

Unrestricted Stone Duality for Markov Processes

Robert Furber*, Dexter Kozen†, Kim Larsen*, Radu Mardare* and Prakash Panangaden‡

*Aalborg University, Denmark

†Cornell University

‡McGill University, Montreal, Canada

Abstract—Stone duality relates logic, in the form of Boolean algebra, to spaces. Stone-type dualities abound in computer science and have been of great use in understanding the relationship between computational models and the languages used to reason about them. Recent work on probabilistic processes has established a Stone-type duality for a restricted class of Markov processes. The dual category was a new notion—Aumann algebras—which are Boolean algebras equipped with countable family of modalities indexed by rational probabilities. In this article we consider an alternative definition of Aumann algebra that leads to dual adjunction for Markov processes that is a duality for many measurable spaces occurring in practice. This extends a duality for measurable spaces due to Sikorski. In particular, we do not require that the probabilistic modalities preserve a distinguished base of clopen sets, nor that morphisms of Markov processes do so. The extra generality allows us to give a perspicuous definition of event bisimulation on Aumann algebras.

I. INTRODUCTION

Dualities in computer science have enjoyed a recent spate of popular interest. Since Plotkin and Smyth’s discovery of a Stone-type duality between the predicate-transformer semantics of Dijkstra and state-transformer semantics [1], [2], it has become increasingly apparent that dualities are ubiquitous in computer science, having appeared in automata and formal language theory, automated deduction, programming language semantics and verification, domain theory, and concurrency theory [3]–[14].

Dualities are important because they establish canonical connections between computational models and the languages used to reason about them. A duality gives an exact characterization of the power of a state transition system by showing how the system determines a corresponding logic or algebra in a canonical way, and vice versa. Moreover, algebra homomorphisms correspond directly to structure-preserving maps or bisimulations of the transition system, allowing mathematical arguments to be transferred in both directions.

The original duality of Stone [15] asserts that the category of Boolean algebras and Boolean algebra homomorphisms is contravariantly equivalent to the category of Stone spaces and continuous maps. Jonsson and Tarski [16] extended Stone’s result to Boolean algebras with modal operators and Stone spaces with transitions. A recent surge of interest in probabilistic systems, due largely to impetus from the artificial intelligence and machine learning communities, has led to the study of various logics with constructs for reasoning about

probabilities of events or expected behaviour. Recent papers on Markovian logic [11], [17], [18] have established completeness and finite model properties for such systems.

In this paper we focus on the duality of a certain class of models of probabilistic computation, namely *Markov transition systems*, with a certain class of Boolean algebras with operators that behave like probabilistic modalities, the *Aumann algebras*. Aumann algebra is the algebraic analogue of Markovian logic; that is, it is to Markovian logic what Boolean algebra is to propositional logic. They were first defined in [19], where a restricted form of the duality was established. The duality was shown to hold only under certain (somewhat artificial) assumptions, to wit:

- The Aumann algebra must be countable.
- The Borel sets of the Markov transition system must be generated by a distinguished countable family of clopen sets, and morphisms must preserve the distinguished clopens.

These assumptions were made in order to apply the *Rasiowa–Sikorski lemma* [20], a lemma of logic that is dual to the *Baire category theorem* of topology. The RSL/BCT implies that certain “bad” ultrafilters (those not satisfying the countably many infinitary defining conditions of countable Aumann algebras) can be deleted from the Stone space without changing the supported algebra of measurable sets, since the “good” ultrafilters are topologically dense.

Although not a perfect duality due to these restrictions, the groundwork laid in that paper nevertheless led to significant advances in the completeness of Markovian logics [21], [22]. Previously, strong completeness theorems had used a powerful infinitary axiom scheme called the *countable additivity rule*, which has uncountably many instances. Moreover, one needs to postulate *Lindenbaum’s lemma* (every consistent set of formulas extends to a maximally consistent set), which for these logics is conjectured but not proven. The duality result of [19] gives rise to a complete axiomatization that does not involve infinitary axiom schemes with uncountably many instances and that satisfies Lindenbaum’s lemma.

In this paper we improve the duality of [19] to a full-fledged Stone-like duality between Markov transition systems and Aumann algebras based on Sikorski’s Stone duality for σ -perfect σ -fields [23], [24] and σ -spatial Boolean algebras. A σ -Boolean algebra is σ -spatial if every element, other than \perp , is contained in an σ -complete ultrafilter; this is the algebraic analogue of Lindenbaum’s lemma. The construction does not use the RSL/BCT, thus the restrictions mentioned

above are no longer necessary. However, we need to change the definition of an Aumann algebra, so this duality is not strictly a generalization of [19]. In particular, the axiom AA8 that we use has uncountably many instances. We say that the duality is *Stone-like* because the duality is no longer an algebraic/topological duality in the strict sense of the word, as the “topological” side is axiomatized in terms of measure-theoretic properties of the state space, not topological properties as with traditional Stone-type results. This is an easy price to pay, because it brings the relevant properties of the state space needed for duality into sharp relief.

Our key results are:

- a version of the duality of [19] for all σ -spatial Aumann algebras, not just countable ones; and
- the removal of the assumptions that Markov processes be countably generated and that maps preserve the distinguished clopens.

The paper is organized as follows. In §II, we briefly review the necessary background material. In §III, we describe Sikorski’s Stone duality for measurable spaces. We do this so as to fix notation and also because it will be used in two different places later in the article. It takes the form of an adjunction that becomes a categorical duality when restricted to objects for which the unit and counit are isomorphisms. In §IV, we recall Halmos’s description of free σ -Boolean algebras and what a presentation of a σ -Boolean algebra is. We then give a presentation of the Borel σ -field of $[0, 1]$ as an abstract σ -Boolean algebra. In §V, we describe an alternative definition of Aumann algebras and prove a duality for Aumann algebras and (discrete-time, continuous-space, time-homogeneous) Markov processes extending the duality for σ -Boolean algebras and measurable spaces in §III. Finally, in §VI we describe the generalization of labelled Markov processes and how event bisimulation is formulated in the setting of Aumann algebras.

