# Erratum: Zero-One Laws in Modal Logic

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Jean-Marie Le Bars [2002] showed that the 0-1 law for frame satisfiability fails for the formula

$$q \wedge \neg p \wedge \Box\Box((p \vee q) \Rightarrow \neg\Diamond(p \vee q)) \wedge \Box\Diamond p. \tag{1}$$

This, unfortunately, contradicts one of the main theorems in "Zero-one laws for modal logic" (*Annals of Pure and Applied Logic* **69**, 1994, pp. 157–193). Checking the paper carefully revealed the error. The problem and what can be salvaged from the proof is briefly described here.

We refer the reader to the paper for the formal definition of $\epsilon$-special structures. It is claimed (Theorem 5.5) that

(a) If $\phi$ is not satisfied in a finite 0-special structure, then the probability that $\phi$ is frame satisfiable is 0.

(b) If $\phi$ is satisfied in a finite 0-special structure, then the probability of that $\phi$ is frame satisfiable is 1.

The proof of (b) is correct; the proof of (a) is "almost" correct in the following sense. It is shown that if the probability that $\phi$ is satisfied is not 0, then there is a fixed structure $M$ which is $\epsilon$-special for $\phi$ for all $\epsilon > 0$ such that $\phi$ is satisfied in $M$. We then claim (Proposition 5.10) "straightforward continuity arguments" show that $M$ is also a 0-special structure for $\phi$. Unfortunately, this is false. It is possible to find a formula $\phi$ and a structure $M$ such that $\phi$ is satisfied in $M$ and $M$ is $\epsilon$-special for $\phi$ for all $\epsilon > 0$ without being 0-special for $\phi$. In fact, as shown below, Le Bars' formula in (1) provides a counterexample. (Note that if our Proposition 5.10 were true, then (a) would follow immediately.)

To summarize, the results in "Zero-one laws for modal logic" do show the following.

(a) If, for some $\epsilon$ with $0 < \epsilon < 1$, $\phi$ is not satisfiable in a finite $\epsilon$-special structure for $\phi$, then the probability that $\phi$ is satisfiable is 0.

(b) If $\phi$ is satisfied in a finite 0-special structure, then the probability of that $\phi$ is satisfiable is 1.

That leaves the formulas that are satisfied in an $\epsilon$-special structure for all $\epsilon > 0$ but are not satisfied in a 0-special structure. Our results do not say anything about these formulas (and Le Bars' example shows that there are good reasons that that should be so).

Here is the counterexample. Let $\phi$ be the formula in (1). Consider the structure $M = (S, \mathcal{R}, \pi)$, where

- $S = \{s_1, s_2, s_3, s_4\}$;
- $\mathcal{R} = \{(s_1, s_2), (s_1, s_3), (s_2, s_2), (s_2, s_3), (s_2, s_4), (s_3, s_1), (s_3, s_2), (s_3, s_3), (s_3, s_4), (s_4, s_2), (s_4, s_3)\}$;
- $\pi$ is such that $(M, s_1) \models q \wedge \neg p$, $(M, s_2) \models \neg p \wedge \neg q$, $(M, s_3) \models \neg p \wedge \neg q$, and $(M, s_4) \models p \wedge \neg q$.

The definition of $\mathcal{R}$ makes it the smallest relation such that $s_2$ and $s_3$ are sinks (there is an edge from every node to both $s_2$ and $s_3$), $\mathcal{R}(s_2) \cap \{s_1, s_4\} = \{s_4\}$ and $\mathcal{R}(s_3) \cap \{s_1, s_4\} = \{s_1, s_4\}$. The reason we want these latter two properties will be clear shortly.

It is easy to check that $(M, s_1) \models \phi$. Let the labeling $f$ be such that $f(s_1) = f(s_2) = f(s_3) = 0$ and $f(s_4) = 1 - (\epsilon/2)$. Let $S_0 = \{s_2, s_3\}$. Again, it is easy to see that this labeling makes $M$ $\epsilon$-special for $\phi$ with respect to $S_0$ for all $\epsilon > 0$, by checking the five requirements:

- SP1: note that the only set to which SP1 applies is $\{s_4\}$, and it clearly holds in that case.

- SP2: the only sets to which SP2 applies are $\{s_4\}$ and $\{s_1, s_4\}$, and as observed earlier, $\mathcal{R}(s_2) \cap (S - S_0) = \{s_4\}$, and $\mathcal{R}(s_3) \cap (S - S_0) = \{s_1, s_4\}$.

- SP3: Holds trivially for $\epsilon > 0$, since there is no set of nodes whose weight is strictly greater than 1.

