

Ambiguous Language and Differences in Beliefs

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Abstract

Standard models of multi-agent modal logic do not capture the fact that information is often *ambiguous*, and may be interpreted in different ways by different agents. We propose a framework that can model this, and consider different semantics that capture different assumptions about the agents' beliefs regarding whether or not there is ambiguity. We consider the impact of ambiguity on a seminal result in economics: Aumann's result saying that agents with a common prior cannot agree to disagree. This result is known not to hold if agents do not have a common prior; we show that it also does not hold in the presence of ambiguity. We then consider the tradeoff between assuming a *common interpretation* (i.e., no ambiguity) and a common prior (i.e., shared initial beliefs).

1 Introduction

In the study of multi-agent modal logics, it is always implicitly assumed that all agents interpret all formulas the same way. While they may have different beliefs regarding whether a formula φ is true, they agree on what φ means. Formally, this is captured by the fact that the truth of φ does not depend on the agent.

Of course, in the real world, there is *ambiguity*; different agents may interpret the same utterance in different ways. For example, consider a public announcement p . Each player i may interpret p as corresponding to some event E_i , where E_i may be different from E_j if $i \neq j$. This seems natural: even if people have a common background, they may still disagree on how to interpret certain phenomena or new information. Someone may interpret a smile as just a sign of friendliness; someone else may interpret it as a "false" smile, concealing contempt; yet another person may interpret it as a sign of sexual interest.

To model this formally, we can use a straightforward approach already used in Halpern (2009) and Grove and Halpern (1993): formulas are interpreted relative to a player. But once we allow such ambiguity, further subtleties arise. Returning to the announcement p , not only can it be interpreted differently by different players, it may not even occur to the players that others may interpret the announcement in a different way. Thus, for example, i may believe that E_i is common knowledge. The assumption that each player

believes that her interpretation is how everyone interprets the announcement is but one assumption we can make about ambiguity. It is also possible that player i may be aware that there is more than one interpretation of p , but believes that player j is aware of only one interpretation. For example, think of a politician making an ambiguous statement which he realizes that different constituencies will interpret differently, but will not realize that there are other possible interpretations. In this paper, we investigate a number of different semantics of ambiguity that correspond to some standard assumptions that people make with regard to ambiguous statements, and investigate their relationship.

Our interest in ambiguity is motivated by a seminal result in game theory: Aumann's 1976 theorem showing that players cannot "agree to disagree". More precisely, this theorem says that agents with a common prior on a state space cannot have common knowledge that they have different posteriors.¹ This result has been viewed as paradoxical in the economics literature. Trade in a stock market seems to require common knowledge of disagreement (about the value of the stock being traded), yet we clearly observe a great deal of trading.

One well known explanation for the disagreement is that we do not in fact have common priors: agents start out with different beliefs. We provide a different explanation here, in terms of ambiguity. It is easy to show that we can agree to disagree when there is ambiguity, even if there is a common prior. We then show that these two explanations of the possibility of agreeing to disagree are closely related, but not identical. We can convert an explanation in terms of ambiguity to an explanation in terms of lack of common priors.² Importantly, however, the converse does not hold; there are models in which players have a common interpretation that cannot in general be converted into an equivalent model with ambiguity and a common prior. In other words, using heterogeneous priors may be too permissive if we are interested in modeling a situation where differences in beliefs are due to differences in interpretation.

Although our work is motivated by applications in eco-

¹We explain this result in more detail in Section 3.

²More precisely, we can convert a model with ambiguity and a common prior to an equivalent model—equivalent in the sense that the same formulas are true—where there is no ambiguity but no common prior.

nomics, ambiguity has long been a concern in linguistics and natural language processing. For example, there has been a great deal of work on word-sense disambiguation (i.e., trying to decide from context which of the multiple meanings of a word are intended); see Hirst 1988 for a seminal contribution, and Navigli 2009 for a recent survey. However, there does not seem to be much work on incorporating ambiguity into a logic. Apart from the literature on the logic of context and on underspecification (see Van Deemter and Peters 1996), the only paper that we are aware of that does this is one by Monz 1999. Monz allows for statements that have multiple interpretations, just as we do. But rather than incorporating the ambiguity directly into the logic, he considers updates by ambiguous statements. There are also connections between ambiguity and vagueness. Although the two notions are different—a term is *vague* if it is not clear what its meaning is, and is *ambiguous* if it can have multiple meanings, Halpern 2009 also used agent-dependent interpretations in his model of vagueness, although the issues that arose were quite different from those that concern us here.

The rest of this paper is organized as follows. Section 2 introduces the logic that we consider. Section 3 investigates the implications of the common-prior assumption when there is ambiguity. Section 4 studies the tradeoff between heterogeneous priors and ambiguity, and Section 5 concludes.

2 Syntax and Semantics

2.1 Syntax

We want a logic where players use a fixed common language, but each player may interpret formulas in the language differently. We also want to allow the players to be able to reason about (probabilistic) beliefs.

The syntax of the logic is straightforward (and is, indeed, essentially the syntax already used in papers going back to Fagin and Halpern 1994). There is a finite, nonempty set $N = \{1, \dots, n\}$ of players, and a countable, nonempty set Φ of primitive propositions. Let $\mathcal{L}_n^C(\Phi)$ be the set of formulas that can be constructed starting from Φ , and closing off under conjunction, negation, the modal operators $\{CB_G\}_{G \subseteq N, G \neq \emptyset}$, and the formation of probability formulas. (We omit the Φ if it is irrelevant or clear from context.) Probability formulas are constructed as follows. If $\varphi_1, \dots, \varphi_k$ are formulas, and $a_1, \dots, a_k, b \in \mathbb{Q}$, then for $i \in N$,

$$a_1 pr_i(\varphi_1) + \dots + a_k pr_i(\varphi_k) \geq b$$

is a probability formula, where $pr_i(\varphi)$ denotes the probability that player i assigns to a formula φ . Note that this syntax allows for nested probability formulas. We use the abbreviation $B_i \varphi$ for $pr_i(\varphi) = 1$, $EB_G^1 \varphi$ for $\wedge_{i \in G} B_i \varphi$, and $EB_G^{m+1} \varphi$ for $EB_G^m EB_G^1 \varphi$ for $m = 1, 2, \dots$. Finally, we take *true* to be the abbreviation for a fixed tautology such as $p \vee \neg p$.

2.2 Epistemic probability structures

There are standard approaches for interpreting this language (Fagin and Halpern 1994), but they all assume that there

is no ambiguity, that is, that all players interpret the primitive propositions the same way. To allow for different interpretations, we use an approach used earlier (Halpern 2009; Grove and Halpern 1993): formulas are interpreted relative to a player.

