_\omega$ actually has order ω^{-2} not ω^{-1} as one might at first expect.

2.3. Bounds on Multivariate Monomials

Assume M_{SAT} is a satisfiable monomial that involves X_i or $\neg X_i$ to order e_i and doesn't involve ℓ of the X_i's at all. Since the total degree of the monomial is k,

$$\|M_{\text{SAT}}\|_\omega \geq 2^\ell \prod_{e_1+\cdots+e_n=k}\left[1 - \frac{e_i(e_i-1)}{\omega^2}\right].$$

Each term in the product is less than 1, so $|M_{\text{SAT}}|$ is minimized when $\ell = 0$. This requires the e_i to all be at least 1. The product is minimized when $e_1 = k - n + 1$ and $e_2 = e_3 = \cdots = e_n = 1$ and maximized when the e_i are all equal. Therefore,

$$\|M_{\text{SAT}}\|_\omega \geq 1 - \frac{(k-n)(k-n+1)}{\omega^2}.$$

This gives us a lower bound on the size of satisfiable monomials.

An upper bound is achieved when ℓ is as large as possible, since the ω-norm of univariate Boolean monomials is bounded above by 1. In this case M_{SAT} is, in fact, a univariate Boolean monomial and $\ell = k-1$. So,

$$2^{k-1} \geq \|M_{\text{SAT}}\|_\omega \geq 1 - \frac{(k-n)(k-n+1)}{\omega^2}. \tag{16}$$

To make the ω-norm of a non-satisfiable formula as large as possible we again use a univariate Boolean monomial to accumulate the maximum number of factors of 2. That is, we use $\delta = 2$, $r = 1$ and $m = k$ in Proposition 2, so

$$\|M_{\text{NSAT}}\|_\omega \leq 2^{n-1} 2 \left(\frac{2k}{\omega}\right)^2 = \frac{k^2 \cdot 2^{n+2}}{\omega^2}. \tag{17}$$

Recall that $N_m^{(1)}$ actually has order ω^{-2} when evaluated at odd integral multiples of π since the ω^{-1} term only contains cosine terms.

2.4. Bounds for Multivariate Polynomials

Equations (16) and (17) of the previous section show that, in the notation of the introduction, we can use $A = (k-n)\cdot(k-n+1)$ and $B = k^2 2^{n+2}$. For the ω norm of satisfiable and non-satisfiable to be greater than $1 - \delta^{-1}$ and less than δ^{-1} respectively, ω should be chosen such that

$$\omega^2 \geq \delta\left((k-n)\cdot(k-n+1) + k 3^k 2^{n+1}\right).$$

In particular, $\omega > \sqrt{\delta} 2^{n+k}$ suffices. We state this as follows.

Proposition 3 *Assume $F(X_1, \ldots, X_n)$ is a Boolean formula in n variables in 3SAT conjunctive normal form with no more than k terms in the conjunction. For any value of δ greater than 1, the quantity $\|F\|_\omega$ is less than δ^{-1} if F is not satisfiable greater than $1-\delta^{-1}$ when F is satisfiable if ω is chosen to be an odd integral multiple of π greater than $\sqrt{\delta}2^{n+k}$.*

Proposition 3 is the main result of this paper. We have proven that there the computation of a multidimensional integral can be used to resolve the satisfiability of a Boolean formula. The values that this integral takes on for satisfiable and non-satisfiable formulas are well separated. Furthermore, the integrand of this integral is smooth, and the size of the numbers used to compute the integrand involve a polynomial number of bits in the size of the satisfiability problem.

3. Integral Evaluation via Monte Carlo Techniques

The computation of multidimensional integrals, such as those of the form (5) is rather difficult by deterministic means. A simple integration method like the trapezoid rule or Simpson's rule requires at least 2 evaluation points at in each dimension or a total of at least 2^n evaluations to compute the integral.

Nonetheless, there are effective randomized algorithms for computing these integrals called *Monte Carlo methods* [7, 10, 9]. At their core, these algorithms are applications of the central limit theorem of probability: When sampled over randomly chosen points within the integration domain, the average value of the integrand times the volume of the integration domain tends towards the value of the integral and the error falls off as the square root of the number of sample points. To be more precise

$$\int_{-1}^{1} \cdots \int_{-1}^{1} g(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \frac{2^n}{N} \sum_{i=1}^{N} g(x_1^{(i)}, \ldots, x_n^{(i)}) + \frac{\mathrm{Var}(g)}{\sqrt{N}}, \quad (18)$$

where the $x_j^{(i)}$ are randomly chosen from the interval $[-1, 1]$. Notice that the error that arises in this method is dependent only on the number of sample points and does not depend upon the number of dimensions in the integral.

