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Chapter 23

Voting

In the previous chapter, we saw a first example of an institution that can synthesize information held by many people, through the ways in which markets serve to aggregate the individual beliefs of investors. We now turn to a second fundamental institution: voting.

23.1 Voting for Group Decision-Making

Like markets, voting systems also serve to aggregate information across a group, and as a result, it's hard to draw a perfectly clear dividing line between these two kind of institutions. But there are definite distinctions between the respective settings in which they are typically applied. A first important distinction is that voting is generally used in situations where a group of people is expressly trying to reach a single decision that in a sense will speak for the group. When a population votes on a set of candidates or ballot initiatives, a legislative body votes on whether to pass a bill, a jury votes on a verdict in a trial, a prize committee votes on the recipient of an award, or a group of critics votes on the top movies of the past century, the resulting decision is a single outcome that stands for the group, and has some kind of binding effect going forward. In contrast, markets synthesize the opinions of a group more indirectly, as investors' beliefs are conveyed implicitly through their transactions in the market — choosing how much to invest or to bet, choosing whether to buy or not to buy, and so forth. The overt goal of the market is to enable these transactions, rather than any broader synthesis or group decision that might in fact arise from the transactions in aggregate.

There are other important distinctions as well. A simple but important one is that the choices in a market are often numerical in nature (how much money to transact in various ways), and the synthesis that takes place generally involves arithmetic on these quantities —

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weighted averages and other measures. Many of the key applications of voting, on the other hand, take place in situations where there's no natural way to "average" the preferences of individuals — since the preferences are over different people, different policy decisions, or different options under a largely subjective criterion. Indeed, as we will see in this chapter, much of the richness of the theory of voting comes from precisely this attempt to combine preferences in the absence of simple metaphors like averaging.

The notion of voting encompasses a broad class of methods for reaching a group decision. For example, the methods to reach a jury verdict, an outcome in a U.S. Presidential Election, or a winner of college football's Heisman Trophy are all distinct, and these distinctions have effects both on the process and the result. Moreover, voting can be used in settings where a single "winner" must be chosen, as well as in situations where the goal is to produce a ranked list. Examples of the latter include the ranking of college sports teams by aggregating multiple polls, or published rankings of the greatest movies, songs, or albums of all time by combining the opinions of many critics.

Voting is often used in situations where the voters disagree because of genuine divergence in their subjective evaluations. For example, film critics who disagree on whether to rank *Citizen Kane* or *The Godfather* as the greatest movie of all time are generally not disagreeing because they lack relevant information about the two movies — we can expect that they are closely familiar with both — but because of differing aesthetic evaluations of them. In other cases, however, voting is used to achieve group decisions where the difficulty is a lack of information — where the members of the group would likely be unanimous if they had all the information relevant to the decision. For example, jury verdicts in criminal trials often hinge on genuine uncertainty as to whether the defendant committed the crime; in such cases one expects that jurors all have approximately the same goal in mind (determining the correct verdict), and the differences are in their access to and processing of the available information. We will consider both of these settings in this chapter.

Ideas from the theory of voting have been adopted in a number of recent on-line applications [140]. Different Web search engines produce different rankings of results; a line of work on *meta-search* has developed tools for combining these rankings into a single aggregate ranking. Recommendation systems for books, music, and other items — such as Amazon's product-recommendation system — have employed related ideas for aggregating preferences. In this case, a recommendation system determines a set of users whose past history indicates tastes similar to yours, and then uses voting methods to combine the preferences of these other users to produce a ranked list of recommendations (or a single best recommendation) for you. Note that in this case, the goal is not a single aggregate ranking for the whole population, but instead an aggregate ranking for each user, based on the preferences of similar users.

Across all of these different contexts in which voting arises, one sees a recurring set of

questions. How should we produce a single ranking from the conflicting opinions provided by multiple voters? Is some version of majority voting a good mechanism? Is there a better one? And ultimately, what does it even mean for a voting system to be good? These are some of the questions we address in this chapter.

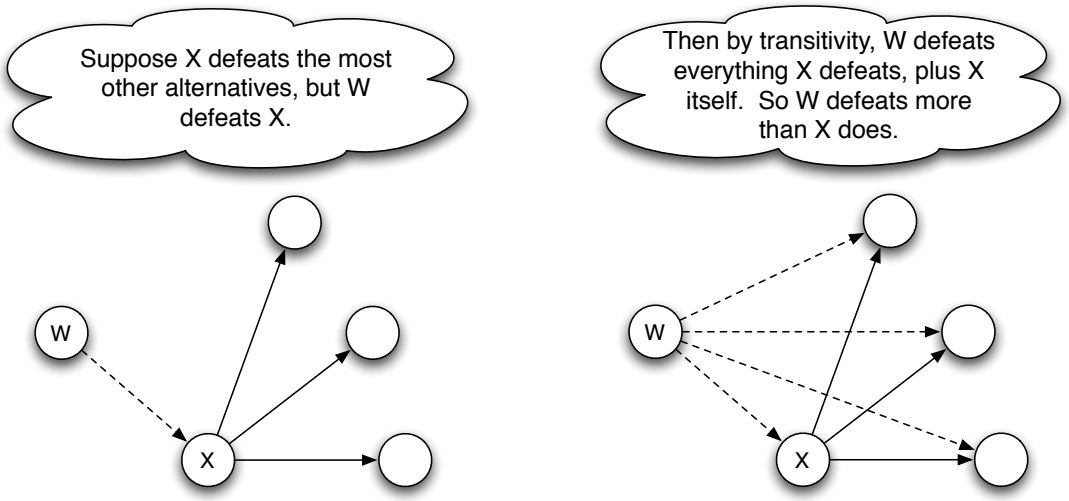
23.2 Individual Preferences

The goal of a voting system, for our purposes, can be described as follows. A group of people is evaluating a finite set of possible *alternatives*; these alternatives could correspond to political candidates, possible verdicts in a trial, amounts of money to spend on national defense, nominees for an award, or any other set of options in a decision. The people involved wish to produce a single *group ranking* that orders the alternatives from best to worst, and that in some sense reflects the collective opinion of the group. Of course, the challenge will be to define what it means to "reflect" the multiple opinions held by members of the group.

To begin with, let's consider how to model the opinions of any one member of the group. We suppose that for each individual, he or she is able to determine a preference between any two alternatives when presented with these two as a pair. If individual i prefers alternative X to alternative Y, then we write $X \succ_i Y$. (Sometimes, for ease of discussion, we will say that X "defeats" Y according to i's preferences.) Thus, for example, if a set of film critics are each given a large list of movies and asked to express preferences, we could write *Citizen Kane* $\succ_i$ *The Godfather* to express the fact that critic i prefers the former movie to the latter. We will sometimes refer to an individual's preferences over all pairs of alternatives, represented by $\succ_i$, as this individual's *preference relation* over the alternatives.

Completeness and Transitivity. We require that individual preferences satisfy two properties. The first is that each person's preferences are *complete*: for each pair of distinct alternatives X and Y, either she prefers X to Y, or she prefers Y to X, but not both. It is possible to extend the theory here to consider the possibility that an individuals has ties in her preferences (i.e. for some pairs of alternatives, she likes them equally), and also the possibility that for some pairs of alternatives, an individual has no preference (perhaps because individual i has no knowledge of one of X or Y). Both of these extensions introduce interesting complications, but for this chapter we focus on the case in which each individual has a preference between each pair of alternatives.

The second requirement is that each individual's preferences be *transitive*: if an individual i prefers X to Y and Y to Z, then i should also prefer X to Z. This seems like a very sensible restriction to impose on preferences, since otherwise we could have situations in which an individual had no apparent favorite alternative. In other words, suppose we were evaluating preferences over flavors of ice cream, and we had an individual i for whom *Chocolate* $\succ_i$



(a) (b)

Figure 23.1: With complete and transitive preferences, the alternative X that defeats the most others in fact defeats all of them. If not, some other alternative W would defeat X (as in (a)), but then by transitivity W would defeat more alternatives than X does (as in (b)).

Vanilla, and *Vanilla* $\succ_i$ *Strawberry* — and, in a violation of transitivity, also *Strawberry* $\succ_i$ *Chocolate*. A simple informal argument for why such preferences seem pathological is the following: if individual i were to walk up to the counter at an ice cream shop and see all three flavors on display, which would she choose? This would in some sense have to be her favorite, despite the fact that each of the three flavors is defeated by some other flavor in her preferences. There has been a long line of work exploring the philosophical and psychological basis for transitive preferences, as well as identifying natural situations in which non-transitive preferences can in fact arise [12, 41, 163]. For our purposes here, we will assume that each individual's preferences are transitive.

Individual Rankings. Thus far we have been expressing an individual's opinions about a set of alternatives in terms of his or her preferences over pairs. An alternate model for opinions would be to imagine that each individual produces a completely ranked list of all the alternatives, ranking them from best to worst.

Notice that from such a ranked list, we could define a preference relation $\succ_i$ very simply: we'd say that $X \succ_i Y$ if alternative X comes before alternative Y in i's ranked list. In this case, we'll say that *the preference relation arises from the ranked list.* It's not hard to see that if a preference relation arises from a ranked list of the alternatives, then it must be complete and transitive: completeness holds since for each pair of alternatives, one precedes

the other in the list; and transitivity holds since if X precedes Y and Y precedes Z in the list, then X must also precede Z.

What is somewhat less obvious is that this fact holds in the opposite direction as well:

> *If a preference relation is complete and transitive, then it arises from some ranked list of the alternatives.*

The way to see why this is true is to consider the following method for constructing a ranked list from a complete and transitive preference relation. First, we identify the alternative X that defeats the most other alternatives in pairwise comparisons — that is, the X so that $X \succ_i Y$ for the most other choices of Y. We claim that this X in fact defeats all the other alternatives: $X \succ_i Y$ for all other Y.

We'll see why this is true in a moment; but first, let's see why this fact lets us construct the ranked list we want. To begin with, having established that X defeats all other alternatives, we can safely put it at the front of the ranked list. We now remove X from the set of alternatives, and repeat exactly the same process on the remaining alternatives. The preferences defined by $\succ_i$ are still complete and transitive on the remaining alternatives, so we can apply our claim again on this smaller set: for the alternative Y that defeats the most others in this set, it in fact defeats every remaining alternative. Hence Y defeats every alternative in the original set except for X, so we can put Y second in the list, remove it too from the set of alternatives, and continue in this way until we exhaust the finite set of alternatives. The way we've constructed the list, each alternative is preferred to all the alternatives that come after it, and so $\succ_i$ arises from this ranked list.

All of this depends on showing that for any complete and transitive preferences over a set of alternatives (including the original preferences, and the ones we get as we remove alternatives), the alternative X that defeats the most others in fact defeats all of them. Here is an argument showing why this is true (also illustrated in Figure 23.1). We suppose, for the sake of a contradiction, that it were not true; then there would be some alternative W that defeats X. But then, for every Y that is defeated by X, we'd have $W \succ_i X$ and $X \succ_i Y$, and so by transitivity $W \succ_i Y$. The conclusion is that W would defeat everything X defeats, and also defeat X — so W would defeat more alternatives than X does. This is a contradiction, since we chose X as the alternative that defeats the most others. Therefore, our assumption that some W defeats X cannot be correct, and so X in fact defeats all other alternatives. This argument justifies our construction of the ranked list.

In view of all this, when we have a complete and transitive preference relation, we can equally well view it as a ranked list. Both of these views will be useful in the discussion to follow.

23.3 Voting Systems: Majority Rule

In the previous section we developed a way to talk about the individual preference relations that we're seeking to combine. We can now describe a *voting system* (also called an *aggregation procedure*) as follows: it is any method that takes a collection of complete and transitive individual preference relations — or equivalently, a collection of individual rankings — and produces a *group ranking*.