An (*epistemic probability*) structure (over Φ) has the form

$$M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi_j)_{j \in N}),$$

where Ω is the state space, and for each $i \in N$, Π_i is a partition of Ω , \mathcal{P}_i is a function that assigns to each $\omega \in \Omega$ a probability space $\mathcal{P}_i(\omega) = (\Omega_{i,\omega}, \mathcal{F}_{i,\omega}, \mu_{i,\omega})$, and π_i is an interpretation that associates with each state a truth assignment to the primitive propositions in Φ . That is, $\pi_i(\omega)(p) \in \{\text{true}, \text{false}\}$ for all ω and each primitive proposition p . Intuitively, π_i describes player i 's interpretation of the primitive propositions. Standard models use only a single interpretation π ; this is equivalent in our framework to assuming that $\pi_1 = \dots = \pi_n$. We call a structure where $\pi_1 = \dots = \pi_n$ a *common-interpretation structure*. Denote by $[[p]]_i$ the set of states where i assigns the value **true** to p . The partitions Π_i are called *information partitions*. While it is more standard in the philosophy and computer science literature to use models where there is a binary relation \mathcal{K}_i on Ω for each agent i that describes i 's accessibility relation on states, we follow the common approach in economics of working with information partitions here, as that makes it particularly easy to define a player's probabilistic beliefs. Assuming information partitions corresponds to the case that \mathcal{K}_i is an equivalence relation (and thus defines a partition). The intuition is that a cell in the partition Π_i is defined by some information that i received, such as signals or observations of the world. Intuitively, agent i receives the same information at each state in a cell of Π_i . Let $\Pi_i(\omega)$ denote the cell of the partition Π_i containing ω . Finally, the probability space $\mathcal{P}_i(\omega) = (\Omega_{i,\omega}, \mathcal{F}_{i,\omega}, \mu_{i,\omega})$ describes the beliefs of player i at state ω , with $\mu_{i,\omega}$ a probability measure defined on the subspace $\Omega_{i,\omega}$ of the state space Ω . The σ -algebra $\mathcal{F}_{i,\omega}$ consists of the subsets of $\Omega_{i,\omega}$ to which $\mu_{i,\omega}$ can assign a probability. (If $\Omega_{i,\omega}$ is finite, we typically take $\mathcal{F}_{i,\omega} = 2^{\Omega_{i,\omega}}$, the set of all subsets of $\Omega_{i,\omega}$.) The interpretation is that $\mu_{i,\omega}(E)$ is the probability that i assigns to event $E \in \mathcal{F}_{i,\omega}$ in state ω .

Throughout this paper, we make the following assumptions regarding the probability assignments \mathcal{P}_i , $i \in N$:

- A1.** For all $\omega \in \Omega$, $\Omega_{i,\omega} = \Pi_i(\omega)$.
 - A2.** For all $\omega \in \Omega$, if $\omega' \in \Pi_i(\omega)$, then $\mathcal{P}_i(\omega') = \mathcal{P}_i(\omega)$.
 - A3.** For all $j \in N, \omega, \omega' \in \Omega$, $\Pi_i(\omega) \cap \Pi_j(\omega') \in \mathcal{F}_{i,\omega}$.
- Furthermore, we make the following joint assumption on players' interpretations and information partitions:
- A4.** For all $\omega \in \Omega$, $i \in N$, and primitive proposition $p \in \Phi$, $\Pi_i(\omega) \cap [[p]]_i \in \mathcal{F}_{i,\omega}$.

These are all standard assumptions. A1 says that the set of states to which player i assigns probability at state ω is just the set $\Pi_i(\omega)$ of worlds that i considers possible at state ω . A2 says that the probability space used is the same at all the worlds in a cell of player i 's partition. Intuitively, this says that player i knows his probability space. Informally,

A3 says that player i can assign a probability to each of j 's cells, given his information. A4 says that primitive propositions (as interpreted by player i) are measurable according to player i .

2.3 Prior-generated beliefs and the common-prior assumption

One assumption that we do not necessarily make, but want to examine in this framework, is the common-prior assumption. The common-prior assumption is an instance of a more general assumption, that beliefs are generated from a prior, which we now define. The intuition is that players start with a prior probability; they then update the prior in light of their information. Player i 's information is captured by her partition Π_i . Thus, if i 's prior is ν_i , then we would expect $\mu_{i,\omega}$ to be $\nu_i(\cdot \mid \Pi_i(\omega))$.

Definition 2.1 An epistemic probability structure $M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi_j)_{j \in N})$ has *prior-generated beliefs* (*generated by* $(\mathcal{F}_1, \nu_1), \dots, (\mathcal{F}_n, \nu_n)$) if, for each player i , there exist probability spaces $(\Omega, \mathcal{F}_i, \nu_i)$ such that

- for all $i, j \in N$ and $\omega \in \Omega$, $\Pi_j(\omega) \in \mathcal{F}_i$;
- for all $i \in N$ and $\omega \in \Omega$, $\mathcal{P}_i(\omega) = (\Pi_i(\omega), \mathcal{F}_i \mid \Pi_i(\omega), \mu_{i,\omega})$, where $\mathcal{F}_i \mid \Pi_i(\omega)$ is the restriction of \mathcal{F}_i to $\Pi_i(\omega)$,³ and $\mu_{i,\omega}(E) = \nu_i(E \mid \Pi_i(\omega))$ for all $E \in \mathcal{F}_i \mid \Pi_i(\omega)$ if $\nu_i(\Pi_i(\omega)) > 0$. (There are no constraints on $\nu_{i,\omega}$ if $\nu_i(\Pi_i(\omega)) = 0$.)

■

It is easy to check that if M has prior-generated beliefs, then M satisfies A1, A2, and A3. More interestingly for our purposes, the converse also holds for a large class of structures. Say that a structure is *countably partitioned* if for each player i , the information partition Π_i has countably many elements, i.e., Π_i is a finite or countably infinite collection of subsets of Ω .

Proposition 2.2 If a structure M has prior-generated beliefs, then M satisfies A1, A2, and A3. Moreover, every countably partitioned structure that satisfies A1, A2, and A3 is one with prior-generated beliefs, with the priors ν_i satisfying $\nu_i(\Pi_i(\omega)) > 0$ for each player $i \in N$ and state $\omega \in \Omega$.

Proof. The first part is immediate. To prove the second claim, suppose that M is a structure satisfying A1–A3. Let \mathcal{F}_i be the unique algebra generated by $\cup_{\omega \in \Omega} \mathcal{F}_{i,\omega}$. To define ν_i , if there are $N_i < \infty$ cells in the partition Π_i , define $\nu_i(\omega) = \frac{1}{N_i} \mu_{i,\omega}(\omega)$. Otherwise, if the collection Π_i is countably infinite, order the elements of Π_i as p_i^1, p_i^2, \dots . Choose some state $\omega_k \in p_i^k$ for each k , with associated probability space $\mathcal{P}_i(\omega_k) = (\Omega_{i,\omega_k}, \mathcal{F}_{i,\omega_k}, \mu_{i,\omega_k})$. By A2, each choice of ω_k in p_i^k gives the same probability measure μ_{i,ω_k} . Define $\nu_i = \sum_k \frac{1}{2^k} \mu_{i,\omega_k}$. It is easy to see that ν_i is a probability measure on Ω , and that M is generated by $(\mathcal{F}_1, \nu_1), \dots, (\mathcal{F}_n, \nu_n)$. ■

³Recall that the restriction of \mathcal{F}_i to $\Pi_i(\omega)$ is the σ -algebra $\{B \cap \Pi_i(\omega) : B \in \mathcal{F}_i\}$.