II. BACKGROUND

See Johnstone [25] for a detailed introduction to Stone duality and its ramifications, as well as an account of several other related mathematical dualities such as Priestley duality for distributed lattices and Gelfand duality for C^* -algebras.

We assume knowledge of basic notions from measure theory and topology such as field of sets, σ -field, measurable set, measurable space, measurable function, measure, topology, open and closed sets, continuous functions, and the Borel algebra of a topology (denoted by $\mathcal{B}o(X)$ for X a topological space in this article). See [26], [27] for a more thorough introduction.

Throughout, we use *σ -Boolean algebra* to refer to a σ -complete Boolean algebra, *i.e.* a Boolean algebra with countable joins and meets, and use *σ -field* to refer to a pair (X, \mathcal{F}) where X is a set, and \mathcal{F} a family of subsets that is a σ -Boolean algebra under set-theoretic operations. To avoid confusion we do not use the term *σ -algebra*.

The natural notion of an ultrafilter on a σ -Boolean algebra is a *σ -ultrafilter*, *i.e.* an ultrafilter that is additionally closed under countable meets. We say that a σ -Boolean algebra is

σ -spatial if every element is contained in a σ -ultrafilter.¹ If we use σ -homomorphisms as morphisms, σ -Boolean algebras form a category $\sigma\text{-BA}$, with σ -spatial algebras forming a full subcategory $\sigma\text{-BA}_{\text{sp}}$.

Define $\mathbb{Q}_0 = \mathbb{Q} \cap [0, 1]$ and $\mathbb{R}^+ = \mathbb{R} \cap [0, \infty)$.

A. Measurable Spaces and Measures

If $\mathcal{F} \subseteq \mathcal{P}M$, the *σ -field generated by \mathcal{F}* is the smallest σ -field containing \mathcal{F} .

In any measurable space (X, Σ) , a point $x \in X$ defines a σ -ultrafilter on Σ by

$$\langle x \rangle = \{S \in \Sigma \mid x \in S\}.$$

Sikorski introduced the term *σ -perfect* for those measurable spaces for which $\langle - \rangle$ is a bijection from X to the set of σ -ultrafilters on Σ [24, pp. 1, 98].

Measurable spaces form a category $\mathcal{M}es$ with measurable maps as morphisms, and σ -perfect measurable spaces form a full subcategory $\mathcal{P}\mathcal{M}es$.

A nonnegative real-valued set function μ is said to be *finitely additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$, and *countably additive* if $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for a countable pairwise-disjoint family of measurable sets. A *measure* on a measurable space (X, Σ) is a countably additive set function $\mu : \Sigma \rightarrow \mathbb{R}^+$. A measure is a *probability measure* if in addition $\mu(X) = 1$, and a *subprobability measure* if $\mu(X) \leq 1$. We use $\mathcal{G}(X, \Sigma)$ to denote the set of subprobability measures on (X, Σ) .

We can view $\mathcal{G}(X, \Sigma)$ as a measurable space by considering the σ -field generated by the sets $\{\mu \in \mathcal{G}(X, \Sigma) \mid \mu(S) \geq r\}$ for $S \in \Sigma$ and $r \in [0, 1]$. This is the least σ -field on $\mathcal{G}(X, \Sigma)$ such that all maps $\mu \mapsto \mu(S) : \mathcal{G}(X, \Sigma) \rightarrow [0, 1]$ for $S \in \Sigma$ are measurable, where the real interval $[0, 1]$ is endowed its Borel σ -field. In the usual way, we extend \mathcal{G} to a functor $\mathcal{M}es \rightarrow \mathcal{M}es$ by defining

$$\mathcal{G}(f)(\mu)(T) = \mu(f^{-1}(T)),$$

where $f : (X, \Sigma) \rightarrow (Y, \Theta)$ is a measurable map, $\mu \in \mathcal{G}(X, \Sigma)$ and $T \in \Theta$. It is worth mentioning that \mathcal{G} is the subprobabilistic Giry monad [28].

B. Markov Processes

Markov processes (MPs) are models of probabilistic systems with a continuous state space and discrete-time probabilistic transitions [28]–[30].

Definition 1 (Markov process). *A Markov process (MP) is a measurable space (X, Σ) equipped with a measurable map $\theta : (X, \Sigma) \rightarrow \mathcal{G}(X, \Sigma)$. A Markov process is said to be σ -perfect iff (X, Σ) is. Maps of Markov processes are “zig-zags”, *i.e.* if $(X_1, \Sigma_1, \theta_1)$ and $(X_2, \Sigma_2, \theta_2)$ are Markov*

¹This terminology is inspired by the theory of locales.

processes, a measurable map $f : (X_1, \Sigma_1) \rightarrow (X_2, \Sigma_2)$ is a map of Markov processes if

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \theta_1 \downarrow & & \downarrow \theta_2 \\ \mathcal{G}(X_1) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(X_2) \end{array}$$

commutes. Then Markov processes and morphisms thereof form a category **Markov**, with σ -perfect Markov processes forming a full subcategory **PMarkov**.

In a Markov process (X, Σ, θ) , and θ is called the *transition function*. For $x \in X$, $\theta(x) : \Sigma \rightarrow [0, 1]$ is a subprobability measure on the state space (X, Σ) . For $S \in \Sigma$, the value $\theta(x)(S) \in [0, 1]$ represents the probability of a transition from x to a state in S .

The condition that θ be a measurable function $X \rightarrow \mathcal{G}(X, \Sigma)$ is equivalent to the condition that for fixed $S \in \Sigma$, the function $x \mapsto \theta(x)(S)$ is a measurable function $X \rightarrow [0, 1]$ (see e.g. [30, Proposition 2.9]).