This is a very general definition, and at this level of generality it may be hard to see what makes for a "reasonable" voting system. So we begin, in this section and the next, by discussing two of the most common classes of voting systems. By considering these, we'll start to identify some of the principles — and pathologies — at work in voting more generally.

Majority Rule and the Condorcet Paradox. When there are only two alternatives, the most widely used voting system — and arguably the most natural — is *majority rule*. Under majority rule, we take the alternative that is preferred by a majority of the voters and rank it first, placing the other alternative second. For this discussion we will assume that the number of voters is odd, so that we won't have to worry about the possibility of majority rule producing ties.

Since majority rule is so natural in the case of two alternatives, it is natural to try designing a voting system based on majority rule when there are more than two alternatives. This, however, turns out to be remarkably tricky. Probably the most direct approach is to first create *group preferences*, by applying majority rule to each pair of alternatives, and then trying to turn these group preferences into a group ranking. That is, we create a group preference relation $\succ$ out of all the individual preferences $\succ_i$ as follows. For each pair of alternatives X and Y, we count the number of individuals for whom $X \succ_i Y$ and the number of individuals for whom $Y \succ_i X$. If the first number is larger than the second, then we say that the group preference $\succ$ satisfies $X \succ Y$, since a majority of the voters prefer X to Y when these two alternatives are considered in isolation. Similarly, we say $Y \succ X$ in the group preference if $Y \succ_i X$ for a majority of the individuals i. Since the number of voters is odd, we can't have equal numbers favoring X and favoring Y. So for every distinct pair of alternatives we will have exactly one of $X \succ Y$ or $Y \succ X$. That is, the group preference relation is complete.

There's no problem getting this far; the surprising difficulty is that *the group preferences may not be transitive, even when each individual's preferences are transitive.* To see how this can happen, suppose that we have three individuals named 1, 2, and 3, and three alternatives named X, Y, and Z. Suppose further that individual 1's ranking is

$$X \succ_1 Y \succ_1 Z, \tag{23.1}$$

College	National Ranking	Average Class Size	Scholarship Money Offered
X	4	40	$3000
Y	8	18	$1000
Z	12	24	$8000

Figure 23.2: When a single individual is making decisions based on multiple criteria, the Condorcet Paradox can lead to non-transitive preferences. Here, if a college applicants wants a school with a high ranking, small average class size, and a large scholarship offer, it is possible for each option to be defeated by one of the others on a majority of the criteria.

individual 2's ranking is

$$Y \succ_2 Z \succ_2 X, \tag{23.2}$$

and individual 3's ranking is

$$Z \succ_3 X \succ_3 Y. \tag{23.3}$$

Then using majority-rule to define group preferences, we'd have $X \succ Y$ (since X is preferred to Y by both 1 and 3), $Y \succ Z$ (since Y is preferred to Z by both 1 and 2), and $Z \succ X$ (since Z is preferred to X by both 2 and 3). This violates transitivity, which would require $X \succ Z$ once we have $X \succ Y$ and $Y \succ Z$.

The possibility of non-transitive group preferences arising from transitive individual preferences is called the *Condorcet Paradox*, after the Marquis de Condorcet, a French political philosopher who discussed it in the 1700s. And there's something genuinely counter-intuitive about it. If we recall our earlier discussion of non-transitive preferences as being somehow "incoherent," the Condorcet Paradox describes a simple scenario in which a set of people, each with perfectly plausible preferences, manages to behave incoherently when forced to express their collective preferences through majority rule. For example, let's return to our example of an individual who prefers Chocolate to Vanilla to Strawberry to Chocolate. Even if we were to assume that no one individually behaves this way, the Condorcet Paradox shows how this can arise very naturally as the group preferences of a set of ice-cream eating friends, when they plan to share a pint of ice cream and decide on which flavor to buy using majority rule.

The Condorcet Paradox has in fact also been used to show how a single person can naturally be led to form non-transitive individual preferences [41, 163]. Consider, for example, a student deciding which college to attend. She prefers to go to a college that is highly ranked, that has a small average class size, and that offers her a significant amount in scholarship money. Suppose she has been admitted to the following three colleges, with characteristics as described in Figure 23.2.

In comparing colleges, the student was planning to decide between pairs of colleges by favoring the one that did better on a majority of these three criteria. Unfortunately, this

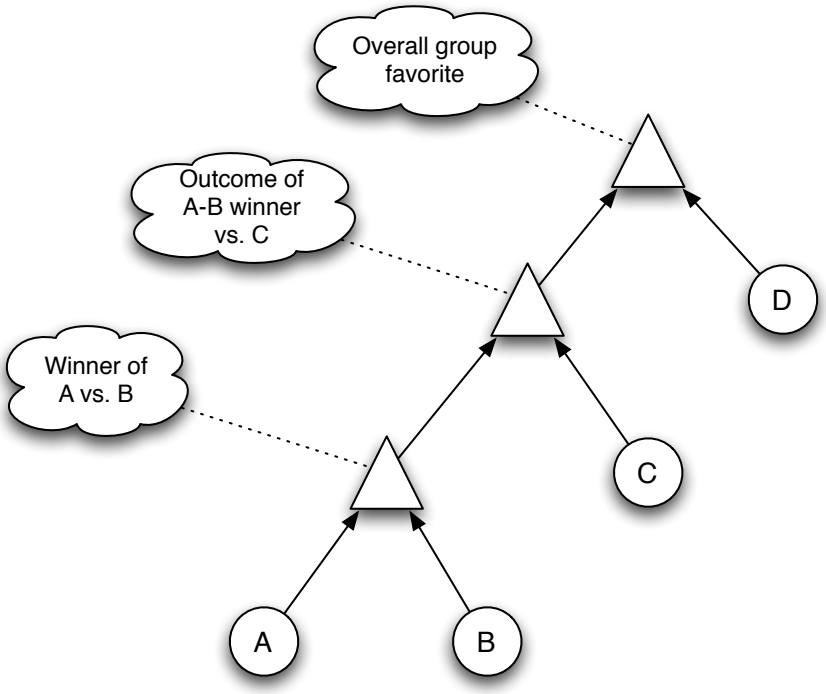
leads to the preferences $X \succ_i Y$ (since X is better than Y on ranking and scholarship money), $Y \succ_i Z$ (since Y is better than Z on ranking and average class size), and $Z \succ_i X$ (since Z is better than X on average class size and scholarship money). It's not hard to see the analogy: each criterion is like a voter, and the student's "individual preference relation" is really the group preference relation synthesized from these three criteria. But it does show some of the complications that arise even when one individual engages in decision-making in the presence of multiple criteria.

Voting Systems based on Majority Rule. The Condorcet Paradox portends trouble for the design of voting systems in general, but given that we need some way to produce an actual group ranking (including an actual top-ranked alternative), it's still worth exploring what can be done using majority rule. We'll focus on methods for selecting a top-ranked alternative, which we'll think of as the "group favorite"; to produce a full ranked list, one could first select a group favorite, remove it from the available alternatives, and then apply the procedure repeatedly to what's left.

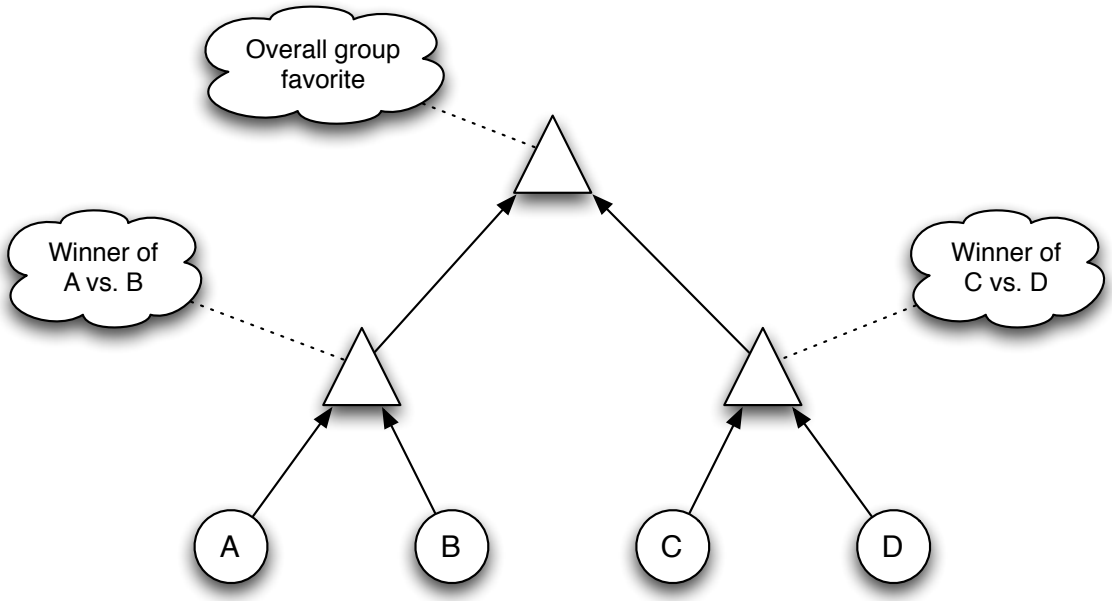
One natural approach for finding a group favorite is as follows. We arrange all the alternatives in some order, and then eliminate them one-by-one in this order using majority rule. Thus, we compare the first two alternatives by majority vote, compare the winner of this vote to the third alternative, then compare the winner of that to the fourth alternative, and so on. The winner of the final comparison is deemed to be the group favorite. We can represent this in a pictorial way as in Figure 23.3(a), showing the sequence of eliminations in a four-alternative example, with alternatives A and B compared first, then the winner compared to C, and the winner of that compared to D. We can think of this as an agenda for a meeting in which pairs of alternatives are proposed to the group, majority votes are taken, and a group favorite emerges from this.

This is an example of a more general strategy for using majority rule over pairs of alternatives to find a group favorite: we can arrange them in any kind of "elimination tournament," in which alternatives are paired off against each other in some fashion, with the winner advancing to a subsequent round while the loser is eliminated. The alternative that eventually emerges as the overall winner of this tournament is declared to be the group favorite. The system we were just discussing, shown in Figure 23.3(a), is one such way to structure an elimination tournament; Figure 23.3(b) depicts another.

Pathologies in Voting Systems based on Majority Rule. These systems do produce a group favorite (and, by repeatedly invoking the system on the remaining alternatives, also produce a group ranking). The Condorcet Paradox, however, can be used to uncover an important pathology that such systems exhibit: their outcomes are susceptible to a kind of *strategic agenda-setting*. Let's go back to our original example of the Condorcet Paradox,

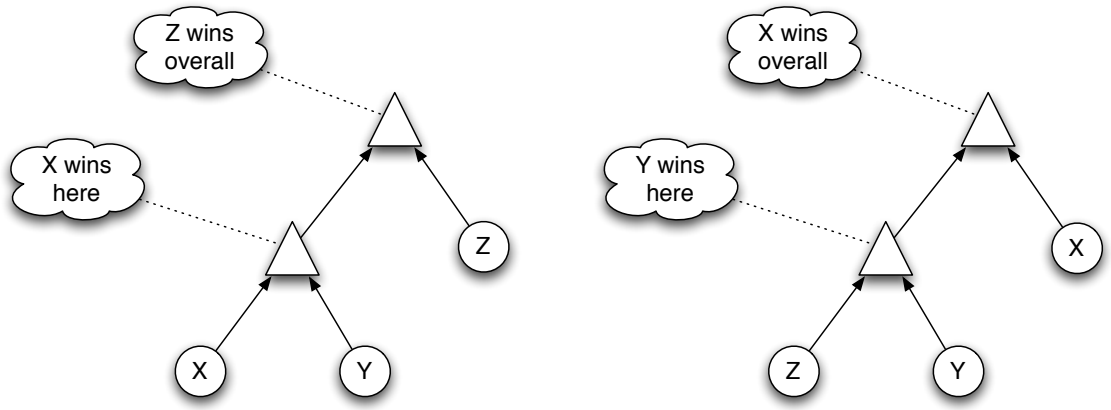


(a) *Introducing new alternatives one at a time.*



(b) *Pairing off alternatives in a different order.*

Figure 23.3: One can use majority rule for pairs to build voting systems on three or more alternatives. The alternatives are considered according to a particular "agenda" (in the form of an elimination tournament), and they are eliminated by pairwise majority vote according to this agenda. This produces an eventual winner that serves as the overall group favorite.