Note that the requirement that that M is countably partitioned is necessary to ensure that we can have $\nu_i(\Pi_i(\omega)) > 0$ for each player i and state ω .

In light of Proposition 2.2, when it is convenient, we will talk of a structure satisfying A1–A3 as being *generated by* $(\mathcal{F}_1, \nu_1), \dots, (\mathcal{F}_n, \nu_n)$.

The common-prior assumption is essentially just the special case of prior-generated beliefs where all the priors are identical. We make one additional technical assumption. To state this assumption, we need one more definition. A state $\omega' \in \Omega$ is *G-reachable* from $\omega \in \Omega$, for $G \subseteq N$, if there exists a sequence $\omega_0, \dots, \omega_m$ in Ω with $\omega_0 = \omega$ and $\omega_m = \omega'$, and $i_1, \dots, i_m \in G$ such that $\omega_\ell \in \Pi_{i_\ell}(\omega_{\ell-1})$. Denote by $R_G(\omega) \subseteq \Omega$ the set of states G -reachable from ω .

Definition 2.3 An epistemic probability structure $M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi_j)_{j \in N})$ satisfies the *common-prior assumption (CPA)* if there exists a probability space $(\Omega, \mathcal{F}, \nu)$ such that M has prior-generated beliefs generated by $((\mathcal{F}, \nu), \dots, (\mathcal{F}, \nu))$, and $\nu(R_N(\omega)) > 0$ for all $\omega \in \Omega$. ■

As shown by Halpern 2002, the assumption that $\nu(R_N(\omega)) > 0$ for each $\omega \in \Omega$ is needed for Aumann's 1976 impossibility result.

2.4 Capturing ambiguity

We use epistemic probability structures to give meaning to formulas. Since primitive propositions are interpreted relative to players, we must allow the interpretation of arbitrary formulas to depend on the player as well. Exactly how we do this depends on what further assumptions we make about what players know about each other's interpretations. There are many assumptions that could be made. We focus on two of them here, ones that we believe arise in applications of interest, and then reconsider them under the assumption that there may be some ambiguity about the partitions.

Believing there is no ambiguity The first approach is appropriate for situations where players may interpret statements differently, but it does not occur to them that there is another way of interpreting the statement. Thus, in this model, if there is a public announcement, all players will think that their interpretation of the announcement is common knowledge. We write $(M, \omega, i) \models^{out} \varphi$ to denote that φ is true at state ω according to player i (that is, according to i 's interpretation of the primitive propositions in φ). The superscript *out* denotes *outermost scope*, since the formulas are interpreted relative to the “outermost” player, namely the player i on the left-hand side of \models^{out} . We define \models^{out} , as usual, by induction.

If p is a primitive proposition,

$$(M, \omega, i) \models^{out} p \text{ iff } \pi_i(\omega)(p) = \text{true}.$$

This just says that player i interprets a primitive proposition p according to his interpretation function π_i . This clause is common to all our approaches for dealing with ambiguity.

For conjunction and negation, as is standard,

$$(M, \omega, i) \models^{out} \neg\varphi \text{ iff } (M, \omega, i) \not\models^{out} \varphi,$$

$$(M, \omega, i) \models^{out} \varphi \wedge \psi \text{ iff } (M, \omega, i) \models^{out} \varphi \text{ and } (M, \omega, i) \models^{out} \psi.$$

Now consider a probability formula of the form $a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b$. The key feature that distinguishes this semantics is how i interprets j 's beliefs. This is where we capture the intuition that it does not occur to i that there is another way of interpreting the formulas other than the way she does. Let

$$[[\varphi]]_i^{out} = \{\omega : (M, \omega, i) \models^{out} \varphi\}.$$

Thus, $[[\varphi]]_i^{out}$ is the event consisting of the set of states where φ is true, according to i . Note that A1 and A3 guarantee that the restriction of $\Omega_{j,\omega}$ to $\Pi_i(\omega)$ belongs to $\mathcal{F}_{i,\omega}$. Assume inductively that $[[\varphi_1]]_i^{out} \cap \Omega_{j,\omega}, \dots, [[\varphi_k]]_i^{out} \cap \Omega_{j,\omega} \in \mathcal{F}_{j,\omega}$. The base case of this induction, where φ is a primitive proposition, is immediate from A3 and A4, and the induction assumption clearly extends to negations and conjunctions. We now define

$$(M, \omega, i) \models^{out} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b \text{ iff } a_1 \mu_{j,\omega}([[\varphi_1]]_i^{out} \cap \Omega_{j,\omega}) + \dots + a_k \mu_{j,\omega}([[\varphi_k]]_i^{out} \cap \Omega_{j,\omega}) \geq b.$$

Note that it easily follows from A2 that $(M, \omega, i) \models^{out} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b$ if and only if $(M, \omega', i) \models^{out} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b$ for all $\omega' \in \Pi_j(\omega)$. Thus, $[[a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b]]_i$ is a union of cells of Π_j , and hence $[[a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b]]_i \cap \Omega_{j,\omega} \in \mathcal{F}_{j,\omega}$.

With this semantics, according to player i , player j assigns φ probability b if and only if the set of worlds where φ holds according to i has probability b according to j . Intuitively, although i “understands” j 's probability space, player i is not aware that j may interpret φ differently from the way she (i) does. That i understands j 's probability space is plausible if we assume that there is a common prior and that i knows j 's partition (this knowledge is embodied in the assumption that i intersects $[[\varphi_k]]_i^{out}$ with $\Omega_{j,\omega}$ when assessing what probability j assigns to φ).⁴

Given our interpretation of probability formulas, the interpretation of $B_j \varphi$ and $EB^k \varphi$ follows. For example,

$$(M, \omega, i) \models^{out} B_j \varphi \text{ iff } \mu_{j,\omega}([[\varphi]]_i^{out}) = 1.$$

For readers more used to belief defined in terms of a possibility relation, note that if the probability measure $\mu_{j,\omega}$ is discrete (i.e., all sets are $\mu_{j,\omega}$ -measurable, and $\mu_{j,\omega}(E) = \sum_{\omega' \in E} \mu_{j,\omega}(\omega')$ for all subsets $E \subset \Pi_j(\omega)$), we can define $\mathcal{B}_j = \{(\omega, \omega') : \mu_{j,\omega}(\omega') > 0\}$; that is, $(\omega, \omega') \in \mathcal{B}_j$ if, in state ω , agent j gives state ω' positive probability. In that case, $(M, \omega, i) \models^{out} B_j \varphi$ iff $(M, \omega', i) \models^{out} \varphi$ for all ω' such that $(\omega, \omega') \in \mathcal{B}_j$. That is, $(M, \omega, i) \models^{out} B_j \varphi$ iff φ is true according to i in all the worlds to which j assigns positive probability at ω .

⁴Note that at state ω , player i will not in general know that it is state ω . In particular, even if we assume that i knows which element of j 's partition contains ω , i will not in general know which of j 's cells describes j 's current information. But we assume that i does know that if the state is ω , then j information is described by $\Omega_{j,\omega}$. Thus, as usual, “ $(M, i, \omega) \models^{out} \varphi$ ” should perhaps be understood as “according to i , φ is true if the actual world is ω ”. This interpretational issue arises even without ambiguity in the picture.