C. Aumann Algebras

Aumann Algebra (AA) [19] is the algebraic analogue of Markovian logic [11], [17], [18]. It is so named in honor of Robert Aumann, who has made fundamental contributions to probabilistic logic [31].

Definition 2 (Aumann algebra). A σ -Aumann algebra is a tuple $(A, (L_r)_{r \in \mathbb{Q}_0})$, where A is a σ -Boolean algebra, $\mathbb{Q}_0 = \mathbb{Q} \cap [0, 1]$ and each $L_r : A \rightarrow A$, such that the following axioms hold, where a, b are arbitrary elements of A and r, s are elements of \mathbb{Q}_0 :

- (AA1) $\top \leq L_0(a)$
- (AA2) $L_r(\perp) \leq \perp$, where $r > 0$
- (AA3) $L_r(a) \leq \neg L_s(\neg a)$ if $r + s > 1$
- (AA4) $L_r(a \wedge b) \wedge L_s(a \wedge \neg b) \leq L_{r+s}(a)$ if $r + s \leq 1$
- (AA5) $\neg L_r(a \wedge b) \wedge \neg L_s(a \wedge \neg b) \leq \neg L_{r+s}(a)$ if $r + s \leq 1$
- (AA6) $a \leq b$ implies $L_r(a) \leq L_r(b)$
- (AA7) $\bigwedge_{r < s} L_r(a) = L_s(a)$
- (AA8) If $r \in \mathbb{Q}_0$ and $r \neq 0$, for all countable descending chains $a_1 \geq a_2 \geq \dots$ such that $\bigwedge_{i=1}^{\infty} a_i = \perp$ we have $\bigwedge_{i=1}^{\infty} L_r(a_i) = \perp$.

TABLE I
AUMANN ALGEBRA

We define a morphism of σ -Aumann algebras $f : (A, (L_r)) \rightarrow (B, (M_r))$ to be a σ -Boolean homomorphism such that $f(L_r(a)) = M_r(f(a))$ for all $a \in A$, i.e. such that the following diagram commutes for all $r \in \mathbb{Q}_0$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ L_r \downarrow & & \downarrow M_r \\ A & \xrightarrow{f} & B \end{array}$$

We define the category **Aumann** to have σ -Aumann algebras as objects and σ -Aumann algebra morphisms as its morphisms, and **AumannSp** to be the full subcategory on σ -spatial Aumann algebras.

The reader may verify that AA1 and AA3-AA6 are the same as in [19, §4.1], and that AA7 and AA8 imply the AA7 of the original definition. Note that AA2 originates in [22, Table 3], and is the small change needed to account for subprobability distributions. Note also that AA2 with $r = 0$ is inconsistent with AA1. Additionally, AA1 is implied by AA7 with $s = 0$. A σ -spatial Aumann algebra is defined to be a σ -Aumann algebra whose underlying σ -Boolean algebra is σ -spatial.

III. DUALITY FOR MEASURABLE SPACES

In this section, we express Sikorski's duality for measurable spaces [24, §24, §32][23, 2.1-2] as an adjunction, and describe how this adjunction can be restricted to an equivalence (see [32, Part 0, Proposition 4.2] for a proof that this is possible for any adjunction).

If (X, Σ) is a measurable space, then Σ is a σ -Boolean algebra. We can use this to define a functor $F : \mathcal{Mes} \rightarrow \sigma\text{-BA}^{\text{op}}$ by defining $F(X, \Sigma) = \Sigma$, and for $f : (X, \Sigma) \rightarrow (Y, \Theta)$ a measurable map and $T \in \Theta$

$$F(f)(T) = f^{-1}(T).$$

This is easily verified to be a functor. The following is easy to verify using the fact that sets can be distinguished by their points.

Lemma 3. For any measurable space (X, Σ) , Σ is a σ -spatial σ -Boolean algebra.

So we can also regard F as having the type $\mathcal{Mes} \rightarrow \sigma\text{-BA}_{\text{sp}}^{\text{op}}$. The analogous functor in Stone duality takes the Boolean algebra of clopens of a Stone space.

The reader familiar with Stone duality will already be expecting ultrafilters to be involved in the definition of a functor the other way. For A a σ -Boolean algebra, we define $\mathcal{U}^\sigma(A)$ to be the set of all σ -ultrafilters on A . Given an element $a \in A$, we define

$$\langle a \rangle = \{u \in \mathcal{U}^\sigma(A) \mid u \ni a\} \subseteq \mathcal{U}^\sigma(A).$$

We can define $\mathcal{F}(A) \subseteq \mathcal{P}(\mathcal{U}^\sigma(A))$ by $\mathcal{F}(A) = \langle A \rangle$, i.e. it is the image of A under the map $\langle \cdot \rangle$.

Lemma 4. $\langle \cdot \rangle$ is a surjective morphism of σ -Boolean algebras $A \rightarrow \mathcal{F}(A)$, and $\mathcal{F}(A)$ is a σ -field [24, §24.1]. We also have that A is σ -spatial iff $\langle \cdot \rangle$ is an isomorphism.

The proof is omitted.

We can now define $G : \sigma\text{-BA}^{\text{op}} \rightarrow \mathcal{PMes}$ on objects as $G(A) = (\mathcal{U}^\sigma(A), \mathcal{F}(A))$, where $\mathcal{F}(A) = \langle A \rangle$. On σ -homomorphisms $f : A \rightarrow B$, $G(f)$ is defined for each $u \in \mathcal{U}^\sigma(B)$ as

$$G(f)(u) = f^{-1}(u).$$

In order to prove that $F \dashv G$, we define the unit and counit of the adjunction. For a measurable space (X, Σ) , for each element $x \in X$, we can define an ultrafilter $\langle x \rangle \in \mathcal{U}^\sigma(\Sigma)$ as

$$\langle x \rangle = \{S \in \Sigma \mid x \in S\}.$$

We define the unit $\eta_X : (X, \Sigma) \rightarrow G(F(X, \Sigma))$ and counit $\epsilon_A : F(G(A)) \rightarrow A$ to be

$$\begin{aligned}\eta_X(x) &= \langle x \rangle \\ \epsilon_A(a) &= \langle a \rangle.\end{aligned}$$

The direction of ϵ_A is reversed because we use $\sigma\text{-BA}^{\text{op}}$.