(a) *An agenda in which Z wins.* (b) *An agenda in which X wins.*

Figure 23.4: With individual rankings as in the Condorcet Paradox, the winner of the elimination tournament depends entirely on how the agenda is set.

where three voters had the individual rankings over alternatives X, Y, and Z given by lists (23.1)–(23.3). They decide to choose a group favorite using a version of the system shown in Figure 23.3(a), scaled down to three alternatives: they will first perform majority vote between two of the alternatives, and then perform majority vote between the winner of this first vote and the remaining alternative.

The question then becomes how to set the agenda for this process. That is, which two of the alternatives X, Y, and Z will be voted on first, and which one will be held until the final vote? Because of the structure of the individual preferences, the choice of agenda in this case has a decisive effect on the outcome. If, as in Figure 23.4(a), alternatives X and Y are paired off first, then X will win this first vote but then be defeated by Z as the group favorite. On the other hand, if alternatives Y and Z are paired off first (as in Figure 23.4(b)), then Y will win this first vote but then be defeated by X as the group favorite. (We could do a similar thing to have Y be the group favorite.)

So with individual preferences as in the Condorcet paradox, the overall winner is determined entirely by how the votes between pairs are sequenced. To put it a different way, if the voter who likes Z best gets to set the agenda, then she can sequence the votes so Z wins; but if the voter who likes X or Y best gets to set the agenda, then he can sequence the votes so his respective favorite wins. The group favorite is thus determined by the individual who controls the agenda. Nor can this be remedied by using a system in which voters can re-introduce alternatives once they've been eliminated: with preferences as in the Condorcet Paradox, there's always an alternative that can be re-introduced to defeat the current candidate for the group favorite, and so a process in which alternatives can be re-introduced for

consideration would never come to an end.

Earlier in this section — using an example of a student choosing colleges based on multiple criteria, in Figure 23.2 — we observed that the Condorcet Paradox can also capture pathologies in the decisions made by a single individual, rather than a group. The problem of agenda-setting has an analogue in the context of such individual decisions as well. Suppose, for example, that the student in our earlier college-choice example makes the natural decision to eliminate choices one at a time as acceptance offers come in. Then if the acceptances arrive in the order X, Y, Z, she will eliminate Y in favor of X when Y arrives (X has a higher ranking and a higher scholarship offer) and then eliminate X in favor of Z when the acceptance from Z arrives (since Z has a smaller average class size and a higher scholarship offer). Each of these makes sense as a pairwise decision, and it leads to Z as her overall decision — but it has the property that Y, which she eliminated first, is in fact a choice she preferred to Z. This is precisely the problem of having a final decision that depends on the agenda by which alternatives are considered.

23.4 Voting Systems: Positional Voting

A different class of voting systems tries to produce a group ranking directly from the individual rankings, rather than building up the group ranking from pairwise comparisons of alternatives. In this type of system, each alternative receives a certain *weight* based on its positions in all the individual rankings, and the alternatives are then ordered according to their total weight. A simple example of such a system is the *Borda Count*, named for Jean-Charles de Borda, who proposed it in 1770. The Borda Count is often used to choose the winners of sports awards, such as the Heisman trophy in college football; a variant of it is used to select the Most Valuable Player in professional baseball; and it is used by the Associated Press and United Press International to rank sports teams.

In the Borda Count, if there are k alternatives in total, then individual i's ranking confers a weight of $k-1$ on her first-ranked alternative, a weight of $k-2$ on her second-ranked alternative, and so on down to a weight of 1 on her second-to-last alternative, and a weight of 0 on her last alternative. In other words, each alternative receives a weight from individual i equal to the number of other alternatives ranked lower by i. The total weight of each alternative is simply the sum of the weights it receives from each of the individuals. The alternatives are then ordered according to their total weights. (We will suppose that if two alternatives receive the same total weight, then some tie-breaking system arranged in advance is used to decide which of these two alternatives to place in front of the other.)

For example, suppose there are four alternatives, named A, B, C, and D, and there are two voters with the individual rankings

$$A \succ_1 B \succ_1 C \succ_1 D$$

and

$$B \succ_2 C \succ_2 A \succ_2 D.$$

Then the weight assigned by the Borda Count to alternative A is $3 + 1 = 4$, the weight assigned to B is 5, the weight assigned to C is 3, and the weight assigned to D is 0. Therefore, sorting the weights in descending order, the group ranking is

$$B \succ A \succ C \succ D.$$

It is easy to create variants of the Borda Count that retain its basic flavor: we can assign any number of "points" to each position in each list, and then rank the alternatives by the total number of points they receive based on their positions in all lists. The Borda Count assigns $k - 1$ points for first, $k - 2$ points for second, and so forth, but one could imagine versions that assign points differently: for example, to make only the top three positions in each individual ranking matter, one could assign 3 points for first, 2 for second, 1 for third, and 0 points for all other positions, with the group ranking still determined by the total number of points the alternatives receive. We refer to any system of this type as a *positional voting system*, since the alternatives receive numerical weights based on their positions on the individual rankings.

A key appealing feature of the Borda Count is that — ignoring ties — it always produces a complete, transitive ranking for a set of alternatives. This is simply by its definition, since it creates a single numerical criterion along which to sort the alternatives (including, as noted previously, a rule for tie-breaking). But the Borda Count also has some fundamental pathologies, as we now discuss.

Pathologies in Positional Voting Systems. Most of the problems with the Borda Count, and with positional voting systems more generally, arise from the fact that competition for top spots in the group ranking can depend critically on the rankings of alternatives that are further down in the list.

Here's a hypothetical scenario illustrating how this can happen. Suppose that a magazine writes a column in which it asks five film critics to discuss their choice for the greatest movie of all time; the two movies discussed in the column are *Citizen Kane* and *The Godfather*, and the column ends with a majority-vote decision on the winner. Critics 1, 2, and 3 favor *Citizen Kane*, while critics 4 and 5 favor *The Godfather*.

At the last minute, however, the editors decide the column needs a more "modern" feel, so they introduce *Pulp Fiction* as a third option that needs to be discussed and evaluated. Since there are now three options, the magazine decides to have each critic produce a ranking, and then use the Borda Count for the overall decision that will serve as the punchline of the column. The first three critics (who all prefer older movies) each report the ranking

$$\textit{Citizen Kane} \succ_i \textit{The Godfather} \succ_i \textit{Pulp Fiction}.$$

Critics 4 and 5 (who only like movies made in the last 40 years) each report the ranking

$$\textit{The Godfather} \succ_i \textit{Pulp Fiction} \succ_i \textit{Citizen Kane}.$$

Applying the Borda Count, we see that *Citizen Kane* receives a weight of 2 from each of the first three critics, and a weight of 0 from the last two, for a total of 6. *The Godfather* receives a weight of 1 from each of the first three critics, and a weight of 2 from the last two, for a total of 7. *Pulp Fiction* receives a weight of 0 from each of the first three critics, and a weight of 1 from the last two, for a total of 2. As a result, the Borda Count produces *The Godfather* as the overall group favorite.

Notice what's happened here. The outcome of the head-to-head comparison between *Citizen Kane* and *The Godfather* remains the same as before — *Citizen Kane* is favored by a vote of three to two. But because a third alternative was introduced, the identity of the group favorite has changed. Moreover, this is not because the group was particularly fond of this new, third alternative — the third alternative loses in a head-to-head vote against *each* of the two existing alternatives. To put the difficulty in another way: *Citizen Kane* fails to rank first in the Borda Count even though it defeats each of the other two alternatives in a head-to-head comparison under majority rule. So what we find is that the outcome in the Borda Count can depend on the presence of alternatives that intuitively seem "irrelevant" — weak alternatives that essentially act as "spoilers" in shifting the outcome from one higher-ranked alternative to another.

The possibility of such a result suggests further difficulties with the Borda Count — specifically, the problem of *strategic misreporting of preferences*. To see how this happens, let's consider a slightly different scenario. Suppose in our previous story that critics 4 and 5 actually had the true ranking

$$\textit{The Godfather} \succ_i \textit{Citizen Kane} \succ_i \textit{Pulp Fiction}.$$

In other words, in this version of the story, all five critics agree that *Pulp Fiction* should be ranked last among these three movies. If we were to run the Borda Count on this set of five individual rankings, the group ranking would place *Citizen Kane* first (it would receive a total weight of $3 \cdot 2 + 2 \cdot 1 = 8$ to *The Godfather*'s $3 \cdot 1 + 2 \cdot 2 = 7$). However, suppose that critics 4 and 5 understand the pathologies that are possible with the Borda Count, and they decide in advance to misreport their rankings as

$$\textit{The Godfather} \succ_i \textit{Pulp Fiction} \succ_i \textit{Citizen Kane}.$$

Then we have the individual rankings from the previous scenario, and *The Godfather* ends up being ranked first.

The underlying point is that voters in the Borda Count can sometimes benefit by lying about their true preferences, particularly so as to downgrade the overall group ranking of an alternative that many other voters will put at the top of their individual rankings.

Examples from U.S. Presidential Elections. Versions of these pathologies are familiar from U.S. Presidential elections as well. The full Presidential Election process in the United States has a complex specification, but if we think about how states choose their electors in the general election — i.e. how they choose which candidate will receive the state's electoral votes — it is generally done using *plurality voting*: the candidate who is top-ranked by the most voters wins. (The U.S. Constitution doesn't require this, and some states have considered other methods and used others in the past, but this is the typical system.)

If we think about it, plurality voting is in fact a positional voting system, since an equivalent way to run it is as follows. We ask each voter to report an individual ranking of all the candidates. Each individual ranking then confers a weight of 1 to the candidate at the top of the ranking, and a weight of 0 to all the other candidates. The candidate with the greatest total weight from these rankings is declared the winner. Note that this is just a different way of saying that the candidate who is top-ranked by the most voters wins, but it makes it clear that this system fits the structure of a positional method.

Plurality voting exhibits difficulties analogous to what we observed for the Borda Count. With only two candidates, plurality voting is the same as majority rule; but with more than two candidates, one sees recurring "third-party" effects, where an alternative that is the favorite of very few people can potentially shift the outcome from one of the two leading contenders to the other. In turn, this causes some voters to make their choices strategically, misreporting their top-ranked choice so as to favor a candidate with a better chance of winning. Such issues have been present in recent U.S. Presidential elections, and their effects in important earlier elections, such as the election of Abraham Lincoln in 1860, have also been studied [384].

23.5 Arrow's Impossibility Theorem

We have now looked at a number of different voting systems, and we've seen that when there are more than two alternatives under consideration, they all exhibit pathological behavior. If we were to consider further voting systems used in practice, we'd find they too suffered from inherent problems in the way they produce a group ranking. At some point, however, it makes sense to step back from specific voting systems and ask a more general question: is there *any* voting system that produces a group ranking for three or more alternatives, and avoids all of the pathologies we've seen thus far?

Making this question concrete requires that we precisely specify all the relevant definitions. We've already discussed the precise definition of a voting system: for a fixed number of voters k, it is any function that takes a set of k individual rankings and produces a group ranking. The other thing we need to do is to specify what it means for a voting system to be free of pathologies. We will do this by specifying two properties that we would like a

reasonable voting system to satisfy. They are the following.