It is important to note that $(M, \omega, i) \models \varphi$ does not imply $(M, \omega, i) \models B_i \varphi$: while $(M, \omega, i) \models^{out} \varphi$ means “ φ is true at ω according to i 's interpretation,” this does not mean that i believes φ at state ω . The reason is that i can be uncertain as to which state is the actual state. For i to believe φ at ω , φ would have to be true (according to i 's interpretation) at all states to which i assigns positive probability.

Finally, we define

$$(M, \omega, i) \models^{out} CB_G \varphi \text{ iff } (M, \omega, i) \models^{out} EB_G^k \varphi \text{ for } k = 1, 2, \dots$$

for any nonempty subset $G \subseteq N$ of players.

Awareness of possible ambiguity We now consider the second way of interpreting formulas. This is appropriate for players who realize that other players may interpret formulas differently. We write $(M, \omega, i) \models^{in} \varphi$ to denote that φ is true at state ω according to player i using this interpretation, which is called *innermost scope*. The definition of \models^{in} is identical to that of \models^{out} except for the interpretation of probability formulas. In this case, we have

$$(M, \omega, i) \models^{in} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b \text{ iff } a_1 \mu_{j,\omega}([[\varphi_1]]_j^{in} \cap \Omega_{j,\omega}) + \dots + a_k \mu_{j,\omega}([[\varphi_k]]_j^{in} \cap \Omega_{j,\omega}) \geq b,$$

where $[[\varphi]]_j^{in}$ is the set of states ω such that $(M, \omega, j) \models^{in} \varphi$. Hence, according to player i , player j assigns φ probability b if and only if the set of worlds where φ holds according to j has probability b according to j . Intuitively, now i realizes that j may interpret φ differently from the way that she (i) does, and thus assumes that j uses his (j 's) interpretation to evaluate the probability of φ . Again, in the case that $\mu_{j,\omega}$ is discrete, this means that $(M, \omega, i) \models^{in} B_j \varphi$ iff $(M, \omega', j) \models^{in} \varphi$ for all ω' such that $(\omega, \omega') \in \mathcal{B}_j$.

Note for future reference that if φ is a probability formula or a formula of the form $CB_G \varphi$, then it is easy to see that $(M, \omega, i) \models^{in} \varphi$ if and only if $(M, \omega, j) \models^{in} \varphi$; we sometimes write $(M, \omega) \models^{in} \varphi$ in this case. Clearly, \models^{out} and \models^{in} agree in the common-interpretation case, and we can write \models .

Ambiguity about information partitions Up to now, we have assumed that players “understand” each other's probability spaces. This may not be so reasonable in the presence of ambiguity and prior-generated beliefs. We want to model the following type of situation. Players receive information, or signals, about the true state of the world, in the form of strings (formulas). Each player understands what signals he and other players receive in different states of the world, but players may interpret signals differently. For instance, player i may understand that j sees a red car if ω is the true state of the world, but i may or may not be aware that j has a different interpretation of “red” than i does. In the latter case, i does not have a full understanding of j 's information structure.

We would like to think of a player's information as being characterized by a formula (intuitively, the formula that describes the signals received). Even if the formulas that

describe each information set are commonly known, in the presence of ambiguity, they might be interpreted differently.

To make this precise, let Φ^* be the set of formulas that is obtained from Φ by closing off under negation and conjunction. That is, Φ^* consists of all propositional formulas that can be formed from the primitive propositions in Φ . Since the formulas in Φ^* are not composed of probability formulas, and thus do not involve any reasoning about interpretations, we can extend the function $\pi_i(\cdot)$ to Φ^* in a straightforward way, and write $[[\varphi]]_i$ for the set of the states of the world where the formula $\varphi \in \Phi^*$ is true according to i .

The key new assumption we make to model players' imperfect understanding of the other players' probability spaces is that i 's partition cell at ω is described by a formula $\varphi_{i,\omega} \in \Phi^*$. But, of course, this formula may be interpreted differently by each player. We want $\Pi_i(\omega)$ to coincide with i 's interpretation of the formula $\varphi_{i,\omega}$. If player j understands that i may be using a different interpretation than he does (i.e., the appropriate semantics are the innermost-scope semantics), then j correctly infers that the set of states that i thinks are possible in ω is $\Pi_i(\omega) = [[\varphi_{i,\omega}]]_i$. But if j does not understand that i may interpret formulas in a different way (i.e., under outermost scope), then he thinks that the set of states that i thinks are possible in ω is given by $[[\varphi_{i,\omega}]]_j$, and, of course, $[[\varphi_{i,\omega}]]_j$ may not coincide with $\Pi_i(\omega)$. In any case, we require that j understand that these formulas form a partition and that ω belongs to $[[\varphi_{i,\omega}]]_j$. Thus, we consider structures that satisfy A5 and A6 (for outermost scope) or A5 and A6' (for innermost scope), in addition to A1–A4.

A5. For each $i \in N$ and $\omega \in \Omega$, there is a formula $\varphi_{i,\omega} \in \Phi^*$ such that $\Pi_i(\omega) = [[\varphi_{i,\omega}]]_i$.

A6. For each $i, j \in N$, the collection $\{[[\varphi_{i,\omega}]]_j : \omega \in \Omega\}$ is a partition of Ω and for all $\omega \in \Omega$, $\omega \in [[\varphi_{i,\omega}]]_j$.

A6'. For each $i \in N$, the collection $\{[[\varphi_{i,\omega}]]_i : \omega \in \Omega\}$ is a partition of Ω and for all $\omega \in \Omega$, $\omega \in [[\varphi_{i,\omega}]]_i$.

Assumption A6 is appropriate for outermost scope: it presumes that player j uses his own interpretation of $\varphi_{i,\omega}$ in deducing the beliefs for i in ω . Assumption A6' is appropriate for innermost scope. Note that A6' is a weakening of A6. While A6 requires the signals for player i to induce an information partition according to every player j , the weaker version A6' requires this to hold only for player i himself.

We can now define analogues of outermost scope and innermost scope in the presence of ambiguous information. Thus, we define two more truth relations, $\models^{out,ai}$ and $\models^{in,ai}$. (The “ai” here stands for “ambiguity of information”.) The only difference between $\models^{out,ai}$ and \models^{out} is in the semantics of probability formulas. In giving the semantics in a structure M , we assume that M has prior-generated beliefs, generated by $(\mathcal{F}_1, \nu_1), \dots, (\mathcal{F}_n, \nu_n)$. As we observed in Proposition 2.2, this assumption is without loss of generality as long as the structure is countably partitioned. However, the choice of prior beliefs is relevant, as we shall see, so we have to be explicit about them. When i evaluates j 's probability at a state ω , instead of using $\nu_{j,\omega}$, player i uses $\nu_j(\cdot | [[\varphi_{j,\omega}]]_i)$. When $i = j$, these two approaches agree, but in general they do not. Thus, assuming that M satisfies A5 and A6 (which

are the appropriate assumptions for the outermost-scope semantics), we have

$$(M, \omega, i) \models^{out,ai} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b \text{ iff} \\ a_1 \nu_j([[\varphi_1]]_i^{out,ai} \mid [[\varphi_{j,\omega}]]_i^{out,ai}) + \dots \\ + a_k \nu_j([[\varphi_k]]_i^{out,ai} \mid [[\varphi_{j,\omega}]]_i^{out,ai}) \geq b,$$

where $[[\psi]]_i^{out,ai} = \{\omega' : (M, \omega', i) \models^{out,ai} \psi\}$.