Theorem 5. *(F, G, η, ϵ) is an adjunction making G a right adjoint to F . In fact, G maps into $\mathcal{P}Mes$ and by restricting F and G to the categories where the unit and counit are isomorphisms, they define an adjoint equivalence $\sigma\text{-BA}_{\text{sp}}^{\text{op}} \simeq \mathcal{P}Mes$.*

In passing, the above theorem shows that $\mathcal{P}Mes$ and $\sigma\text{-BA}_{\text{sp}}$ are reflective [33, §IV.3] subcategories of Mes and $\sigma\text{-BA}$, respectively.

The reader might object to the definition of a σ -perfect measurable space as only attempting to “solve” a problem by defining it out of existence. To address this potential criticism, we show that there are many σ -perfect measurable spaces occurring in practice using a theorem of Hewitt. First, we recall that on a topological space X , the Baire σ -field $\mathcal{B}a(X)$ can be defined to be the σ -field generated by the zero sets, the subsets of X of the form $f^{-1}(0)$ for some continuous map $X \rightarrow \mathbb{R}$. If X is metrizable, $\mathcal{B}a(X)$ is the same as the Borel σ -field. We also need to refer to the concept of a realcompact space. We omit the definition [34, §5.9], but we only need the fact that every σ -compact Hausdorff space and every separably metrizable space² is realcompact, as is shown in [34, §8.2].

Theorem 6 (Hewitt). *For a completely regular space X , $(X, \mathcal{B}a(X))$ is σ -perfect iff X is realcompact.*

See [35, Theorem 16] for the proof³.

Therefore the Borel σ -field of any separable metric space, e.g. a Polish space or analytic space, and the Baire σ -field of any compact Hausdorff space are σ -perfect. We warn the reader that it is not the case that the Baire σ -field of a locally compact space is σ -perfect, nor the Borel σ -field of an unmetrizable compact Hausdorff space, and σ -perfectness is not preserved under σ -subfields (even though σ -spatiality is preserved under subalgebras).

For example, consider an uncountable set X . Take Σ to consist of countable sets and their complements (co-countable sets), the countable co-countable σ -field. Then we define u to be the set of co-countable sets. This is a non-principal σ -ultrafilter on Σ , so (X, Σ) is a measurable space that is not σ -perfect. But note that if we take $X = \mathbb{R}$, this is a σ -subfield of the Borel σ -field.

One thing to note is that the smallest cardinality of a set X such that $(X, \mathcal{P}(X))$ is not σ -spatial is strongly inaccessible,

²Not requiring completeness.

³Hewitt uses Q -space to mean realcompact space.

i.e. it is not possible to produce this set by taking powersets or unions of strictly smaller families of strictly smaller sets. This was first proven by Ulam [36, Lemma 1 and Satz 4]. It is therefore not possible to prove the existence, or even the consistency, of a non- σ -spatial discrete set [37, Theorem 12.12].

IV. PRESENTATIONS OF σ -BOOLEAN ALGEBRAS

In this section we describe Halmos’s construction of the free σ -Boolean algebra on a set, how to give presentations of σ -fields in terms of generators and relations, how to define measurable maps in terms of presentations, and give a presentation of $\mathcal{B}o([0, 1])$. We note that this point that presentations can be defined in a more general setting of universal algebra for theories with operations of countable arity, and Lemma 8 and Proposition 9 (i) could have been given as a reference to such a general theorem rather than being proved in this special setting. However, we will stick with a presentation closer to measure theory than universal algebra.

We have the usual forgetful functor $U : \sigma\text{-BA} \rightarrow \mathbf{Set}$.

Proposition 7 (Halmos). *The functor U has a left adjoint H , given on objects by*

$$H(X) = \mathcal{B}a(2^X),$$

where 2^X is given the product topology. This is also a left adjoint to the restriction of U to $\sigma\text{-BA}_{\text{sp}}$.

For the proof, see [38, §23, Theorem 14].

We have already seen the definition of a σ -ultrafilter. A σ -ideal in a σ -Boolean algebra A is a subset $I \subseteq A$ that is downward closed (i.e. if $b \leq a$ and $a \in I$, then $b \in I$) and closed under countable joins. As any σ -ideal is an ideal, we can define A/I to be the set of equivalence classes of elements of A modulo the relation $a \sim b \Leftrightarrow a \Delta b \in I$, where Δ is the symmetric difference, as usual, and σ -Boolean operations are well-defined with respect to this equivalence relation.

If K is a subset of a σ -Boolean algebra A , we can define the sub- σ -Boolean algebra B generated by K to be the smallest σ -Boolean algebra in A containing K . This can equivalently be defined as either the intersection of all σ -Boolean subalgebras of A that contain K , or by building up elements of B as σ -Boolean combinations of elements of K using countable ordinals, as in the Borel hierarchy [39, §II.3]. If $B = A$, we say that A is generated by K . Note that K will not necessarily generate A as a Boolean algebra, in general countable operations will be necessary.

Recall that $H : \mathbf{Set} \rightarrow \sigma\text{-BA}$ is the free (σ -spatial) σ -Boolean algebra on a set, from Proposition 7, in the following lemma.

Lemma 8. *A set $K \subseteq A$ generates A iff the universal map $\tilde{i} : H(K) \rightarrow A$:*

$$\begin{array}{ccc} K & \xrightarrow{\eta_K} & U(H(K)) & & H(K) \\ & \searrow i & \downarrow U\tilde{i} & & \vdots \tilde{i} \\ & & U(A) & & A \end{array}$$

is surjective, where i is the inclusion morphism.