- First, if there is any pair of alternatives X and Y for which $X \succ_i Y$ in the rankings of every individual i, then the group ranking should also have $X \succ Y$. This is a very natural condition, known as the *Pareto Principle*, or *Unanimity*; it simply requires that if everyone prefers X to Y, then the group ranking should reflect this. One can think of Unanimity as ensuring that the group ranking be responsive to the individual rankings in at least a minimal way.

- Second, we require that for each pair of alternatives, the ordering of X and Y in the group ranking should depend only on how each individual ranks X and Y relative to each other. In other words, suppose we have a set of individual rankings that produces a group ranking in which $X \succ Y$. If we then take some third alternative Z and shift its position in some of the individual rankings, while leaving the relative ordering of X and Y unchanged, then the voting system should still produce $X \succ Y$ for this new set of individual rankings.

  This condition is called *Independence of Irrelevant Alternatives (IIA)*, since it requires that the group ranking of X and Y should depend only on voter preferences between X and Y, not on how they evaluate other alternatives. IIA is more subtle than Unanimity, but the failure of IIA is in fact responsible for most of the pathological behavior we saw in our earlier discussions of specific voting systems. It was clearly at work in the strategic misreporting of preferences for the Borda Count, since there the shift in ranking of a third alternative Z was sufficient to change the outcome between two other alternatives X and Y. It also plays a role in the problem of strategic agenda-setting for elimination systems based on majority rule: the key idea there was to choose an agenda that eliminated one alternative X early, before it could be paired against an alternative Y that it would in fact defeat.

Voting Systems that Satisfy Unanimity and IIA. Since Unanimity and IIA are both reasonable properties, it's natural to ask what voting systems satisfy them. When there are only two alternatives, majority rule clearly satisfies both: it favors X to Y when all voters do, and — since there are only two alternatives — the group ranking of X and Y clearly does not depend on any other alternatives.

When there are three or more alternatives, it's trickier to find a voting system that satisfies these two properties: neither the positional systems nor the systems based on majority rule that we've considered will work. There is, however, a voting system that satisfies the two properties: dictatorship. That is, we pick one of the individuals i, and we simply declare the group ranking to be equal to the ranking provided by individual i. Notice that there are really k different possible voting systems based on dictatorship — one in which each of the

k possible voters is chosen as the dictator.

We can easily check that each of these k dictatorship systems satisfies Unanimity and IIA. First, if everyone prefers X to Y, then the dictator does, and hence the group ranking does. Second, the group ranking of X and Y depends only how the dictator ranks X and Y, and does not depend on how any other alternative Z is ranked.

Arrow's Theorem. In the 1950s, Kenneth Arrow proved the following remarkable result [22, 23], which clarifies why it's so hard to find voting systems that are free of pathological behavior.

> *Arrow's Theorem: If there are at least three alternatives, then any voting system that satisfies both Unanimity and IIA must correspond to dictatorship by one individual.*

In other words, dictatorship is the only voting system that satisfies both Unanimity and IIA.

Since dictatorship is generally viewed as an undesirable property too, Arrow's Theorem is often phrased as an impossibility result. That is, suppose we say a voting system satisfies *non-dictatorship* if there is no individual i for which the group ranking always coincides with i's ranking. Then we can phrase Arrow's Theorem as follows.

> *Arrow's Theorem (equivalent version): If there are at least three alternatives, then there is no voting system that satisfies Unanimity, IIA, and non-dictatorship.*

Ultimately, what Arrow's Theorem shows us is not that voting is necessarily "impossible," but that it is subject to unavoidable trade-offs — that any system we choose will exhibit certain forms of undesirable behavior. It therefore helps to focus discussions of voting on how to manage these trade-offs, and to evaluate different voting systems in light of them.

23.6 Single-Peaked Preferences and the Median Voter Theorem

Condorcet's Paradox and Arrow's Theorem are facts of nature; we cannot make them go away. However, a common approach when faced with an impossibility result is to consider reasonable special cases of the problem where the underlying difficulties do not arise. There has been a long line of research in voting that follows this direction.

The starting point for this line of research is the observation that there's something a bit unusual about the individual rankings used in the set-up of the Condorcet Paradox. Recall that with three alternatives X, Y, and Z, and three voters 1, 2, and 3, we had

$$X \succ_1 Y \succ_1 Z$$

$$Y \succ_2 Z \succ_2 X$$

$$Z \succ_3 X \succ_3 Y$$

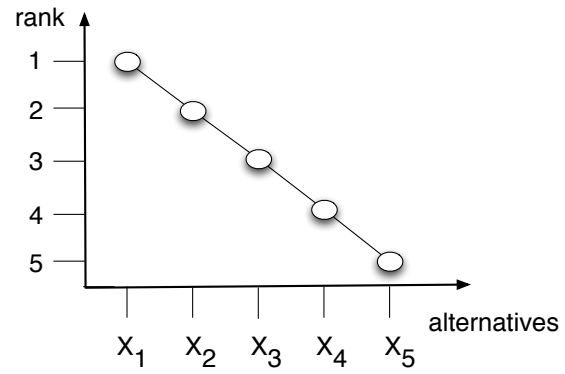
Suppose that X, Y, and Z correspond to amounts of money to spend on education or national defense, with X corresponding to a small amount, Y to a medium amount, and Z to a large amount. Then the preferences of voter 1 make sense: she is happiest with the smallest amount, and second-happiest with a medium amount. The preferences of voter 2 also make sense: he is happiest with a medium amount, but if not medium, then he prefers a large amount. The preferences of voter 3, on the other hand, are harder to justify in a simple way: he prefers a large amount, but his second choice is a small amount, with medium coming last. In other words, the first two voters have preferences that can be explained by proximity to a fixed number: each of them has an "ideal" amount that they'd like, and they evaluate the alternatives by how close they come to this ideal. The third voter's preferences can't be explained this way: there's no "ideal" quantity such that both large and small are close to it, but medium isn't. This is not to say that a person couldn't hold these preferences (e.g. "if we're not willing to invest enough in education to do it right, we shouldn't spend anything at all"), but they're more unusual.

Similar reasoning would apply if X, Y, and Z were political candidates arranged in order on a political spectrum, with X the liberal candidate, Y the moderate candidate, and Z the conservative candidate. In this case, voter 1 prefers more liberal candidates; voter 2 prefers moderates and leans conservative when forced to choose between extremes; but voter 3 favors the conservative candidate, followed next by the liberal candidate, with the moderate candidate last. Again, the preferences of voters 1 and 2 can be explained by assuming that each evaluates candidates by their proximity to a personal "ideal" point on the political spectrum, but voter 3's preferences are less natural in that they can't be explained this way.

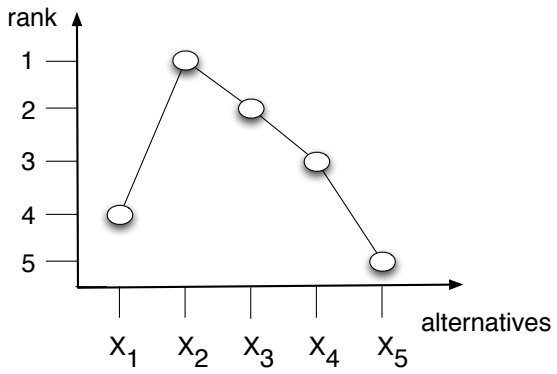
We now describe a way to formalize the "unusualness" in voter 3's ranking, and we then show that for rankings that do not contain this structure, the Condorcet Paradox cannot arise.

Single-peaked preferences. For alternatives corresponding to numerical quantities or linear orderings like a political spectrum, it is reasonable to assume that individual preferences tend to look like the preferences of voters 1 and 2 in our example: each has a particular favorite point in the range of alternatives, and they evaluate alternatives by their proximity to this point. In fact, for our discussion here, it is enough to assume something weaker: simply that each voter's preferences "fall away" consistently on both sides of their favorite alternative.

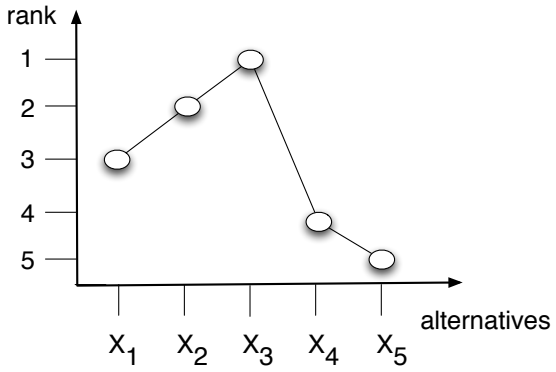
To make this precise, let's assume that the k alternatives are named $X_1, X_2, \ldots, X_k$, and that voters all perceive them as being arranged in this order. (Again, we'll think of the examples of numerical quantities or candidates on a political spectrum.) We say that a



(a) *Voter 1's ranking.*



(b) *Voter 2's ranking.*



(c) *Voter 3's ranking.*

Figure 23.5: With single-peaked preferences, each voter's ranking of alternatives decreases on both sides of a "peak" corresponding to her favorite choice.

voter has *single-peaked preferences* if there is no alternative X_s for which both neighboring alternatives X_{s-1} and X_{s+1} are ranked above X_s. In other words, a voter never prefers two options that lie on opposite sides of a middle option. (Since we are assuming voters have complete and transitive preferences, we will also refer to single-peaked preferences as single-peaked rankings.)

Such preferences are called single-peaked because the condition we impose is equivalent to the following one: each voter i has a top-ranked option X_t, and her preferences fall off on both sides of X_t:

$$X_t \succ_i X_{t+1} \succ_i X_{t+2} \succ_i \cdots$$

and

$$X_t \succ_i X_{t-1} \succ_i X_{t-2} \succ_i \cdots$$

Pictorially, this can be represented as in Figure 23.5. The example shown there has three

voters with preferences

$$X_1 \succ_1 X_2 \succ_1 X_3 \succ_1 X_4 \succ_1 X_5$$

$$X_2 \succ_2 X_3 \succ_2 X_4 \succ_2 X_1 \succ_2 X_5$$

$$X_3 \succ_3 X_2 \succ_3 X_1 \succ_3 X_4 \succ_3 X_5$$

and each of the three plots shows one of these sets of individual preferences: In the plots, there is an oval for each alternative, and its height corresponds to its position in the list. As drawn, the single peak in an individual's ranking emerges visually as a peak in the plot.

Majority Rule with Single-Peaked Preferences. Single-peaked preferences are natural as a model for many kinds of rankings, but their significance in the theory of voting lies in the following observation, made by Duncan Black in 1948 [61].

Recall our first, most basic attempt at synthesizing a group ranking from a set of individual rankings, back in Section 23.3: we would compare each pair of alternatives X and Y to each other, using majority rule to produce a group preference of the form $X \succ Y$ or $Y \succ X$ (depending on which alternative is preferred by more voters). As before, we'll suppose that the number of voters is odd, so that we don't have to worry about the possibility of ties. Our hope was that the resulting group preference relation $\succ$ would be complete and transitive, so that we could produce a group ranking from it. Unfortunately, the Condorcet Paradox showed that this hope was in vain: transitive individual preferences can give rise to group preferences that are non-transitive.

But here's the point of the framework we've developed in this section: with single-peaked preferences, our original plan works perfectly. This is the content of the following result.

> *Claim: If all individual rankings are single-peaked, then majority rule applied to all pairs of alternatives produces a group preference relation $\succ$ that is complete and transitive.*

It is not initially clear why this striking fact should be true, but in fact it follows for an intuitively natural reason, as we now describe.

The Median Individual Favorite. As in other attempts to construct group rankings, we start by figuring out how to identify a group favorite — an alternative that can be placed at the top of the ranking — and then proceed to fill in further slots of the ranking. Finding a group favorite is the crux of the problem, since that requires us to identify an alternative that defeats every other alternative in a pairwise majority vote.