For simplicity we suppress mention of the number of dimensions as follows

$$\int_\Omega g(x)\, dV = \frac{1}{N} \sum_{i=1}^{N} g(x^{(i)}) + \frac{\mathrm{Var}(g)}{\sqrt{N}}.$$

The magnitude of the variation of g limits the applicability of this method:

$$\mathrm{Var}(g)^2 = \int_\Omega g^2(x)\, dV - \left(\int_\Omega g(x)\, dV\right)^2.$$

For univariate monomials, explicit computation shows that the variation of satisfiable univariate monomials grows like

$$\mathrm{Var}(X^r) = \frac{\omega}{2\sqrt{2r+1}} - \frac{r+8}{8\sqrt{2r+1}\omega} + \cdots.$$

Since ω will be exponentially large, the error term in the straightforward Monte Carlo technique will also grow exponentially as a function of ω.

The source of the exponential growth in the above integral is the rapid oscillation of the cosine terms in the integral. There are wide variety of techniques for dealing with problems like this in the literature. Whether they suffice to yield a true random polynomial time algorithm for propositional satisfiability remains to be seen.

4. Conclusions

This paper demonstrates that satisfiability problems can be answered by computing the value of multidimensional integrals. The integrands of these integrals are smooth and easy to evaluate, and the values of these integrals corresponding to satisfiable and non-satisfiable Boolean formulas are well separated. It does not appear to be possible to compute these integrals in polynomial time by direct determinstic means. Monte Carlo techniques do look promising, although the simplest methods do not suffice.

The formula (5) should be compared with the technique suggested by Babai and Fortnow [1] and used many of the recent results on interactive proof, *e.g.*, [18]. They use a slightly different conversion of a Boolean formula to a polynomial and note that

$$\sum_{x_1=0}^{1} \cdots \sum_{x_n=0}^{1} F_\varphi(x_1, \ldots, x_n)$$

counts the number of satisfying assignments of the Boolean formula and in particular is only positive when F is satisfiable.

Simply converting the summations to integrals, unfortunately does not give a formula that separates satisfiable and non-satisfiable formulas. The values in the interior of the unit cube can be too large for non-satisfiable Boolean formula. Algebraically, this is due to the small difference between the size of the integral of the largest non-satisfiable Boolean monomials and the integral of smallest satisfiable Boolean monomials. One way to avoid this problem would be to weight the integrand towards the vertices of the unit. This is not what we have done, but instead examine the spectral content of the Boolean polynomial.

The results of this paper suggest that there may be a closer relationship between the complexity classes *NP* and *RP* that was previously expected. If sufficiently clever variation reduction techniques for Monte Carlo integration can be devised, then problems in *NP* may be solved in random polynomial time. However, this result says nothing about the relationship between *NP* and *P*. In fact, the difficulty of computing the integrals by direct means only strengthens the supposition that *NP* is different from *P*. However, if it turns out that *NP* is equal to *RP*, then the importance of the class *P* would diminish.

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A. Appendix: Tables

The following tables give the exact value of the indicator of satisfiable and unsatisfiable univariate monomials as a function of ω, assuming ω is an odd integral multiple of π.

$$
\begin{aligned}
\mathscr{I}(1) &= 2,\\
\mathscr{I}(T) &= 1,\\
\mathscr{I}(T^2) &= 1 - \frac{4}{\omega^2},\\
\mathscr{I}(T^2) &= 1 - \frac{6}{\omega^2},\\
\mathscr{I}(T^4) &= 1 - \frac{12}{\omega^2} + \frac{48}{\omega^4},\\
\mathscr{I}(T^5) &= 1 - \frac{20}{\omega^2} + \frac{120}{\omega^4},\\
\mathscr{I}(T^6) &= 1 - \frac{30}{\omega^2} + \frac{360}{\omega^4} - \frac{1440}{\omega^6},\\
\mathscr{I}(T^7) &= 1 - \frac{42}{\omega^2} + \frac{840}{\omega^4} - \frac{5040}{\omega^6},\\
\mathscr{I}(T^8) &= 1 - \frac{56}{\omega^2} + \frac{1680}{\omega^4} - \frac{20160}{\omega^6} + \frac{80640}{\omega^8}.
\end{aligned}
$$