That is, at $\omega \in \Omega$, player j receives the information (a string) $\varphi_{j,\omega}$, which he interprets as $[[\varphi_{j,\omega}]]_j$. Player i understands that j receives the information $\varphi_{j,\omega}$ in state ω , but interprets this as $[[\varphi_{j,\omega}]]_i$. This models a situation such as the following. In state ω , player j sees a red car, and thinks possible all states of the world where he sees a car that is red (according to j). Player i knows that at world ω player j will see a red car (although she may not know that the actual world is ω , and thus does not know what color of car player j actually sees). However, i has a somewhat different interpretation of “red car” (or, more precisely, of j seeing a red car) than j ; i 's interpretation corresponds to the event $[[\varphi_{j,\omega}]]_i$. Since i understands that j 's beliefs are determined by conditioning her prior ν_j on her information, i can compute what she believes j 's beliefs are.

We can define $\models^{in,ai}$ in an analogous way. Thus, the semantics for formulas that do not involve probability formulas are as given by \models^{in} , while the semantics of probability formulas is defined as follows (where M is assumed to satisfy A5 and A6', which are the appropriate assumptions for the innermost-scope semantics):

$$(M, \omega, i) \models^{in,ai} a_1 pr_j(\varphi_1) + \dots + a_k pr_j(\varphi_k) \geq b \text{ iff} \\ a_1 \nu_j([[\varphi_1]]_j^{in,ai} \mid [[\varphi_{j,\omega}]]_j^{in,ai}) + \dots \\ + a_k \nu_j([[\varphi_k]]_j^{in,ai} \mid [[\varphi_{j,\omega}]]_j^{in,ai}) \geq b.$$

Note that although we have written $[[\varphi_{j,\omega}]]_i^{in,ai}$, since $\varphi_{j,\omega}$ is a propositional formula, $[[\varphi_{j,\omega}]]_i^{in,ai} = [[\varphi_{j,\omega}]]_i^{out,ai} = [[\varphi_{j,\omega}]]_i^{out} = [[\varphi_{j,\omega}]]_i^{in}$. It is important that $\varphi_{j,\omega}$ is a propositional formula here; otherwise, we would have circularities in the definition, and would somehow need to define $[[\varphi_{j,\omega}]]_i^{in,ai}$.

Again, here it may be instructive to consider the definition of $B_j \varphi$ in the case that $\mu_{j,\omega}$ is discrete for all ω . In this case, B_j becomes the set $\{(\omega, \omega') : \nu_j(\omega' | [[\varphi_{j,\omega}]]_j^{in,ai}) > 0\}$. That is, state ω' is considered possible by player j in state ω if agent j gives ω' positive probability after conditioning his prior ν_j on (his interpretation of) the information $\varphi_{j,\omega}$ he receives in state ω . With this definition of B_j , we have, as expected, $(M, \omega, i) \models^{in,ai} B_j \varphi$ iff $(M, \omega', i) \models^{in,ai} \varphi$ for all ω' such that $(\omega, \omega') \in B_j$.

The differences in the different semantics arise only when we consider probability formulas. If we go back to our example with the red car, we now have a situation where player j sees a red car in state ω , and thinks possible all states where he sees a red car. Player i knows that in state ω , player j sees a car that he (j) interprets to be red, and that this determines his posterior. Since i understands j 's notion of seeing a red car, she has a correct perception of j 's posterior in each state of the world. Thus, the semantics for $\models^{in,ai}$ are identical to

those for \models^{in} (restricted to the class of structures with prior-generated beliefs that satisfy A5 and A6'), though the information partitions are not predefined, but rather generated by the signals.

Note that, given an epistemic structure M satisfying A1–A4, there are many choices for ν_i that allow M to be viewed as being generated by prior beliefs. All that is required of ν_j is that for all $\omega \in \Omega$ and $E \in \mathcal{F}_{j,\omega}$ such that $E \subseteq [[\varphi_{j,\omega}]]_j^{out,ai}$, it holds that $\nu_j(E \cap [[\varphi_{j,\omega}]]_j^{out,ai})/\nu_j([[\varphi_{j,\omega}]]_j^{out,ai}) = \mu_{j,\omega}(E)$. However, because $[[\varphi_{j,\omega}]]_i^{out,ai}$ may not be a subset of $[[\varphi_{j,\omega}]]_j^{out,ai} = \Pi_j(\omega)$, we can have two prior probabilities ν_j and ν'_j that generate the same posterior beliefs for j , and still have $\nu_j([[\varphi_k]]_i^{out,ai} \mid [[\varphi_{j,\omega}]]_i^{out,ai}) \neq \nu'_j([[\varphi_k]]_i^{out,ai} \mid [[\varphi_{j,\omega}]]_i^{out,ai})$ for some formulas φ_k . Thus, we must be explicit about our choice of priors here.

3 The common-prior assumption revisited

This section applies the framework developed in the previous sections to understand the implications of assuming a common prior when there is ambiguity. The application in Section 3.1 makes use of the outermost- and innermost-scope semantics, while Section 3.2 considers a setting with ambiguity about information partitions.

3.1 Agreeing to Disagree

The first application we consider concerns the result of Aumann 1976 that players cannot “agree to disagree” if they have a common prior. As we show now, this is no longer true if players can have different interpretations. But exactly what “agreeing to disagree” means depends on which semantics we use.

Example 3.1 [Agreeing to Disagree] Consider a structure M with a single state ω , such that $\pi_1(\omega)(p) = \text{true}$ and $\pi_2(\omega)(p) = \text{false}$. Clearly M satisfies the CPA. The fact that there is only a single state in M means that, although the players interpret p differently, there is perfect understanding of how p is interpreted by each player. Specifically, taking $G = \{1, 2\}$, we have that $(M, \omega) \models^{in} CB_G(B_1 p \wedge B_2 \neg p)$. Thus, with innermost scope, according to each player, there is common belief that they have different beliefs at state ω ; that is, they agree to disagree.

With outermost scope, we do not have an agreement to disagree in the standard sense, but the players do disagree on what they have common belief about. Specifically, $(M, \omega, 1) \models^{out} CB_G p$ and $(M, \omega, 2) \models^{out} CB_G \neg p$. That is, according to player 1, there is common belief of p ; and according to player 2, there is common belief of $\neg p$. To us, it seems that we have modeled a rather common situation here! ■

Note that in the model of Example 3.1, there is maximal ambiguity: the players disagree with probability 1. We also have complete disagreement. As the following result shows, the less disagreement there is in the interpretation of events, the closer the players come to not being able to agree to disagree. Suppose that M satisfies the CPA, where ν is the

common prior, and that $\varphi \in \Phi^*$ (so that φ is a propositional formula). Say that φ is *only ϵ -ambiguous in M* if the set of states where the players disagree on the interpretation of φ has ν -measure at most ϵ ; that is,

$$\nu(\{\omega : \exists i, j((M, \omega, i) \models \varphi \text{ and } (M, \omega, j) \models \neg\varphi)\}) \leq \epsilon.$$

We write \models here because, as we observed before, all the semantic approaches agree on propositional formulas, so this definition makes sense independent of the semantic approach used. Note that if players have a common interpretation, then all formulas are 0-ambiguous.