Let's consider the top-ranked alternative for each voter, and sort this set of individual favorites from left to right, along our linear order. Notice that if several voters have the same alternative as their respective individual favorite, then this alternative will appear multiple

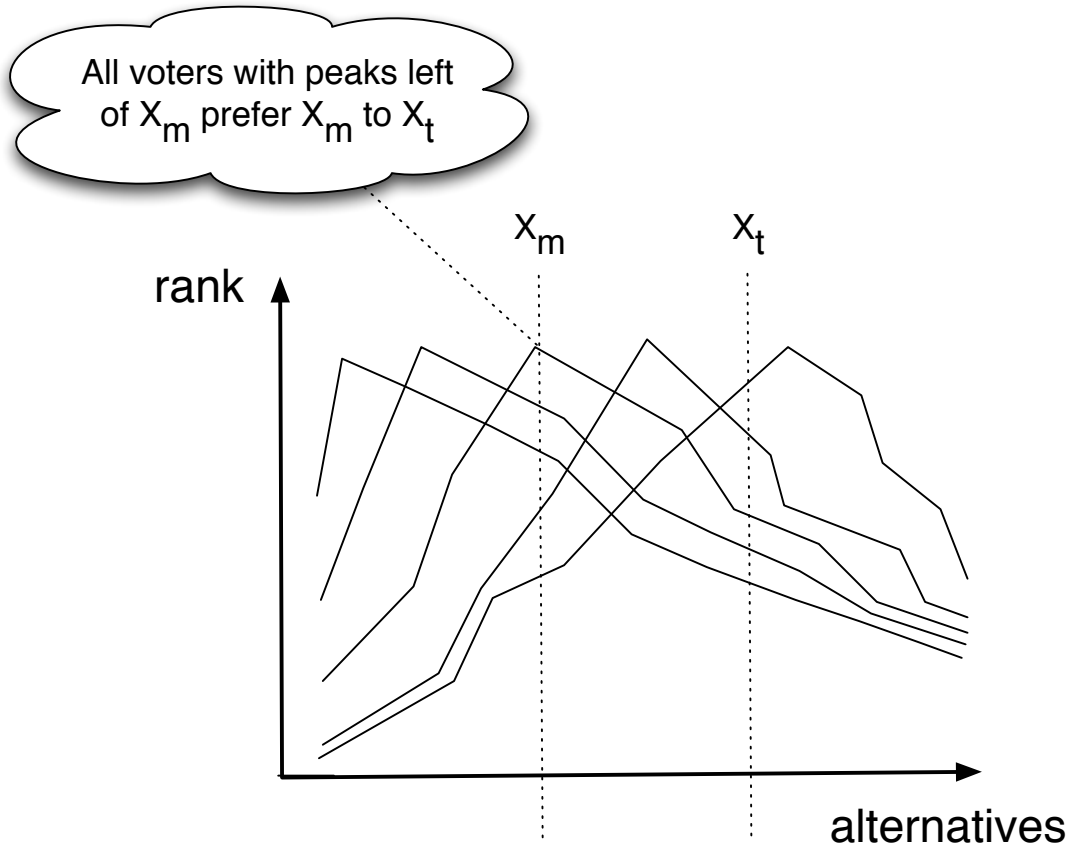


Figure 23.6: The proof that the median individual favorite X_m defeats every other alternative X_t in a pairwise majority vote: if X_t is to the right of X_m, then X_m is preferred by all voters whose peak is on X_m or to its left. (The symmetric argument applies when X_t is to the left of X_m.)

times in the sorted list: it is fine for the list to have repetitions. Now consider the individual favorite that forms the *median* of this list — that is, the individual favorite that lies exactly at the halfway point in the sorted order. For example, in the preferences from Figure 23.5, the sorted list of individual favorites would be X_1, X_2, X_3, and so the median is X_2. With more voters, if the individual favorites were (for example) $X_1, X_1, X_2, X_2, X_3, X_4, X_5$, then the median would also be X_2, since we are considering the median of the list with all the repetitions included.

The median individual favorite is a natural idea to consider as a potential group favorite, since it naturally "compromises" between more extreme individual favorites on either side. And in fact it works very well for this purpose:

The Median Voter Theorem: With single-peaked rankings, the median individual

favorite defeats every other alternative in a pairwise majority vote.

To see why this is true, let X_m be the median individual favorite, and let X_t be any other alternative. Let's suppose that X_t lies to the right of X_m — that is, $t > m$. (The case in which it lies to the left has a completely symmetric argument.) Let's also order the voters in the sorted order of their individual favorites.

The argument is now depicted schematically in Figure 23.6. The number of voters k is odd, and we know that — since it is the median — X_m is in position $(k+1)/2$ of the sorted list of individual favorites. This means that for everyone in the first $(k+1)/2$ positions, X_m is either their favorite, or their favorite lies to the left of X_m. For each voter in this latter group, X_m and X_t are both on the right-hand "down-slope" of this voter's preferences, but X_m is closer to the peak than X_t is, so X_m is preferred to X_t. It follows that everyone in the first $(k+1)/2$ positions prefers X_m to X_t. But this is a strict majority of the voters, and so X_m defeats X_t in a pairwise majority vote.

To put it succinctly: the median individual favorite X_m can always count on gathering a majority of support against any other alternative X_t, because for more than half the voters, X_m lies between X_t and each of their respective favorites.

From this fact about the median individual favorite, it is easy to see why majority rule among all pairs produces a complete and transitive group ranking: we simply build up the group ranking by identifying group favorites one at a time. That is, we start by finding the median individual favorite and placing it at the top of the group ranking. This is safe to do since the Median Voter Theorem guarantees that it defeats all other alternatives that will come later in the list. Now we remove this alternative from each individual ranking. Notice that when we do this, the rankings all remain single-peaked: essentially, we have simply "decapitated" the peak from each ranking, and the second item in each voter's ranking becomes their new peak. We now have a version of the same problem we faced before, with single-peaked rankings on a set of alternatives that is one smaller. So we find the median individual favorite on the remaining alternatives, place it second in the group ranking, and continue in this way until we exhaust the finite set of alternatives.

For example, applying this to the three voters in Figure 23.5, we would identify X_2 as the median individual favorite, and we'd place it first in the group ranking. Once we remove this alternative, we have three single-peaked rankings on the alternatives X_1, X_3, X_4, and X_5. The individual favorites in this reduced set are X_1, X_3, and X_3, so X_3 is the new median individual favorite, and we place it second in the group ranking. Proceeding in this way, we end up with the group ranking

$$X_2 \succ X_3 \succ X_1 \succ X_4 \succ X_5.$$

Since voter 2 was the original "median voter" in the sense of having the original median individual favorite, the start of the group ranking necessarily agrees with the start of voter

2's individual ranking: they both place X_2 first. However, the full group ranking does not coincide with voter 2's full individual ranking: for example, voters 1 and 3 both prefer X_1 to X_4, even through voter 2 doesn't, and the group ranking reflects this.

23.7 Voting as a Form of Information Aggregation

Thus far, we have focused primarily on situations in which voting is used to aggregate fundamentally and genuinely different opinions within a group of people. But there are contexts in which voting is used by a group of people who have a shared goal — where it is reasonable to suppose that there is a true best ranking of the alternatives, and the purpose of the voting system is to discover it. This is unlikely to be appropriate for the ranking of political candidates or works of art, but it can be a good model of jury deliberations in cases where the decision hinges on genuine uncertainty about the facts. It can also be a good model for decisions made by a board of advisers to a company, evaluating business plans that each yield an uncertain future payoff.

In settings like these, where we imagine that there is a true best ranking, it's reasonable to suppose that the individual rankings differ only because they are based on different information, or based on different evaluations of the available information. If everyone in such a case had the same information and evaluated it in the same way, they would have the same ranking.

We will see that these considerations can lead to some potentially complex effects in the way individuals reason about their votes. As a starting baseline, we begin with a simple model in which individuals vote simultaneously, and based purely on their own individual rankings. We then discuss what happens in situations where one or both of these assumptions do not hold — when voting is done sequentially, or when knowledge of other rankings would cause an individual to change her own ranking.

Simultaneous, Sincere Voting: The Condorcet Jury Theorem. We begin with a simple setting in which there are two alternatives X and Y; one of these two is genuinely the best choice, and each voter will cast a vote for what she believes to be this best choice.

To model the idea that voters possess different but uncertain information, we use a general framework that worked well in Chapter 16 on information cascades. We suppose first that there is a *prior probability* of X being the best choice, and that this is known to all voters. For simplicity, we'll take this prior probability to be $1/2$ in our analysis here; this means that initially, X and Y are equally likely to be the best choice. Then, each voter receives an independent, private *signal* about which of X or Y is better. For some value $q > 1/2$, signals favoring the best choice occur at a rate of q; writing this in terms of conditional probability

as in Chapter 16, we have

$$\Pr\left[X\text{-}signal\ is\ observed \mid X\ is\ best\right] = q$$

and

$$\Pr\left[Y\text{-}signal\ is\ observed \mid Y\ is\ best\right] = q.$$

(We can imagine each voter's signal as behaving like the flip of a biased coin: for each voter, it lands on the side indicating the better alternative with probability q.)

Unlike the case in Chapter 16, all the votes in our current analysis are being made simultaneously: no voter is able to see the decisions made by any other voter before reaching her own decision. Also, we assume that everyone is voting *sincerely*: each voter will choose the alternative she believes to be better, based on the information she has available (in the form of her private signal). To model sincere voting by an individual, we can use conditional probabilities just as in Chapter 16. When a voter observes a signal favoring X, she first evaluates the conditional probability

$$\Pr\left[X\ is\ best \mid X\text{-}signal\ is\ observed\right].$$

She then decides to vote in favor of X if this probability is greater than $1/2$, and in favor of Y if this probability is less than $1/2$. The analogous reasoning applies if she observes a signal favoring Y; we focus just on the case of an X-signal since the analysis is symmetric in the two cases.

We can evaluate the conditional probability underlying the voter's decision using Bayes' Rule, by strict analogy with the calculations we did in Section 16.3. We have

$$\Pr\left[X\ is\ best \mid X\text{-}signal\ is\ observed\right] = \frac{\Pr\left[X\ is\ best\right] \cdot \Pr\left[X\text{-}signal\ is\ observed \mid X\ is\ best\right]}{\Pr\left[X\text{-}signal\ is\ observed\right]}.$$

By our assumption about the prior probability, we know that $\Pr\left[X\ is\ best\right] = 1/2$. By the definition of the signals, we know that $\Pr\left[X\text{-}signal\ is\ observed \mid X\ is\ best\right] = q$. Finally, there are two ways for an X-signal to be observed: if X is best, or if Y is best. Therefore,

$$\begin{aligned}\Pr\left[X\text{-}signal\ is\ observed\right] &= \Pr\left[X\ is\ best\right] \cdot \Pr\left[X\text{-}signal\ is\ observed \mid X\ is\ best\right] + \\ &\quad \Pr\left[Y\ is\ best\right] \cdot \Pr\left[X\text{-}signal\ is\ observed \mid Y\ is\ best\right] \\ &= \frac{1}{2} \cdot q + \frac{1}{2}(1-q) = \frac{1}{2}.\end{aligned}$$

Putting all this together, we get

$$\Pr\left[X\ is\ best \mid X\text{-}signal\ is\ observed\right] = \frac{(1/2)q}{1/2} = q.$$

The conclusion — which is completely natural — is that the voter will favor the alternative that is reinforced by the signal she receives. In fact, the calculation using Bayes' Rule gives

us more than just this conclusion; it also shows how much probability she should assign to this favored alternative based on the signal.