We say a σ -ideal $I \subseteq A$ is *generated* by a subset $R \subseteq I$ if I is the smallest σ -ideal containing R , equivalently if

$$I = \left\{ a \in A \mid \exists (b_i)_{i \in \mathcal{N}}. \forall i \in \mathcal{N}. b_i \in R \text{ and } a \leq \bigvee_{i=1}^{\infty} b_i \right\}.$$

Note that R does not necessarily generate I as an ideal, as countable joins may be necessary to produce every element of I . Also note that the σ -ideal generated by a set and the σ -Boolean algebra generated by a set are not necessarily the same.

A σ -ideal is *principal* if it is generated by one element, and if $(b_i)_{i \in \mathcal{N}}$ is a countable set of generators for a σ -ideal I , then I is generated by $\bigvee_{i \in \mathcal{N}} b_i$, i.e. every countably generated σ -ideal is principal.

A *presentation* of a σ -Boolean algebra A is a pair (K, R) where $K \subseteq A$ generates the σ -Boolean algebra A and R generates the σ -ideal $\tilde{i}^{-1}(\perp)$ in $H(K)$, where $i : K \rightarrow A$ is the inclusion morphism. We call the elements of R *relations*. This agrees with the usual notion of a presentation of a group or ring in terms of generators and relations. In view of Theorem 5, we can define a presentation of a σ -perfect measurable space (X, Σ) to be a presentation of Σ .

Once we have a presentation of a σ -Boolean algebra, we can define homomorphisms by giving their values on the generators and checking that the relations are satisfied.

Proposition 9.

- (i) Let A and B be σ -Boolean algebras, (K, R) a presentation of A , and $f : K \rightarrow B$ a function. There exists a σ -homomorphism $g : A \rightarrow B$ such that $g|_K = f$ iff $\tilde{f}(r) = \perp$ for all $r \in R$.
- (ii) Let (X, Σ) and (Y, Θ) be measurable spaces, where (X, Σ) is σ -perfect, and let (K, R) be a presentation of (X, Σ) . Let $f : K \rightarrow \Theta$ be a function. There exists a measurable map $g : (Y, \Theta) \rightarrow (X, \Sigma)$ such that $g^{-1}|_K = f$ iff $\tilde{f}(r) = \emptyset$ for all $r \in R$.

In the special case that there are only countably many relations, we can verify that a set of relations is sufficient to define a presentation in another way.

Lemma 10. Let (X, Σ) be a σ -perfect measurable space, $K \subseteq \Sigma$ a set of generators with inclusion morphism $i : K \rightarrow \Sigma$, and $(r_j)_{j \in J}$ a countable set of relations, i.e. elements of $H(K)$ such that $\tilde{i}(r_j) = \emptyset$. The following condition implies that $(K, (r_j)_{j \in J})$ is a presentation of (X, Σ) : For all $u \in \mathcal{U}^\sigma(H(K))$ such that $\forall j \in J. r_j \notin u$, we have that there exists $x \in X$ such that $\tilde{i}^{-1}(\langle x \rangle) = u$.

We can now give a presentation of $\mathcal{B}o([0, 1])$ for later use. This presentation is related to a presentation of $\mathcal{B}o([-\infty, \infty])$ given by Sikorski [23, II.3 Lemma].

Define

$$K = \{[r, 1] \mid r \in \mathbb{Q}_0\} \quad (1)$$

This is a countable family of closed subsets of $[0, 1]$. To define the relations, we write

$$B_r = \eta_K([r, 1])$$

Then we define the relations as

$$R = \{(B_r \wedge B_s) \triangle B_s\}_{r < s} \cup \left\{ \left(\bigwedge_{r < s} B_r \right) \triangle B_s \right\}_{s \in \mathbb{Q}_0}, \quad (2)$$

where r and s are understood to range over \mathbb{Q}_0 .

Lemma 11. The above (K, R) , as in (1) and (2), define a presentation of $([0, 1], \mathcal{B}o([0, 1]))$.

V. DUALITY FOR MARKOV PROCESSES

In this section, we extend the adjunction and duality from Section III to Markov processes. Recall that the category of Markov processes is called **Markov** and the category of Aumann algebras is **Aumann**. We define a σ -perfect Markov process to be one whose underlying measurable space is σ -perfect, forming the full subcategory **PMarkov**. Likewise, an Aumann algebra is called σ -spatial if its underlying σ -Boolean algebra is σ -spatial, and these form a full subcategory **AumannSp**.

As σ -perfectness of a Markov process only depends on the underlying measurable space, our remarks at the end of Section III imply that the usual Markov processes defined on the Borel σ -fields of Polish spaces or analytic spaces are σ -perfect, so the duality works as a categorical equivalence in these cases.

When defining the adjunction, it is useful to recall Girya's definition of p_S , where (X, Σ) is a measurable space and $S \in \Sigma$

$$p_S : \mathcal{G}(X, \Sigma) \rightarrow [0, 1] \\ p_S(\nu) = \nu(S).$$

The σ -field of $\mathcal{G}(X, \Sigma)$ is defined to be the coarsest such that p_S is measurable, equivalently that generated by $p_S^{-1}(B)$ as S varies over all $S \in \Sigma$ and B varies over the Borel sets of $[0, 1]$, or equivalently any family of sets generating $\mathcal{B}o([0, 1])$.

As in the case of $\mathcal{M}es$ and σ -**BA**, we define $F : \mathbf{Markov} \rightarrow \mathbf{AumannSp}^{\text{op}}$ based on $F : \mathcal{M}es \rightarrow \sigma\text{-BA}_{\text{sp}}^{\text{op}}$ and $G : \mathbf{Aumann}^{\text{op}} \rightarrow \mathbf{PMarkov}$ based on $G : \sigma\text{-BA}^{\text{op}} \rightarrow \mathcal{P}\mathcal{M}es$.