- SP4: Every state in $M$ satisfies $(p \vee q) \Rightarrow \neg\Diamond(p \vee q)$, and hence $\Box((p \vee q) \Rightarrow \neg\Diamond(p \vee q))$ and $\Box\Box((p \vee q) \Rightarrow \neg\Diamond(p \vee q))$. The only states that satisfy $\Diamond\Box\neg p$ $(= \neg\Box\Diamond p)$, $\Diamond(p \vee q)$ $(= \neg\Box\neg(p \vee q))$, and $\Diamond p$ are $s_2$ and $s_3$; since $(s_2, s_4) \in \mathcal{R}$, $(s_3, s_4) \in \mathcal{R}$, $(M, s_4) \models \Box\neg p$, and $(M, s_4) \models p$, SP4 holds for $\Diamond\Box\neg p$, $\Diamond(p \vee q)$, and $\Diamond p$.

- SP5: If $(M, s) \models \Box\psi$ for any formula $\psi$, then $(M, t) \models \psi$ for $t \in S_0$, since both $s_2$ and $s_3$ are sinks.

However, if $\epsilon = 0$, then $M$ is not 0-special with respect to $\{s_2, s_3\}$. It fails SP1 for $\{s_4\}$. Moreover, as we now show, there is no 0-special structure satisfying $\phi$.

For suppose that $M' = (S, \mathcal{R}, \pi)$ is 0-special with respect to some set $S_0 \subseteq S$ and satisfies $\phi$. Let $f$ be the labeling that makes $M'$ 0-special. Let $s_0 \in S$ be such that $(M', s_0) \models \phi$. Let $T = \{t \in S : (M, t) \models p, \exists s' \in S_0((s', t) \in \mathcal{R})\}$. Since $(M', s_0) \models \Box\Box((p \vee q) \Rightarrow \neg\Diamond(p \vee q))$, by SP5, $(M', s') \models \Box((p \vee q) \Rightarrow \neg\Diamond(p \vee q))$ for all $s' \in S_0$, so $(M', t) \models (p \vee q) \Rightarrow \neg\Diamond(p \vee q)$ for all $t \in T$. Since $(M', t) \models p$ for all $t \in T$, it follows that $(M', t) \models \neg\Diamond(p \vee q)$ for all $t \in T$. Thus, $(t, t') \notin \mathcal{R}$ for all $t, t' \in T$.

We claim that $\sum_{t \in T} f(t) < 1$. To see this, let $T' = \{t \in T - S_0 : f(t) > 0\}$. Note that $\sum_{(t,t') \in T' \times T' - \mathcal{R}} f(t) \cdot f(t') = (\sum_{t \in T'} f(t))^2$. Since, by SP1,

$$\sum_{t \in T'} f(t) > \sum_{(t,t') \in T' \times T' - \mathcal{R}} f(t) \cdot f(t'),$$

it follows that $\sum_{t \in T} f(t) = \sum_{t \in T'} f(t) < 1$.

Next, let $U = (S - S_0) - T$. Let $T'' = T \cap (S - S_0)$. Clearly $T'' \subseteq S - S_0$ and (since $T'' \subseteq T$) $\sum_{t \in T''} f(t) < 1$. Finally, note that $U = (S - S_0) - T''$.

By SP2, there must be some $s \in S_0$ such that $\mathcal{R}(s) \cap (S - S_0) = U$. Since $(M', s_0) \models \Box\Diamond p$, it follows that $(M, s) \models \Diamond p$ for all $s \in S_0$. By SP4, there must be some $t \in \mathcal{R}(s)$ such that $t \in S - S_0$ and $(M, t) \models p$. This means that $t \in \mathcal{R}(s) \cap T'' \neq \emptyset$. But that contradicts the assumption that $\mathcal{R}(s) \cap (S - S_0) = U$. This contradiction proves the result.

We close by briefly noting a few other typos in the text:

- In the statement of Theorem 5.5, $\mu(\phi)$ denotes the probability that $\phi$ is satisfiable. whereas in the discussion in Section 2, $\mu(\phi)$ denotes the probability that $\phi$ is valid. (This switch does not affect the proof.)
- In the statement of Lemma 5.6, it should be "$F_j(k, \delta), j = 1, 2, 3$", rather than $F_k(k, \delta), k = 1, 2, 3$".
- In the displayed equation in the second paragraph of the proof of Lemma 5.6 (which starts "For F2"), $T_1$ and $T_{k_2}$ should be $t_1$ and $t_{k_2}$.
- In the statement of P1 just before Lemma 5.11, it should say that $\gamma^{-1}(t)$ $\mathcal{R}'$-covers $\gamma^{-1}(s)$ (not $\mathcal{R}$ covers).

# References

Le Bars, J.-M. (2002). The 0-1 law fails for frame satisfiability of propositional modal logic. In *Proc. 17th IEEE Symp. on Logic in Computer Science*, pp. 225–234.