Proposition 3.2 *If M satisfies the CPA and φ is only ϵ -ambiguous in M , then there cannot exist players i and j , numbers b and b' with $b' > b + \epsilon$, and a state ω such that all states are G -reachable from ω and*

$$(M, \omega) \models^{in} CB_G((pr_i(\varphi) < b) \wedge (pr_j(\varphi) > b')).$$

Proof. Essentially the same arguments as those used by Aumann 1976 can be used to show that if all states are reachable from ω and $(M, \omega) \models^{in} CB_G(pr_i(\varphi) < b)$, then it must be the case that $\nu([[\varphi]]_i) < b$, where ν is the common prior. Similarly, $\nu([[\varphi]]_j) > b'$. This contradicts the assumption that φ is only ϵ -ambiguous in M . ■

3.2 Understanding differences in beliefs

Since our framework separates meaning from message, it is worth asking what happens if players receive the same message, but interpret it differently. Aumann 1987 has argued that “people with different information may legitimately entertain different probabilities, but there is no rational basis for people who have always been fed precisely the same information to do so.” Here we show that this is no longer true when information is ambiguous, even if players have a common prior and fully understand the ambiguity that they face, except under strong assumptions on players’ beliefs about the information that others receive. This could happen if players with exactly the same background and information can interpret things differently, and thus have different beliefs.

We assume that information partitions are generated by signals, which may be ambiguous. That is, in each state of the world ω , each player i receives some signal $\sigma_{i,\omega}$ that determines the states of the world he thinks possible; that is, $\Pi_i(\omega) = [[rec_i(\sigma_{i,\omega})]]_i$, where $rec_i(\sigma_{i,\omega}) \in \Phi^*$ is “ i received $\sigma_{i,\omega}$.” As usual, we restrict attention to structures with prior-generated beliefs that satisfy A5 and A6’ when considering innermost-scope semantics and A5 and A6 when considering outermost-scope semantics.

In any given state, the signals that determine the states that players think are possible may be the same or may differ across players. Following Aumann 1987, we are particularly interested in the former case. Formally, we say that σ_ω is a *common signal* in ω if $\sigma_{i,\omega} = \sigma_\omega$ for all $i \in N$. For example, if players have a common interpretation, and all players observe a red car in state ω , then σ_ω is “red car”, while $rec_i(\sigma_\omega)$ is “ i observes a car that is red.” The fact that “red car” is a common signal in ω means that all players in

fact observe a red car in state ω . But assuming that players have received a common signal does not imply that they have the same posteriors, as the next example shows:

Example 3.3 There are two players, 1 and 2, and three states, labeled $\omega_1, \omega_2, \omega_3$. The common prior gives each state equal probability, and players have the same interpretation. In ω_1 , both players receive signal σ ; in ω_2 , only 1 does; in ω_3 , only 2 receives σ . The primitive proposition p is true in ω_1 and ω_2 , and the primitive proposition q is true in ω_1 and ω_3 . In state ω_1 , both players receive signal σ , but player 1 assigns probability 1 to p and probability $\frac{1}{2}$ to q , while 2 gives probability $\frac{1}{2}$ to p and probability 1 to q . ■

Thus, players who receive a common signal can end up having a different posterior over formulas, even if they have a common prior and the same interpretation. The problem is that even though players have received the same signal, they do not know that the other has received it, and they do not know that the other knows they have received it, and so on. That is, the fact that players have received a common signal in ω does not imply that the signal is common knowledge in ω . We say that signal σ is a *public signal at state ω* if $(M, \omega) \models CB_N(\wedge_{i \in N} rec_i(\sigma))$: it is commonly believed at ω that all players received σ .

For the remainder of this section, we will be considering structures with a common prior ν . To avoid dealing with topological issues, we assume that ν is a discrete measure. Of course, if the common prior ν is discrete, then so are all the measures $\mu_{i,\omega}$. Let $Supp(\mu)$ denote the support of the probability measure μ . If μ is discrete, then $Supp(\mu) = \{\omega : \mu(\omega) \neq 0\}$.

Even though common signals are not sufficient for players to have the same beliefs, as Example 3.3 demonstrates, Aumann's claim does hold for common-interpretation structures if players receive a public signal (provided that they started with a common prior):

Proposition 3.4 *If M is a common-interpretation structure with a common prior, and σ is a public signal at ω , then players' posteriors are identical at ω : for all $i, j \in N$ and $E \in \mathcal{F}$,*

$$\mu_{i,\omega}(E \cap \Pi_i(\omega)) = \mu_{j,\omega}(E \cap \Pi_j(\omega)).$$

In particular, for any formula φ ,

$$\mu_{i,\omega}([[\varphi]] \cap \Pi_i(\omega)) = \mu_{j,\omega}([[\varphi]] \cap \Pi_j(\omega)).$$

Proof. Let ν be the common prior in M . By assumption, $\Pi_i(\omega) = [[rec_i(\sigma)]]$ for all players $i \in N$. Since σ is public, we have that $(M, \omega) \models B_i(rec_j(\sigma))$. Thus, $(M, \omega') \models rec_j(\sigma)$ for all $\omega' \in Supp(\mu_{i,\omega})$. It follows that $Supp(\mu_{i,\omega}) \subseteq \Pi_j(\omega)$ for all players i and j . Since $\mu_{i,\omega}(Supp(\mu_{i,\omega})) = 1$, we have that $\nu(\Pi_j(\omega) | \Pi_i(\omega)) = \nu(\Pi_i(\omega) | \Pi_j(\omega)) = 1$. Thus, for all $E \in \mathcal{F}$, we must have $\nu(E | \Pi_i(\omega)) = \nu(E | \Pi_j(\omega))$. The result now follows immediately. ■

There is another way of formalizing the assumption that (it is commonly believed that) players are “fed the same information”; namely, we say that if one player i receives a signal σ then so do all others. Formally,

a signal σ is a *shared signal* at state ω if $(M, \omega) \models \wedge_{i,j \in N} CB_N(rec_i(\sigma) \Leftrightarrow rec_j(\sigma))$. If there is no ambiguity, a signal is shared iff it is public; we leave the straightforward proof (which uses ideas from the proof of Proposition 3.4) to the reader.

Proposition 3.5 *If M is a common-interpretation structure, and σ_ω is received at state ω by all players, then σ_ω is a public signal at ω iff σ is a shared signal at ω .*

The assumption that signals are public or shared is quite strong: one requires common belief that a particular signal is received (and so precludes any uncertainties about what one player believes that other players believe that others have received), while the other requires common belief that different players always receive the same signal (and, similarly, precludes uncertainties about what is received).