For a Markov process (X, Σ, θ) , and a morphism of Markov processes $f : (X, \Sigma, \theta) \rightarrow (Y, \Theta, \lambda)$ we define F as

$$F(X, \Sigma, \theta) = (\Sigma, (L_r)_{r \in \mathbb{Q}_0}) \\ F(f) = f^{-1},$$

where L_r is defined, for $S \in \Sigma$, as

$$L_r(S) = \{x \in X \mid \theta(x)(S) \geq r\} \quad (3)$$

Proposition 12. F defines a functor $\mathbf{Markov} \rightarrow \mathbf{AumannSp}^{\text{op}}$.

We can now define the Markov process arising from a σ -Aumann algebra, defining the functor $G : \mathbf{Aumann}^{\text{op}} \rightarrow \mathbf{PMarkov}$ on objects.

Given an Aumann algebra $(A, (L_r)_{r \in \mathbb{Q}_0})$ we define, using Proposition 9, for each $a \in A$ a measurable map $\theta_a : \mathcal{U}^\sigma(A) \rightarrow [0, 1]$ such that

$$\theta_a^{-1}([r, 1]) = \llbracket L_r(a) \rrbracket, \quad (4)$$

and then define $\theta : \mathcal{U}^\sigma(A) \rightarrow \mathcal{G}(\mathcal{U}^\sigma(A))$ as

$$\theta(u)(\llbracket a \rrbracket) = \theta_a(u). \quad (5)$$

Proposition 13. *If $(A, (L_r)_{r \in \mathbb{Q}_0})$ is a σ -Aumann algebra, $(\mathcal{U}^\sigma(A), \mathcal{F}(A), \theta)$ is a σ -perfect Markov process.*

We can now show that this defines a functor $G : \mathbf{Aumann}^{\text{op}} \rightarrow \mathbf{PMarkov}$. On objects, we should have

$$G(A, (L_r)_{r \in \mathbb{Q}_0}) = (\mathcal{U}^\sigma(A), \mathcal{F}(A), \theta),$$

as described above. Given a map of σ -Aumann algebras $g : (A, (L_r)_{r \in \mathbb{Q}_0}) \rightarrow (B, (M_r)_{r \in \mathbb{Q}_0})$ we define $G(g)$ exactly as for σ -Boolean algebras, i.e. $G(g)(u) = g^{-1}(u)$.

Proposition 14. *With the above definition, G is a functor $\mathbf{Aumann}^{\text{op}} \rightarrow \mathbf{PMarkov}$.*

Proof. By Proposition 13, it is defined correctly on objects. If we have a morphism of σ -Aumann algebras $(A, (L_r)_{r \in \mathbb{Q}_0}) \rightarrow (B, (M_r)_{r \in \mathbb{Q}_0})$, by Theorem 5, this defines a measurable map $G(f) : \mathcal{U}^\sigma(B) \rightarrow \mathcal{U}^\sigma(A)$, and the identity map and composition are preserved. Therefore we only need to show that $G(f)$ is a map of Markov processes, i.e. that the diagram:

$$\begin{array}{ccc} \mathcal{U}^\sigma(B) & \xrightarrow{G(g)} & \mathcal{U}^\sigma(A) \\ \lambda \downarrow & & \downarrow \theta \\ \mathcal{G}(\mathcal{U}^\sigma(B)) & \xrightarrow{G(G(g))} & \mathcal{G}(\mathcal{U}^\sigma(A)) \end{array}$$

commutes, where θ and λ are the morphisms defining the Markov processes on $\mathcal{U}^\sigma(A)$ and $\mathcal{U}^\sigma(B)$ respectively.

The bottom left path is

$$\mathcal{G}(G(g))(\lambda(u))(\llbracket a \rrbracket) = \lambda(u)(G(g)^{-1}(\llbracket a \rrbracket))$$

Now,

$$\begin{aligned} G(g)^{-1}(\llbracket a \rrbracket) &= \{u \in \mathcal{U}^\sigma(B) \mid u \in G(g)^{-1}(\llbracket a \rrbracket)\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid G(g)(u) \in \llbracket a \rrbracket\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid a \in G(g)(u)\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid a \in g^{-1}(u)\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid g(a) \in u\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid u \in \llbracket g(a) \rrbracket\} \\ &= \llbracket g(a) \rrbracket, \end{aligned}$$

so the bottom left path is equal to

$$\lambda(u)(\llbracket g(a) \rrbracket) = \lambda_{g(a)}(u). \quad (6)$$

The top right path is equal to:

$$\theta(G(g)(u))(\llbracket a \rrbracket) = \theta_a(G(g)(u)) \quad (7)$$

To show that the right hand sides of (6) and (7) are equal, we will show that $\lambda_{g(a)} = \theta_a \circ G(g)$ using Theorem 5. Let $r \in \mathbb{Q}_0$. Then

$$\begin{aligned} \lambda_{g(a)}^{-1}([r, 1]) &= \llbracket M_r(g(a)) \rrbracket && \text{definition of } \lambda_{g(a)} \\ &= \llbracket (L_r(a)) \rrbracket && g \text{ a } \sigma\text{-AA morphism} \\ &= G(g)^{-1}(\llbracket L_r(a) \rrbracket) && \text{naturality (Theorem 5)} \\ &= G(g)^{-1}(\theta_a^{-1}([r, 1])) && \text{definition of } \theta_a \\ &= (\theta_a \circ G(g))^{-1}([r, 1]). \end{aligned}$$

As intervals of the form $\llbracket [r, 1] \rrbracket_{r \in \mathbb{Q}_0}$ generate $\mathcal{B}o([0, 1])$ (Lemma 11), we have

$$\lambda_{g(a)}^{-1}(S) = (\theta_a \circ G(g))^{-1}(S)$$

for all Borel subsets S of $[0, 1]$. As $\lambda_{g(a)}$ and $\theta_a \circ G(g)$ are both maps $(\mathcal{U}^\sigma(B), \mathcal{F}(B)) \rightarrow ([0, 1], \mathcal{B}o([0, 1]))$, i.e. measurable maps between σ -perfect measurable spaces, we can apply the categorical duality from Theorem 5 to deduce $\lambda_{g(a)} = \theta_a \circ G(g)$ from the equation above, and therefore the diagram commutes. \square