$$\begin{aligned}
\mathscr{I}(T\wedge(\neg T)) &= \frac{4}{\omega^2},\\
\mathscr{I}(T^2\wedge(\neg T)) &= \frac{2}{\omega^2},\\
\mathscr{I}(T^3\wedge(\neg T)) &= \frac{6}{\omega^2}-\frac{48}{\omega^4},\\
\mathscr{I}(T^4\wedge(\neg T)) &= \frac{8}{\omega^2}-\frac{72}{\omega^4},\\
\mathscr{I}(T^5\wedge(\neg T)) &= \frac{10}{\omega^2}-\frac{240}{\omega^4}+\frac{1440}{\omega^6},\\
\mathscr{I}(T^6\wedge(\neg T)) &= \frac{12}{\omega^2}-\frac{480}{\omega^4}+\frac{3600}{\omega^6},\\
\mathscr{I}(T^7\wedge(\neg T)) &= \frac{14}{\omega^2}-\frac{840}{\omega^4}+\frac{15120}{\omega^6}-\frac{80640}{\omega^8},\\
\mathscr{I}(T^8\wedge(\neg T)) &= \frac{16}{\omega^2}-\frac{1344}{\omega^4}+\frac{40320}{\omega^6}-\frac{282240}{\omega^8}.
\end{aligned}$$

$$\begin{aligned}
\mathscr{I}(T^2\wedge(\neg T)^2) &= -\frac{4}{\omega^2}+\frac{48}{\omega^4},\\
\mathscr{I}(T^3\wedge(\neg T)^2) &= -\frac{2}{\omega^2}+\frac{24}{\omega^4},\\
\mathscr{I}(T^4\wedge(\neg T)^2) &= -\frac{2}{\omega^2}+\frac{168}{\omega^4}-\frac{1440}{\omega^6},\\
\mathscr{I}(T^5\wedge(\neg T)^2) &= -\frac{2}{\omega^2}+\frac{240}{\omega^4}-\frac{2160}{\omega^6},\\
\mathscr{I}(T^6\wedge(\neg T)^2) &= -\frac{2}{\omega^2}+\frac{360}{\omega^4}-\frac{11520}{\omega^6}+\frac{80640}{\omega^8},\\
\mathscr{I}(T^7\wedge(\neg T)^2) &= -\frac{2}{\omega^2}+\frac{504}{\omega^4}-\frac{25200}{\omega^6}+\frac{201600}{\omega^8}.
\end{aligned}$$

$$\begin{aligned}
\mathscr{I}(T^3\wedge(\neg T)^3) &= -\frac{144}{\omega^4}+\frac{1440}{\omega^6},\\
\mathscr{I}(T^4\wedge(\neg T)^3) &= -\frac{72}{\omega^4}+\frac{720}{\omega^6},\\
\mathscr{I}(T^5\wedge(\neg T)^3) &= -\frac{120}{\omega^4}+\frac{9360}{\omega^6}-\frac{80640}{\omega^8},\\
\mathscr{I}(T^6\wedge(\neg T)^3) &= -\frac{144}{\omega^4}+\frac{13680}{\omega^6}-\frac{120960}{\omega^8},\\
\mathscr{I}(T^6\wedge(\neg T)^3) &= -\frac{168}{\omega^4}+\frac{25200}{\omega^6}-\frac{967680}{\omega^8}+\frac{7257600}{\omega^{10}}.
\end{aligned}$$

$$\begin{aligned}
\mathscr{I}(T^4\wedge(\neg T)^4) &= \frac{48}{\omega^4}-\frac{8640}{\omega^6}+\frac{80640}{\omega^8},\\
\mathscr{I}(T^5\wedge(\neg T)^4) &= \frac{24}{\omega^4}-\frac{4320}{\omega^6}+\frac{40320}{\omega^8},\\
\mathscr{I}(T^6\wedge(\neg T)^4) &= \frac{24}{\omega^4}-\frac{11520}{\omega^6}+\frac{846720}{\omega^8}-\frac{7257600}{\omega^{10}},\\
\mathscr{I}(T^5\wedge(\neg T)^5) &= \frac{7200}{\omega^6}-\frac{806400}{\omega^8}+\frac{7257600}{\omega^{10}},\\
\mathscr{I}(T^6\wedge(\neg T)^5) &= \frac{3600}{\omega^6}-\frac{403200}{\omega^8}+\frac{3628800}{\omega^{10}}.
\end{aligned}$$