What happens if we introduce ambiguity? If the signal itself is a propositional formula (which is the case in many cases of interest), then players may interpret the signal differently; that is, we may have $[[\sigma_\omega]]_i \neq [[\sigma_\omega]]_j$ for $i \neq j$. Moreover, players may have a different interpretation of observing a given signal, i.e., it is possible that $[[rec_i(\sigma_\omega)]]_i \neq [[rec_i(\sigma_\omega)]]_j$. Going back to our example of the red car, different players may interpret “red car” differently, and they may interpret the notion of observing a red car differently. In addition, it is now possible that players have the same posteriors over events, but not over formulas, or vice versa, given that they may interpret formulas differently.

If we assume that players are not aware that there is ambiguity, then we retain the equivalence between shared signals and common signals, and players' posteriors over formulas coincide after receiving a public signal. However, they may have different beliefs over events:

Proposition 3.6 *If M is a structure satisfying A5 and A6, and σ is received at state ω by all players, then σ is a public signal at ω iff σ is a shared signal at ω under outermost-scope semantics. Moreover, if M has a common prior, and σ is a public signal at ω , then players' posteriors on formulas are identical at ω ; that is, for all formulas ψ , we have*

$$\mu_{i,\omega}([[[\psi]]_i^{out,ai} \cap \Pi_i(\omega))] = \mu_{j,\omega}([[[\psi]]_j^{out,ai} \cap \Pi_j(\omega))).$$

However, players' posteriors on events may differ; that is, there may exist some E such that

$$\mu_{i,\omega}(E \cap \Pi_i(\omega)) \neq \mu_{j,\omega}(E \cap \Pi_j(\omega)).$$

We leave the straightforward arguments to the reader.

The situation for innermost scope presents an interesting contrast. A first observation is that public signals and shared signals are no longer equivalent:

Example 3.7 Consider a structure M with two players, where $\Omega = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$. Suppose that $[[rec_1(\sigma)]]_1 = [[rec_2(\sigma)]]_1 = \{\omega_{11}, \omega_{12}\}$, and $[[rec_1(\sigma)]]_2 = [[rec_2(\sigma)]]_2 = \{\omega_{11}, \omega_{21}\}$. Assume that the beliefs in M are generated by a common prior that gives each state probability $1/4$. Clearly $(M, \omega_{11}) \models^{in,ai} CB_N(rec_1(\sigma) \Leftrightarrow rec_2(\sigma)) \wedge \neg CB_N(rec_1(\sigma))$. ■

The problem in Example 3.7 is that, although the signal is shared, the players don't interpret receiving the signal the same way. It is not necessarily the case that player 1 received σ from player 1's point of view iff player 2 received σ from player 2's point of view. The assumption that players receive shared signals is not strong enough to ensure that they have identical posteriors, either over formulas or over events. In Example 3.7, for example, players clearly have different posteriors on the event $\{\omega_{11}, \omega_{12}\}$ in state ω_{11} ; similarly, it is not hard to show that players can have different posteriors over formulas. Say that σ is *strongly shared at state ω* if

- $(M, \omega) \models^{in, ai} \wedge_{i,j} CB_N(rec_i(\sigma) \Leftrightarrow rec_j(\sigma));$ and
- $(M, \omega) \models^{in, ai} \wedge_{i,j} CB_N(B_i(rec_i(\sigma)) \Leftrightarrow B_j(rec_j(\sigma))).$

The second clause says that it is commonly believed at ω that each player believes that he has received σ iff each of the other players believes that he has received σ . This clause is implied by the first in common-interpretation structures and with outermost scope, but not with innermost scope.

Proposition 3.8 *If M is a structure satisfying A5 and A6', and σ is received at ω by all players, then σ is a public signal at ω iff σ is a strongly shared signal at ω under the innermost-scope semantics. If M is a structure with a common prior and σ is a public signal at ω , then players' posteriors over events are identical at ω : for all $i, j \in N$ and all $E \in \mathcal{F}$,*

$$\mu_{i,\omega}(E \cap \Pi_i(\omega)) = \mu_{j,\omega}(E \cap \Pi_j(\omega)).$$

However, players' posteriors on formulas may differ; that is, for some formula ψ , we could have that

$$\mu_{i,\omega}([\psi]_i^{in, ai} \cap \Pi_i(\omega)) \neq \mu_{j,\omega}([\psi]_j^{in, ai} \cap \Pi_j(\omega)).$$

These results emphasize the effect of ambiguity on shared and public signals.

4 Common priors or common interpretations?

As Example 3.1 shows, we can have agreement to disagree with ambiguity under the \models^{in} semantics (and thus, also the $\models^{in, ai}$ semantics). We also know that we can do this by having heterogeneous priors. As we now show, structures with ambiguity that satisfy the CPA have the same expressive power as common-interpretation structures that do not necessarily satisfy the CPA (and common-interpretation structures, by definition, have no ambiguity). On the other hand, common-interpretation structures with heterogeneous priors are more general than structures with ambiguity and common priors.

To make this precise, we consider what formulas are valid in structures with or without ambiguity or a common prior. To define what it means for a formula to be valid, we need some more notation. Fix a nonempty, countable set Ψ of primitive propositions, and let $\mathcal{M}(\Psi)$ be the class of all structures that satisfy A1–A4 and that are defined over some

nonempty subset Φ of Ψ such that $\Psi \setminus \Phi$ is countably infinite.⁵ Given a subset Φ of Ψ , a formula $\varphi \in \mathcal{L}_n^C(\Phi)$, and a structure $M \in \mathcal{M}(\Psi)$ over Φ , we say that φ is *valid in M according to outermost scope*, and write $M \models^{out} \varphi$, if $(M, \omega, i) \models^{out} \varphi$ for all $\omega \in \Omega$ and $i \in N$. Given $\varphi \in \Psi$, say that φ is *valid according to outermost scope* in a class $\mathcal{N} \subseteq \mathcal{M}(\Psi)$ of structures, and write $\mathcal{N} \models^{out} \varphi$, if $M \models^{out} \varphi$ for all $M \in \mathcal{N}$ defined over a set $\Phi \subset \Psi$ of primitive propositions that includes all the primitive propositions that appear in φ .

We get analogous definitions by replacing \models^{out} by \models^{in} , $\models^{out, ai}$ and $\models^{in, ai}$ throughout (in the latter two cases, we have to restrict \mathcal{N} to structures that satisfy A5 and A6 or A6', respectively, in addition to A1–A4). Finally, given a class of structures \mathcal{N} , let \mathcal{N}_c be the subclass of \mathcal{N} in which players have a common interpretation. Thus, $\mathcal{M}_c(\Psi)$ denotes the structures in $\mathcal{M}(\Psi)$ with a common interpretation. Let $\mathcal{M}^{ai}(\Psi)$ denote all structures in $\mathcal{M}(\Psi)$ with prior-generated beliefs that satisfy A5 and A6 (where we assume that the prior ν that describes the initial beliefs is given explicitly).⁶ Finally, let $\mathcal{M}^{cpa}(\Psi)$ (resp., $\mathcal{M}^{cpa, ai}(\Psi)$) consist of the structures in $\mathcal{M}(\Psi)$ (resp., $\mathcal{M}^{ai}(\Psi)$) satisfying the CPA.