The Marquis de Condorcet wrote about this type of scenario in 1785. In his version, he took as a given the assumption that each voter chooses the best alternative with some probability $q > 1/2$, rather than deriving it from the assumption of a private signal — but the model based on either of these starting assumptions is effectively the same. Condorcet's interest was in showing that majority rule is effective when there are many voters who favor the better of two choices at a rate slightly better than half. His probabilistic formulation of individuals' decisions was a novel step — probability was still a relatively new idea in his time — and his main observation, now known as the *Condorcet Jury Theorem*, is the following. Suppose that X is the best alternative (the case for Y being symmetric). Then as the number of voters increases, the fraction of voters choosing X will converge almost surely to the probability of receiving an X-signal, which is $q > 1/2$. In particular, this means that the probability of the majority reaching a correct decision converges to 1 as the number of voters grows. In this sense, Condorcet's Jury Theorem is one of the oldest explicit formulations of the "wisdom of crowds" idea: aggregating the estimates of many people can lead to a decision of higher quality than that of any individual expert.

23.8 Insincere Voting for Information Aggregation

One of the assumptions behind the Condorcet Jury Theorem in the previous section is that all individuals are voting sincerely: each is choosing the alternative he or she believes to be best, given the information available. On the surface, this seems like a mild assumption. If the voters could share all their signals, they would reach a unanimous evaluation of the best alternative; but since they aren't able to communicate with each other and have access only to their own private signals, why should a voter do anything anything other than follow her best guess based on her own signal?

In fact, however, there are very natural situations in which an individual should actually choose to vote insincerely — favoring the alternative she believes to be worse — *even though her goal is to maximize the probability that the group as a whole selects the best alternative.* This is clearly a counter-intuitive claim, and its underlying basis has only been elucidated relatively recently [30, 159, 160]. To explain how this phenomenon can arise, we begin with a hypothetical experiment, modeled on a scenario described by Austen-Smith and Banks [30].

An Experiment that Encourages Insincere Voting. Here's how the experiment works. An experimenter announces that an urn with 10 marbles will be placed at the front of a room; there is a 50% chance the urn will contain ten white marbles, and a 50% chance that it will contain nine green marbles and one white marble. (We describe the first kind of urn as

"pure" and the second kind as "mixed.")

The experimenter asks a group of three people to collectively guess which kind of urn it is. Their group decision will be made according to the following protocol. First, each of the three people is allowed to draw one marble from the urn, look at it (without showing it to the other two), and then replace it in the urn. Then, the three people are asked to cast simultaneous votes, without communicating, each guessing which type of urn they think it is. If a majority of the votes are for the correct type of urn, then all three people win a monetary prize; if a majority of the votes are for the wrong type of urn, then all three people get nothing. (Note that each person gets nothing if the majority is wrong, even if they personally voted for the correct alternative.)

One can see that the experiment is designed to create a set of independent private signals for the voters: the color of the marble drawn by each voter is her private signal, and she cannot communicate this signal to any of the other voters. Rather, the group decision must be reached by majority vote, with each voter having access to these different and potentially conflicting probabilistic signals.

We now ask how an individual should reason about the conditional probabilities of the different urn types based on the marble she draws; after this, we'll consider how she should actually vote.

Conditional Probabilities and Decisions about Voting. First, suppose that you (as one of the three people in the experiment) draw a white marble. While we won't go through the precise calculations, it's not hard to work out, using Bayes' Rule just as we did in the previous section, that the urn in this case is significantly more likely to be pure than mixed. (Intuitively, if you see a white marble, it's much more likely to have been from the all-white pure urn than to be the single white marble in the mixed urn.) On the other hand, if you draw a green marble, then in fact you know for certain that the urn is mixed — since green marbles are only found in mixed urns.

Therefore, if you were to vote sincerely, you would vote "pure" on drawing a white marble and "mixed" if you draw a green marble. But suppose you knew that the other two people in the group were going to vote sincerely, and you wanted to choose your vote to maximize the chance that the majority among the three of you produced the right answer. Then a useful question to ask yourself is, "In what situations does my vote actually affect the outcome?" If you think about it, your vote only affects the outcome when the other two (sincere) votes are split — when there is one vote for pure and one vote for mixed. In this case, however, one of your two partners in the experiments actually drew a green marble, and so the urn must be mixed. Here's the conclusion from this reasoning: whenever your vote actually matters to the outcome, the urn is mixed!

So if you know your two partners will be voting sincerely, you can best help the group

by always voting "mixed," so as to give a single draw of a green marble the chance to sway the majority outcome. In other words, you're manipulating the group choice by voting strategically. You're not doing this to take advantage of the other voters; indeed, you're doing it to make it more likely the group will make the best choice. But nonetheless, it is not optimal for you to vote sincerely in this case, if your two partners are voting sincerely.

Interpretations of the Voting Experiment. Once we appreciate what's going on here, it becomes natural to think of voting with a shared objective in terms of game theory. The voters correspond to players, their possible strategies are the possible ways of choosing votes based on private information, and they receive payoffs based on the votes selected by everyone. The experiment we've just considered constructs a scenario in which sincere voting is not an equilibrium. Notice that while our analysis has therefore ruled out the most natural candidate for an equilibrium, it hasn't actually determined what an equilibrium looks like for this game. In fact, there are multiple equilibria, some of which are a bit complicated to compute, and we won't try to work them out here.

There are a few further points worth reflecting on from this discussion. First, the experiment presents the phenomenon of insincere voting in a very clean and stylized form, which has the advantage of clearly exposing what's going on. But versions of this scenario arise in real-world situations as well, when a highly symmetric decision process like majority vote clashes with a pair of alternatives that has an asymmetric structure — like the pure and mixed urns here. Suppose, for example, that a corporate advisory board has to decide between a risky and a safe course of action for the company, and they decide to use majority vote. Suppose further that board members have their own private evidence in favor of one option or the other, and in fact if anyone were to have genuine evidence in favor of the risky option, then it would be clearly the better choice. If you're a board member in this case, and you know that the other board members will be voting sincerely, then your vote will only matter in the case when half of the rest of the board has evidence in favor of the risky option — in which case, the risky option is the better idea. So the group would be better served if you voted insincerely in favor of the risky option, to improve its chances of being chosen when it should be. Of course, viewing the process of voting as a game, you should appreciate that the situation is in fact more complicated: rather than assuming that the other board members will vote sincerely, you may want to assume that they are also going through this reasoning. Determining how to behave, given this, is a complex problem.

Finally, it's worth highlighting a key methodological point in this analysis — the underlying principle in which you evaluate the consequences of your actions only in the cases where they actually affect the outcome. This was the clarifying insight that exposed why insincere voting was the right decision. Researchers have observed that the use of this principle for voting forms a parallel with reasoning in other game-theoretic contexts as well, including

the "winner's curse" for auctions that we saw in Chapter 9 [159]. There, when many people bid on an item that has a common value (such as oil-drilling rights for a tract of land), the value of your bid only matters if you win, in which case your estimate of the true value of the item is more likely to be an over-estimate than an under-estimate. Hence you should take this into account when bidding, and bid lower than your estimate of the true value. This type of insincerity in bidding is analogous to the insincerity in voting that we've been discussing here; in both cases, they arise because you're evaluating your decision contingent on its actually affecting the outcome, which provides additional implicit information that needs to be taken into account.

23.9 Jury Decisions and the Unanimity Rule

Jury decisions in criminal trials were an important initial example to motivate this discussion: they form a natural class of instances where a group of voters (the jurors) agree in principle that there is a "best" decision for the group — the defendant should be convicted if guilty and acquitted if innocent — and they want to aggregate their individual opinions to try arriving at this best decision. Given what we've just seen, it is natural to ask: can insincere voting arise in this case, and if so, what are its consequences? As Feddersen and Pesendorfer have argued, insincere voting in fact can arise naturally as a strategy for jurors who want their vote to contribute to the best overall group decision [160]. We describe the basic structure of their analysis here.

Verdicts, Unanimity, and Private Signals. If we compare jury decisions in criminal trials with the set-up for the Condorcet Jury Theorem from Section 23.7, we notice two basic differences, both of which arise from institutional features of the criminal-justice system designed to help avoid convicting innocent defendants.

The first difference is that it generally requires a unanimous vote in order to convict a defendant. So if we have k jurors, and the two options *acquittal* and *conviction*, each juror votes for one of these options, and conviction is chosen by the group only if each juror votes for it. The second difference is in the criterion that jurors are asked to use for evaluating the two alternatives. In the model from Section 23.7, if each voter could observe all the available information, she would choose alternative X if

$$\Pr\left[X \textit{ is best} \mid \textit{all available information}\right] > \frac{1}{2}.$$

In a criminal trial, however, the instructions to a jury are not, "The defendant should be convicted if he is more likely to be guilt than innocent," but instead "The defendant should be convicted if he is guilty beyond a reasonable doubt." This means that jurors should not

be asking whether

$$\Pr\left[\textit{defendant is guilty} \mid \textit{all available information}\right] > \frac{1}{2},$$

but whether

$$\Pr\left[\textit{defendant is guilty} \mid \textit{all available information}\right] > z$$

for some larger number z.

We now consider how to model the information available to each juror. Following the framework used for the Condorcet Jury Theorem in Section 23.7, we assume that each juror receives an independent private signal suggesting guilt (a G-signal) or innocence (an I-signal). The defendant, in reality, is of course either guilty or innocent, and we assume that signals favoring the truth are more abundant than signals favoring the wrong answer: for some number $q > \frac{1}{2}$, we have

$$\Pr\left[\textit{G-signal} \mid \textit{defendant is guilty}\right] = q$$

and

$$\Pr\left[\textit{I-signal} \mid \textit{defendant is innocent}\right] = q.$$

A juror who observes a G-signal is interested in the conditional probability of guilt given the signal, namely $\Pr\left[\textit{defendant is guilty} \mid \textit{G-signal}\right]$. Let's assume a prior probability of $1/2$ that the defendant is guilty — i.e., in the absence of any signals. Then the argument using Bayes' Rule from Section 23.7 (with conviction and acquittal playing the roles of the two alternatives X and Y from that section) applies directly here, showing that

$$\Pr\left[\textit{defendant is guilty} \mid \textit{G-signal}\right] = q,$$

and similarly that

$$\Pr\left[\textit{defendant is innocent} \mid \textit{I-signal}\right] = q.$$

The conclusions of the analysis to follow would remain essentially the same, with slightly different calculations, if we were to assume any prior probability between 0 and 1.

Before proceeding with the analysis, it's fair to ask whether the modeling assumption that jurors receive independent, private signals about guilt or innocence is reasonable — after all, they sit through the trial together, and they all see the same evidence being presented. Clearly, the assumption of private signals is a simplified approximation, but it is also clear that jurors in real trials can and do form widely divergent views of the facts in a case. This is natural: despite seeing the same evidence, jurors form different interpretations and inferences based on their own personal intuitions and decision-making styles — things that cannot necessarily be transmitted as facts from one person to another [160]. So in this case we can think of the private signals as representing private *interpretations* of the information presented, rather than some personal source of additional information A rational juror is thus guided by her own signal, but she would also be influenced by knowledge of the signals of others — i.e. by knowledge that others had interpreted things the same or differently.

Modeling a Juror's Decision. As noted above, the unanimity rule is designed to make it hard for an innocent defendant to be convicted, since such a result would require every single juror to "erroneously" favor conviction. On the surface, this informal principle makes sense — but as we saw in Section 23.8, reasoning about such principles can become subtle when we assume that individuals are choosing their votes with the overall group decision in mind.