Theorem 15. *F is a left adjoint to G , and when restricted they define adjoint equivalences $\mathbf{AumannSp}^{\text{op}} \simeq \mathbf{PMarkov}$.*

Proof. Recall the natural transformations $\llbracket - \rrbracket : A \rightarrow F(G(A))$ and $\langle - \rangle : (X, \Sigma) \rightarrow G(F(X, \Sigma))$ from Theorem 5. If we show that $\llbracket - \rrbracket$ is a morphism of Aumann algebras and $\langle - \rangle$ a morphism of Markov processes, then the commutativity of the naturality diagrams and the triangle diagrams defining an adjunction follows from the proofs in Theorem 5, and we have shown F is a left adjoint to G .

We first show that $\llbracket - \rrbracket$ is a σ -Aumann algebra morphism. That is to say, we want to show that for all $a \in A$ and $r \in \mathbb{Q}_0$ that $\llbracket L_r(a) \rrbracket = M_r(\llbracket a \rrbracket)$, where $(M_r)_{r \in \mathbb{Q}_0}$ is the Aumann algebra structure on $F(G(A))$. Well,

$$\begin{aligned} L_r(\llbracket a \rrbracket) &= \{u \in \mathcal{U}^\sigma(A) \mid \theta(u)(\llbracket a \rrbracket) \geq r\} && \text{see (3)} \\ &= \{u \in \mathcal{U}^\sigma(A) \mid \theta_a(u) \geq r\} && \text{see (5)} \\ &= \theta_a^{-1}([r, 1]) \\ &= \llbracket L_r(a) \rrbracket && \text{see (4)}. \end{aligned}$$

We now show that $\langle - \rangle$ is a morphism of Markov processes, i.e. the following diagram commutes

$$\begin{array}{ccc} (X, \Sigma) & \xrightarrow{\langle - \rangle} & G(F(X, \Sigma)) \\ \theta \downarrow & & \downarrow \lambda \\ \mathcal{G}(X, \Sigma) & \xrightarrow{G(\langle - \rangle)} & \mathcal{G}(G(F(X, \Sigma))), \end{array}$$

where λ is map making $G(F(X, \Sigma))$ a Markov process. In equations, what we want to show is that $\mathcal{G}(\langle - \rangle) \circ \theta = \lambda \circ \langle - \rangle$. Recall that the σ -field on $G(F(X, \Sigma))$ consists of elements

of the form $\langle S \rangle$ for $S \in \Sigma$, so we want to show that, for all $x \in X$ and $S \in \Sigma$,

$$\mathcal{G}(\langle - \rangle)(\theta(x))(\langle S \rangle) = \lambda(\langle x \rangle)(\langle S \rangle). \quad (8)$$

If we expand the definition of \mathcal{G} on the left hand side, we get

$$\mathcal{G}(\langle - \rangle)(\theta(x))(\langle S \rangle) = \theta(x)(\langle - \rangle^{-1}(\langle S \rangle)).$$

Now, we can simplify the argument of $\theta(x)$ as follows

$$\begin{aligned} \langle - \rangle^{-1}(\langle S \rangle) &= \{x \in X \mid \langle x \rangle \in \langle S \rangle\} \\ &= \{x \in X \mid S \in \langle x \rangle\} \\ &= \{x \in X \mid x \in S\} = S, \end{aligned}$$

so, all together, the left hand side of (8) is $\theta(x)(S)$. For the right hand side, we can expand the definition

$$\lambda(\langle x \rangle)(\langle S \rangle) = \lambda_S(\langle x \rangle)$$

according to (5). We now prove that $\lambda_S(\langle x \rangle) = \theta(x)(S)$ by showing that, for all $r \in \mathbb{Q}_0$, $\lambda_S(\langle x \rangle) \geq r$ iff $\theta(x)(S) \geq r$.

$$\begin{aligned} \lambda_S(\langle x \rangle) \geq r &\Leftrightarrow \langle x \rangle \in \lambda_S^{-1}([r, 1]) \\ &\Leftrightarrow \langle x \rangle \in \langle L_r(S) \rangle \quad (4) \\ &\Leftrightarrow L_r(S) \in \langle x \rangle \\ &\Leftrightarrow x \in L_r(S) \\ &\Leftrightarrow \theta(x)(S) \geq r \quad (3). \end{aligned}$$

Because every real is the supremum of the rationals below it, this implies that $\theta(x)(S) = \lambda_S(\langle x \rangle)$, and therefore that (8) holds.

As we explained at the start of the proof, this suffices to show that F is a left adjoint to G . We can show that F and G define an adjoint equivalence $\mathbf{PMarkov} \simeq \mathbf{AumannSp}$ by showing that when $\langle - \rangle$ and $\langle - \rangle^{-1}$ are, respectively, σ -Boolean algebra isomorphisms and measurable isomorphisms, they are σ -Aumann algebra isomorphisms and Markov process isomorphisms.

We do so as follows. We first need to show that $\langle - \rangle^{-1}$ is a σ -Aumann algebra homomorphism. First, we observe that the equality $M_r(\langle a \rangle) = \langle L_r(a) \rangle$ implies

$$\begin{aligned} \langle - \rangle^{-1}(M_r(\langle a \rangle)) &= L_r(a) \\ &= L_r(\langle - \rangle^{-1}(\langle a \rangle)). \end{aligned}$$

As every element of the algebra $F(G(A))$ is of the form $\langle a \rangle$ for some $a \in A$, we have shown that $\langle - \rangle^{-1}$ is an Aumann algebra isomorphism.