In particular, things become complicated for the following reason. Suppose that you're one of the k jurors, and you received an I-signal. At first, it seems clear that you should vote to acquit: after all, your I-signal on its own gives you a conditional probability of $q > \frac{1}{2}$ that the defendant is innocent. But then you remember two things. First, the criterion for conviction by the group is

$$\Pr[\textit{defendant is guilty} \mid \textit{available information}] > z,$$

which means that in principle the unobserved signals of everyone else — if only you knew what they were — could be enough to push the conditional probability of guilt above z, despite your I-signal. Second, you ask yourself the key question from Section 23.8: "In what situations does my vote actually affect the outcome?" Given the unanimity rule, your vote only affects the outcome when every juror but you is voting to convict. If you believe that everyone else's vote will reflect the signal they received, then you can work out exactly what the full set of signals is in the event that your vote affects the outcome: it consists of $k-1$ G-signals and your one I-signal.

What is the probability the defendant is guilty in this case? We can use Bayes' Rule to say that

$$\begin{aligned}&\Pr[\textit{defendant is guilty} \mid \textit{you have the only } I\textit{-signal}]\\ &\quad = \frac{\Pr[\textit{defendant is guilty}] \cdot \Pr[\textit{you have the only } I\textit{-signal} \mid \textit{defendant is guilty}]}{\Pr[\textit{you have the only } I\textit{-signal}]}.\end{aligned}$$

Our assumption is that $\Pr[\textit{defendant is guilty}] = 1/2$, and since the G-signals are independent, we have $\Pr[\textit{you have the only } I\textit{-signal} \mid \textit{defendant is guilty}] = q^{k-1}(1-q)$. (For this latter calculation, there is a probability of q^{k-1} that each of the $k-1$ other jurors gets a G-signal, times a probability of $1-q$ that you get an I-signal.) Finally, as usual in Bayes' Rule calculations, we determine the two different ways in which all jurors but you receive G-signals: if the defendant is guilty, or if he is innocent:

$$\begin{aligned}&\Pr[\textit{you have the only } I\textit{-signal}]\\ &\quad = \Pr[\textit{defendant is guilty}] \cdot \Pr[\textit{you have the only } I\textit{-signal} \mid \textit{defendant is guilty}] +\\ &\qquad \Pr[\textit{defendant is innocent}] \cdot \Pr[\textit{you have the only } I\textit{-signal} \mid \textit{defendant is innocent}]\\ &\quad = \frac{1}{2} \cdot q^{k-1}(1-q) + \frac{1}{2}(1-q)^{k-1}q.\end{aligned}$$

(The second term in the last expression arises from an analogous calculation to what we used for the first term: if the defendant is innocent, there is a probability of $(1-q)^{k-1}$ that each of the $k-1$ jurors other than you gets a G-signal, times a probability of q that you get an I-signal.) Putting these quantities together, we have

$$\begin{aligned}\Pr\left[\textit{defendant is guilty} \mid \textit{you have the only } I\textit{-signal}\right] &= \frac{\frac{1}{2}q^{k-1}(1-q)}{\frac{1}{2}q^{k-1}(1-q)+\frac{1}{2}(1-q)^{k-1}q} \\ &= \frac{q^{k-2}}{q^{k-2}+(1-q)^{k-2}},\end{aligned}$$

where the second quality follows just by canceling $q(1-q)/2$ from both the numerator and denominator.

Now, since $q > 1/2$, the term $(1-q)^{k-2}$ represents an arbitrarily small portion of the total denominator as the jury size k goes to infinity — and so in particular

$$\Pr\left[\textit{defendant is guilty} \mid \textit{you have the only } I\textit{-signal}\right]$$

converges to 1 as k goes to infinity. Hence if the jury size k is large enough, it follows that $\Pr\left[\textit{defendant is guilty} \mid \textit{you have the only } I\textit{-signal}\right] > z$.

We conclude from this that if you believe everyone else is voting their signals, and if there are enough other jurors, then in the only case where your vote to acquit affects the outcome, the defendant is in fact guilty beyond a reasonable doubt. So if you were to vote with the actual instructions to the jury in mind, you should ignore your signal and vote to convict. Of course, you should do this with even more confidence in the event that you receive a G-signal, and so we can summarize the conclusion even more starkly: if you believe everyone else is voting their signals, and the jury size is large enough, you should always ignore your signal and vote to convict.

Intuitively, what's going on is that you only affect the outcome of a unanimous vote when everyone else holds the opposite opinion; on the assumption that everyone else is as well-informed as you are, and voting their true opinion, the conclusion is that they're probably (collectively) right, and you're wrong. As with our earlier example in Section 23.8, this serves as an interesting reminder that when you design a procedure or protocol for a group of people to follow, you should expect that they'll adapt their behavior in light of the rules you define. Here, the voting system based on unanimity was designed to help prevent erroneous convictions, but in fact it creates an incentive for people to disregard signals that the defendant is innocent.

Equilibria for Voting under Unanimity and Other Systems. As in Section 23.8, we've shown that (for large enough juries) voting your signal is not an equilibrium — if everyone else is doing it, then you should always vote to convict. In their analysis of this

problem, Feddersen and Pesendorfer went further and worked out what the equilibria for jury voting in this model actually look like.

First, there's an equilibrium that's easy to find but a bit pathological: if everyone decides to ignore their signals and vote to acquit, this is an equilibrium. To see why, notice that no juror can affect the outcome by changing her behavior, and hence there is no incentive for any juror to change what she is doing.

More interestingly, there is a unique equilibrium with the properties that (i) all jurors use the same strategy, and (ii) each juror's behavior actually depends on the signal she receives. This is a mixed-strategy equilibrium, in which each juror always votes to convict on a G-signal, and votes to convict with some probability between 0 and 1 on an I-signal. The idea is that each juror with an I-signal may randomly choose to disregard it, effectively correcting for the possibility that she is wrong. One can show that when jurors follow this equilibrium, the probability that their group decision convicts an innocent defendant is a positive number that does not converge to zero as the size of the jury goes to infinity. This forms a sharp contrast to the Condorcet Jury Theorem, where the probability of a correct decision is converging to 1 as the number of voters grows. The problem here is that the unanimity rule encourages voters to "over-correct" so strongly for the chance that they might be wrong, it leads to a noticeable probability that the group as a whole reaches the wrong decision.

Moreover, the unanimity rule is particularly bad in this regard. Specifically, with further analysis, we can study voting systems in which convicting a defendant requires only that an f fraction of the jurors vote for conviction, for different values of f with $0 < f < 1$. For a given choice of f, we'll call such a system the *f-majority rule*. There is still an equilibrium here in which jurors employ randomization, sometimes disregarding their signals to correct for the possibility that they are wrong. But with the f-majority rule, a juror's vote affects the outcome when the remaining jurors are divided between convicting and acquitting in a ratio of f to $(1 - f)$ — a much less extreme split than under the unanimity rule, where a juror's vote affects the outcome only in the event that she is singular in her opposition to convicting. As a result of this, the randomized correction used by jurors is correspondingly less extreme, and one can show that as the jury size goes to infinity, the probability of the group decision being wrong goes to 0 [160].

This result offers a further reason to question the appropriateness of the unanimity rule — the result suggests that a decision rule for juries requiring conviction by a wide majority, rather than a unanimous vote, might actually induce behavior in which there is a lower probability of erroneous convictions. It is again an indication of the subtle issues that arise when one evaluates the trade-offs between different social institutions in light of the behaviors they induce in the people who take part in them.

23.10 Sequential Voting and the Relation to Information Cascades

Let's return to the original formulation of the Condorcet Jury Theorem, with individuals who vote simultaneously and sincerely over two alternatives X and Y. In the last two sections, we've examined what happens when we remove the assumption of sincerity. It's also interesting to instead remove the assumption of simultaneity and see what happens. We'll keep sincerity in this discussion, since this simplifies the analysis by only changing one aspect of the model at a time. So each voter will cast a vote for the alternative she believes to be the best choice.

When we assume that voters act sincerely but sequentially, we have a model that closely aligns with the formulation of information cascades from Chapter 16. In our model for information cascades, we assumed that voters make choices sequentially: they are able to observe the choices (but not the private signals) of earlier voters; and they can choose to disregard their own signals if it increases the chance that they personally choose the better alternative. Note that in this model of cascades, voters are still behaving sincerely in the sense that they are trying to choose the alternative that is more likely to be correct, based on everything they are able to observe.

Aside from this distinction between simultaneous and sequential voting, the set-up for the Condorcet Jury Theorem from Section 23.7 is otherwise quite similar to the model for information cascades from Chapter 16. In both models, there is a given prior probability for X to be correct, and there are private signals favoring the correct alternative with probability greater than $1/2$. Therefore, we can invoke our analysis from Section 16.5 to argue that if voters act sequentially, two initial votes in favor of X will cause a cascade in which all subsequent votes are for X as well — regardless of whether X is the correct decision. More generally, once the number of votes for one alternative first exceeds the number of votes for the other alternative by at least two, a cascade will form in which all subsequent voters will strategically choose to disregard their own signals.

The fact that cascades begin when one alternative leads the other by exactly two votes depends on the specific structure of our simplified model from Chapter 16. The broader principle, however, is quite general. In sequential voting of the type we're describing, a cascade will eventually develop. And cascades can be wrong: even if Y is the best alternative, a cascade for X can develop. Moreover, increasing the number of voters does essentially nothing to stop this cascade. So the principle behind the Condorcet Jury Theorem does not apply in this setting: There is no reason to expect that a large crowd of sequential voters will get the answer right.

Profile 1:

Individual	Ranking	Ranking restricted to X and Y
1	$W \succ X \succ Y \succ Z$	$X \succ Y$
2	$W \succ Z \succ Y \succ X$	$Y \succ X$
3	$X \succ W \succ Z \succ Y$	$X \succ Y$

Profile 2:

Individual	Ranking	Ranking restricted to X and Y
1	$X \succ Y \succ W \succ Z$	$X \succ Y$
2	$Z \succ Y \succ X \succ W$	$Y \succ X$
3	$W \succ X \succ Y \succ Z$	$X \succ Y$

Figure 23.7: The two profiles above involve quite different rankings, but for each individual, her ranking restricted to X and Y in the first profile is the same as her ranking restricted to X and Y in the second profile. If the voting system satisfies IIA, then it must produce the same ordering of X and Y in the group ranking for both profiles.

23.11 Advanced Material: A Proof of Arrow's Impossibility Theorem

In this section we give a proof of Arrow's Theorem [22, 23], which was stated in Section 23.5. The proof we present is not Arrow's original one; instead, we follow a shorter proof found more recently by John Geanakoplos [179].

Let's begin by stating the theorem in a language that will help in discussing the ideas in the proof. We start with a finite set of alternatives. We have a set of k individuals, whom we can assume to be numbered $1, 2, 3, \ldots, k$; each individual has a ranking of the possible alternatives. We'll call the collection of all k rankings a *profile*. In this terminology, a *voting system* is simply a function that takes a profile and produces a *group ranking*: a single ranking of the alternatives.[1] The voting system satisfies *Unanimity* if it puts $X \succ Y$ in the group ranking whenever $X \succ_i Y$ according to the ranking of each individual i. The voting system satisfies the *Independence of Irrelevant Alternatives* (abbreviated IIA) if the ordering of alternatives X and Y in the group ranking depends only on the ordering of X and Y in each individual ranking, and not on their position relative to any other alternatives.

Here's a slightly different but equivalent way to describe IIA, which will be useful in our discussion. Consider a profile of rankings, and any two alternatives X and Y. We say that an individual's ranking *restricted to X and Y* consists of a copy of his or her ranking in which we erase all the alternatives other than X and Y. A profile *restricted to X and Y* is

[1] As in earlier sections of this chapter, we will consider the case in which individual rankings have no ties, and the voting system is required to produce a group ranking that has no ties either.

the profile consisting of all individual rankings restricted to X and Y. Then, as illustrated in Figure 23.7, if a voting system satisfies IIA, it must produce the same ordering of X and Y for any two profiles that are the same when restricted to X and Y. (In other words, the profile restricted to X and Y is the only "data" the voting system can look at in ordering X and Y in the group ranking.)