It is a generally true fact that a measurable isomorphism that is a map of Markov processes is an isomorphism of Markov processes, but we give the special case that $\langle - \rangle^{-1}$ is a morphism of Markov processes here:

$$\begin{aligned} \mathcal{G}(\langle - \rangle) \circ \theta &= \lambda \circ \langle - \rangle \Leftrightarrow \mathcal{G}(\langle - \rangle) \circ \theta \circ \langle - \rangle^{-1} = \lambda \\ &\Leftrightarrow \theta \circ \langle - \rangle^{-1} = \mathcal{G}(\langle - \rangle^{-1}) \circ \lambda. \end{aligned}$$

□

Note that by Theorem 6 this duality can be applied to any of the Stone-Markov processes with countable base considered in [19], although the dual Aumann algebra will be the Borel sets, not the base of clopens.

VI. EVENT BISIMULATION AND DUALITY FOR LABELLED MARKOV PROCESSES

Given a measurable space (X, Σ) , a *labelled Markov process* is a tuple $(X, \Sigma, (\theta^e)_{e \in E})$, where E is a set of *labels* and for each $e \in E$, $\theta^e : X \rightarrow \mathcal{G}(X, \Sigma)$ is a measurable function. If $(X, \Sigma, (\theta^e)_{e \in E})$ and $(Y, \Theta, (\lambda^e)_{e \in E})$ are labelled Markov processes with the same label set E , we say that a measurable function $f : (X, \Sigma) \rightarrow (Y, \Theta)$ is a *morphism of labelled Markov processes* if it is a morphism of Markov processes $f : (X, \Sigma, \theta_e) \rightarrow (Y, \Theta, \lambda_e)$ for each $e \in E$. For any set of labels, we have a category $\mathbf{LabMarkov}_E$ of E -labelled Markov processes and their morphisms. It should now be obvious how to define the full subcategory of σ -perfect labelled Markov processes, $\mathbf{PLabMarkov}_E$.

We can define a *labelled σ -Aumann algebra* to be $(A, (L_r^e)_{e \in E, r \in \mathbb{Q}_0})$, such that for each label $e \in E$, $(A, (L_r^e)_{r \in \mathbb{Q}_0})$ is a σ -Aumann algebra. A morphism of E -labelled σ -Aumann algebras $(A, (L_r^e)) \rightarrow (B, (M_r^e))$ is a σ -Boolean homomorphism $A \rightarrow B$ that is a σ -Aumann algebra homomorphism for each $e \in E$. For each set of labels E , we have categories $\mathbf{LabAumann}_E$ and $\mathbf{LabAumannSp}_E$ defined in the familiar way. By working with each $e \in E$ independently, we can define F, G and a duality as in Theorem 15.

In the context of labelled Markov processes, an *event bisimulation* on $(X, \Sigma, (\theta^e)_{e \in E})$ is defined to be a sub σ -field $\Lambda \subseteq \Sigma$ such that $(X, \Lambda, (\theta^e|_\Lambda)_{e \in E})$ is a labelled Markov process, where for each $e \in E$, $\theta^e|_\Lambda : X \rightarrow \mathcal{G}(X, \Lambda)$ is the function such that for each $x \in X$, $\theta^e|_\Lambda(x)$ is the restriction of $\theta^e(x)$ to Λ . This notion was originally defined in [40, Definition 4.3], as a version of the notion of probabilistic bisimulation [41] that is more adapted to probabilistic logics.

Theorem 16. *Let $(X, \Sigma, (\theta^e)_{e \in E})$ be a labelled Markov process. A σ -field $\Lambda \subseteq \Sigma$ is an event bisimulation iff it is a σ -Aumann subalgebra of $F(X, \Sigma, (\theta^e)_{e \in E})$, i.e. iff it is preserved by the Aumann algebra operations.*

Proof. We start with the only if direction, which is to say we show that an event bisimulation Λ is also an Aumann subalgebra of the Aumann algebra $F(X, \Sigma, (\theta^e))$.

Since an event bisimulation is a σ -field supporting a labelled Markov process, when F is applied to it becomes an Aumann algebra. As $\theta^e|_\Lambda$ is the restriction of θ^e to Λ at all points $x \in X$, we get that Λ and Σ agree on the effect of the L_r^e operators for all $r \in \mathbb{Q}_0$. This shows that Λ is an Aumann subalgebra of Σ .

For the other direction, suppose A_0 is an Aumann subalgebra of $F(X, \Sigma, (\theta^e)_{e \in E})$. We prove that \mathcal{A}_0 is an event bisimulation of $(X, \Sigma, (\theta^e)_{e \in E})$ as follows. We need to show that for each $e \in E$,

$$\theta^e|_{\mathcal{A}_0} : (X, \mathcal{A}_0) \rightarrow \mathcal{G}(X, \mathcal{A}_0)$$

is measurable. This is equivalent to showing that for each $a \in \mathcal{A}_0$ and each $B \in \mathcal{B}o([0, 1])$,

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}(B)) \in \mathcal{A}_0.$$

To do this, it is sufficient to prove that for any rational $r \leq 1$,

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}([r, 1])) \in \mathcal{A}_0.$$

But since $a \in \mathcal{A}_0$, we get

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}([r, 1])) = L_r^e a \in \mathcal{A}_0$$

and this concludes the proof. \square

Therefore we can, if we like, define event bisimulations directly on σ -Aumann algebras, by taking them to be σ -Aumann subalgebras.

VII. CONCLUSION

We have given a general Stone-like duality between spatial Aumann algebras and certain Markov processes, improving a similar duality of [19] of a more restricted form. We have also shown how the improved version captures the notion of event bisimulation for Markov processes.

Strictly speaking, the result of [19] is not a special case of the result of this paper, because we have amended the definition of Aumann algebras to assume countable completeness: all countable joins are assumed to exist, not just the definable ones. This result is probably not the last word on the subject, as it may be possible to derive an even more general version of the duality parameterized by the class of joins that are assumed to exist that would subsume both the results of [19] and those of this paper. We leave such investigations to future work.

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