Recall from Section 23.5 that a voting system can satisfy both Unanimity and IIA via *dictatorship*: it selects some individual j in advance, and for any profile of individual rankings, it simply declares the group ranking to be j's ranking. There are k different dictatorship procedures, depending on which of the k individuals is chosen in advance to be the dictator. Arrow's Theorem is that the k dictatorship procedures are the only voting systems that satisfy Unanimity and IIA. This is the statement we prove here.

The challenge in proving Arrow's Theorem is that the Unanimity and IIA conditions are both quite simple, and hence give us relatively little to work with. Despite this, we need to take an arbitrary voting system satisfying these two properties, and show that it in fact coincides with dictatorship by a single individual.

Our proof will consist of three main steps. First we show the following interesting fact; its utility in the proof is not immediately apparent, but it plays a crucial role. Let's call X a *polarizing alternative* if it is ranked either first or last by every individual. Profiles P and P' in Figure23.8 are examples of profiles in which X is a polarizing alternative. We'll show that if a voting system satisfies Unanimity and IIA, then it must place any polarizing alternative in either first or last place in the group ranking. In other words, such a voting system can't find a way to "average" and place a polarizing alternative somewhere in the middle of the group ranking. Note that many profiles don't contain a polarizing alternative; this fact only applies to those that do. In the second step of the proof, we then use this fact to identify a natural candidate for the role of dictator, and in the third step, we prove that this individual is in fact a dictator.

First Step: Polarizing Alternatives. For the remainder of the proof, let F be a voting system satisfying Unanimity and IIA. We will use P to denote a profile of individual rankings, and use $F(P)$ to denote the group ranking that F produces, as a function of this profile. We will work toward identifying an individual j with the property that F simply consists of dictatorship by j.

First, let P be a profile in which X is a polarizing alternative, and suppose by way of contradiction that F does not place X in either first or last place in the group ranking $F(P)$. This means that there are other alternatives Y and Z so that $Y \succ X \succ Z$ in the group ranking $F(P)$.

Now, for any individual ranking that puts Y ahead of Z, let's change it by sliding Z to the position just ahead of Y. This produces a new profile P', as sketched in Figure 23.8. Since

Profile P:

Individual	Ranking
1	$X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$
2	$X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$
3	$\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$

Profile P':

Individual	Ranking
1	$X \succ \cdots \succ Z \succ Y \succ \cdots$
2	$X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$
3	$\cdots \succ Z \succ Y \succ \cdots \succ X$

Figure 23.8: A polarizing alternative is one that appears at the beginning or end of every individual ranking. A voting system that satisfies IIA must put such an alternative at the beginning or end of the group ranking as well. The figure shows the key step in the proof of this fact, based on rearranging individual rankings while keeping the polarizing alternative in its original position.

X is a polarizing alternative, the relative order of X and Z does not change in any individual ranking when we do this, nor does the relative order of X and Y. Therefore, by IIA, we still have $Y \succ X \succ Z$ in the group ranking $F(P')$. But in P', alternative Z is ahead of alternative Y in every individual ranking, and so by Unanimity we have $Z \succ Y$ in the group ranking $F(P')$. Putting these together, the group ranking $F(P')$ has $Y \succ X \succ Z \succ Y$, which contradicts the fact that the voting system F always produces a transitive group ranking.

This contradiction shows that our original assumption of alternatives Y and Z with $Y \succ X \succ Z$ in $F(P)$ cannot be correct, and so X must appear in either the first or last position in the group ranking $F(P)$.

Second Step: Identifying a Potential Dictator. In the next step, we create a sequence of profiles with the property that each differs from the next by very little, and we watch how the group ranking (according to F) changes as we move through this sequence. As we track these changes, a natural candidate for the dictator will emerge.

Here is how the sequence of profiles is constructed. We pick one of the alternatives, X, and we start with any profile P_0 that has X at the end of each individual ranking. Now, one individual ranking at a time, we move X from last place to first place while leaving all other parts of the rankings the same, as shown in Figure 23.9. This produces a sequence of rankings $P_0, P_1, P_2, \ldots, P_k$, where P_i

(i) has X at the front of the individual rankings of $1, 2, \ldots, i$;

Profile P_0:

Individual	Ranking
1	$\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$
2	$\cdots \succ Z \succ \cdots \succ Y \succ \cdots \succ X$
3	$\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$

Profile P_1:

Individual	Ranking
1	$X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$
2	$\cdots \succ Z \succ \cdots \succ Y \succ \cdots \succ X$
3	$\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$

Profile P_2:

Individual	Ranking
1	$X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$
2	$X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$
3	$\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$

Profile P_3:

Individual	Ranking
1	$X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$
2	$X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$
3	$X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$

Figure 23.9: To find a potential dictator, one can study how a voting system behaves when we start with an alternative at the end of each individual ranking, and then gradually (one person at a time) move it to the front of people's rankings.

(ii) has X at the end of the individual rankings of $i+1, i+2, \ldots, k$; and

(iii) has the same order as P_0 on all other alternatives.

So in other words, P_{i-1} and P_i differ only in that individual i ranks X last in P_{i-1}, and he ranks it first in P_i.

Now, by Unanimity, X must be last in the group ranking $F(P_0)$, and it must be first in the group ranking $F(P_k)$. So somewhere along this sequence there is a first profile in which X is not in last place in the group ranking; suppose this first profile is P_j. Since X is a polarizing alternative in P_j, and it is not in last place in $F(P_j)$, it must be in first place.

So individual j has a huge amount of power over the outcome for alternative X, at least in this sequence of rankings: by switching her own ranking of X from last to first, she causes

X to move from last to first in the group ranking. In the final step of the proof, we will show that j is in fact a dictator.

Third Step: Establishing that j is a Dictator. The key argument in showing that j is a dictator is to show that for any profile Q, and any alternatives Y and Z that are different from X, the ordering of Y and Z in the group ranking $F(Q)$ is the same as the ordering of Y and Z in j's individual ranking in Q. After that we'll show that the same also holds for pairs of alternatives in which one of the alternatives is X. In this way, we'll have established that the ordering of each pair is determined entirely by j's ordering, and hence j is a dictator.

So let Q be any profile, and let Y and Z be alternatives not equal to X such that j ranks Y ahead of Z. We will show that $F(Q)$ puts Y ahead of Z as well.

We create an additional profile Q' that is a variant of Q; this new profile will help us understand how j controls the ordering of Y and Z. First, we take Q, move X to the front of the individual rankings of $1, 2, \ldots, j$, and move X to the end of the individual rankings of $j+1, j+2, \ldots, k$. Then, we move Y to the front of j's individual ranking (just ahead of X). We call the resulting profile Q'.

Now, we make the following observations.

- We know that X comes first in the group ranking $F(P_j)$. Since Q' and P_j are the same when restricted to X and Z, it follows from Independence of Irrelevant Alternatives that $X \succ Z$ in $F(Q')$.
- We know that X comes last in the group ranking $F(P_{j-1})$. Since Q' and P_{j-1} are the same when restricted to X and Y, it follows from IIA that $Y \succ X$ in $F(Q')$.
- By transitivity, we conclude that $Y \succ Z$ in $F(Q')$.
- Q and Q' are the same when restricted to Y and Z, since we produced Q' from Q without ever swapping the order of Y and Z in any individual ranking. By IIA, it follows that $Y \succ Z$ in $F(Q)$.
- Since Q was any profile, and Y and Z were any alternatives (other than X) subject only to the condition that j ranks Y ahead of Z, it follows that the ordering of Y and Z in the group ranking is always the same as j's.

Thus we've shown that j is a dictator over all pairs that do not involve X. We're almost done; we just have to show that j is also a dictator over all pairs involving X as well.

To show this, first observe that we can run the argument thus far with respect to any other alternative W different from X, and thereby establish that there is also an individual ℓ who is a dictator over all pairs not involving W. Suppose that ℓ is not equal to j. Now, for X and some third alternative Y different from X and W, we know that the profiles P_{j-1}

and P_j differ only in j's individual ranking, yet the ordering of X and Y is different between the group rankings $F(P_{j-1})$ and $F(P_j)$. In one of these two group rankings, the ordering of X and Y must therefore differ from the ordering of X and Y in ℓ's individual ranking, contradicting the fact that ℓ is a dictator for the pair X and Y. Hence our assumption that ℓ is different from j must be false, and thus j is in fact a dictator over all pairs.

23.12 Exercises

1. In this chapter, we discussed how voting systems based on majority rule are susceptible to strategic agenda-setting. Let's explore how one might do this on some basic examples.

   (a) Suppose there are four alternatives, named A, B, C, and D. There are three voters who have the following individual rankings:

   $$B \succ_1 C \succ_1 D \succ_1 A$$

   $$C \succ_2 D \succ_2 A \succ_2 B$$

   $$D \succ_3 A \succ_3 B \succ_3 C$$

   You're in charge of designing an agenda for considering the alternatives in pairs and eliminating them using majority vote, via an elimination tournament in the style of the examples shown in Figure 23.3.

   You would like alternative A to win. Can you design an agenda (i.e. an elimination tournament) in which A wins? If so, describe how you would structure it; if not, explain why it is not possible.

   (b) Now, consider the same question, but for a slightly different set of individual rankings in which the last two positions in voter 3's ranking have been swapped. That is, we have:

   $$B \succ_1 C \succ_1 D \succ_1 A$$

   $$C \succ_2 D \succ_2 A \succ_2 B$$

   $$D \succ_3 A \succ_3 C \succ_3 B$$

   We now ask the same question: Can you design an agenda in which A wins? If so, describe how you would structure it; if not, explain why it is not possible.

2. The Borda Count is susceptible to strategic misreporting of preferences. Here are some examples to practice how this works.

(a) Suppose you are one of three people voting on a set of four alternatives named A, B, C, and D. The Borda Count will be used as the voting system. The other two voters have the rankings

$$D \succ_1 C \succ_1 A \succ_1 B$$

$$D \succ_2 B \succ_2 A \succ_2 C$$

You are voter 3 and would like alternative A to appear first in the group ranking, as determined by the Borda Count. Can you construct an individual ranking for yourself so that this will be the result? If so, explain how you would choose your individual ranking; if not, explain why it is not possible.

(b) Let's consider the same question, but with different rankings for the other two voters, as follows:

$$D \succ_1 A \succ_1 C \succ_1 B$$

$$B \succ_2 D \succ_2 A \succ_2 C$$

Again, as voter 3, you would like alternative A to appear first in the group ranking determined by the Borda Count. Can you construct an individual ranking for yourself so that this will be the result? If so, explain how you would choose your individual ranking; if not, explain why it is not possible.

3. In Section 23.6, we considered a setting in which alternatives are arranged on a line. Each voter has an "ideal" point on the line, and she ranks alternatives by the distance of these alternatives to her ideal point. An interesting property of this setting is that the Condorcet Paradox cannot arise; more strongly, majority vote over pairs of alternatives always produces group preferences that are complete and transitive.

Suppose we try to generalize this by allowing alternatives and voters to be positioned in two dimensions rather than one. That is, suppose that each alternative corresponds to a point in two-dimensional space. (For example, perhaps the alternatives are different versions of a piece of legislation, and they differ in two distinct characteristics, corresponding to the two dimensions.) As before, each voter has an "ideal" point in the two-dimensional plane where the alternatives reside, and she evaluates the alternatives by their respective distances (in the plane) to this ideal point.

Unfortunately, the desirable properties that applied to one-dimensional preferences no longer hold here. Show how to construct a set of three alternatives in two dimensions, and a set of three voters, each with an ideal point, so that the resulting set of individual preferences produces the preferences that we saw in the Condorcet Paradox.