Proposition 4.1 *For all formulas $\varphi \in \mathcal{L}_n^C(\Psi)$, the following are equivalent:*

- (a) $\mathcal{M}_c(\Psi) \models \varphi;$
- (b) $\mathcal{M}(\Psi) \models^{out} \varphi;$
- (c) $\mathcal{M}(\Psi) \models^{in} \varphi;$
- (d) $\mathcal{M}_c^{ai}(\Psi) \models \varphi;$
- (e) $\mathcal{M}^{ai}(\Psi) \models^{out, ai} \varphi;$
- (f) $\mathcal{M}^{ai}(\Psi) \models^{in, ai} \varphi.$

Proof. Since the set of structures with a common interpretation is a subset of the set of structures, it is immediate that (c) and (b) both imply (a). Similarly, (e) and (f) both imply (d). The fact that (a) implies (b) is also immediate. For suppose that $\mathcal{M}_c(\Psi) \models \varphi$ and that $M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi_j)_{j \in N}) \in \mathcal{M}(\Psi)$ is a structure over a set $\Phi \subset \Psi$ of primitive propositions that contains the primitive propositions that appear in φ . We must show that $M \models^{out} \varphi$. Thus, we must show that $(M, \omega, i) \models^{out} \varphi$ for all $\omega \in \Omega$ and $i \in N$. Fix $\omega \in \Omega$ and $i \in N$, and let $M'_i = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi'_j)_{j \in N})$, where $\pi'_j = \pi_i$ for all j . Thus, M'_i is a common-interpretation structure over Φ , where the interpretation coincides with i 's interpretation in M . Clearly M'_i satisfies A1–A4, so $M'_i \in \mathcal{M}_c(\Psi)$. It is easy to check that $(M, \omega, i) \models^{out} \varphi$ if and only if $(M'_i, \omega, i) \models \varphi$.

⁵Most of our results hold if we just consider the set of structures defined over some fixed set Φ of primitive propositions. However, for one of our results, we need to be able to add fresh primitive propositions to the language. Thus, we allow the set Φ of primitive propositions to vary over the structures we consider, but require $\Psi \setminus \Phi$ to be countably infinite so that there are always “fresh” primitive propositions that we can add to the language.

⁶For ease of exposition, we assume A6 even when dealing with innermost scope. Recall that A6 implies A6', which is actually the appropriate assumption for innermost scope.

for all states $\omega \in \Omega$ and all formulas $\psi \in \mathcal{L}_n^C(\Phi)$. Since $M'_i \models \varphi$, we must have that $(M, \omega, i) \models^{out} \varphi$, as desired. ■

To see that (a) implies (c), given a structure $M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, (\pi_j)_{j \in N}) \in \mathcal{M}(\Psi)$ over some set $\Phi \subset \Psi$ of primitive propositions and a player $j \in N$, let Ω_j be a disjoint copy of Ω ; that is, for every state $\omega \in \Omega$, there is a corresponding state $\omega_j \in \Omega_j$. Let $\Omega' = \Omega_1 \cup \dots \cup \Omega_n$. Given $E \subseteq \Omega$, let the corresponding subset $E_j \subseteq \Omega_j$ be the set $\{\omega_j : \omega \in E\}$, and let E' be the subset of Ω' corresponding to E , that is, $E' = \{\omega_j : \omega \in E, j \in N\}$.

Define $M' = (\Omega', (\Pi'_j)_{j \in N}, (\mathcal{P}'_j)_{j \in N}, (\pi'_j)_{j \in N})$, where $\Omega' = \Omega_1 \cup \dots \cup \Omega_n$ and, for all $\omega \in \Omega$ and $i, j \in N$, we have

- $\Pi'_i(\omega_j) = (\Pi_i(\omega))'$;
- $\pi_i(\omega_j)(p) = \pi_j(\omega)(p)$ for a primitive proposition $p \in \Phi$;
- $\mathcal{P}'_i(\omega_j) = (\Omega'_{i, \omega_j}, \mathcal{F}'_{i, \omega_j}, \mu'_{i, \omega_j})$, where $\Omega'_{i, \omega_j} = \Omega'_{i, \omega}$, $\mathcal{F}'_{i, \omega_j} = \{E_\ell : E \in \mathcal{F}_{i, \omega}, \ell \in N\}$, $\mu'_{i, \omega_j}(E_i) = \mu_{i, \omega}(E)$, $\mu'_{i, \omega_j}(E_\ell) = 0$ if $\ell \neq i$.

Thus, $\pi_1 = \dots = \pi_n$, so that M' is a common-interpretation structure; on a state ω_j , these interpretations are all determined by π_j . Also note that the support of the probability measure μ'_{i, ω_j} is contained in Ω_i , so for different players i , the probability measures μ'_{i, ω_j} have disjoint supports. Now an easy induction on the structure of formulas shows that $(M', \omega_j) \models \psi$ if and only if $(M, \omega, j) \models^{in} \psi$ for any formula $\psi \in \mathcal{L}_n^C(\Phi)$. It easily follows that if $M' \models \varphi$, then $M \models^{in} \varphi$ for all $\varphi \in \mathcal{L}_n^C(\Phi)$.

The argument that (d) implies (e) is essentially identical to the argument that (a) implies (b); similarly, the argument that (d) implies (f) is essentially the same as the argument that (a) implies (c). Since $\mathcal{M}_c^{ai}(\Psi) \subseteq \mathcal{M}_c(\Psi)$, (a) implies (d). To show that (d) implies (a), suppose that $\mathcal{M}_c^{ai}(\Psi) \models \varphi$ for some formula $\varphi \in \mathcal{L}_n^C(\Psi)$. Given a structure $M = (\Omega, (\Pi_j)_{j \in N}, (\mathcal{P}_j)_{j \in N}, \pi) \in \mathcal{M}_c(\Psi)$ over a set $\Phi \subset \Psi$ of primitive propositions that includes the primitive propositions that appear in φ , we want to show that $(M, \omega, i) \models \varphi$ for each state $\omega \in \Omega$ and player i . Fix ω . Recall that $R_N(\omega)$ consists of the set of states N -reachable from ω . Let $M' = (R_N(\omega), (\Pi'_j)_{j \in N}, (\mathcal{P}'_j)_{j \in N}, \pi')$, with Π'_j and \mathcal{P}'_j the restriction of Π_j and \mathcal{P}_j , respectively, to the states in $R_N(\omega)$, be a structure over a set Φ' of primitive propositions, where Φ' contains Φ and new primitive propositions that we call $p_{i, \omega}$ for each player i and state $\omega \in R_N(\omega)$.⁷ Note that there are only countably many information sets in $R_N(\omega)$, so Φ' is countable. Define π' so that it agrees with π (restricted to $R_N(\omega)$) on the propositions in Φ , and so that $[[p_{i, \omega}]]_i = \Pi_i(\omega)$. Thus, M' satisfies A5 and A6. It is easy to check that, for all $\omega' \in R_N(\omega)$ and all formulas $\psi \in \mathcal{L}_n^C(\Phi)$, we have that $(M, \omega', i) \models \psi$ iff $(M', \omega', i) \models \psi$. Since $M' \models \varphi$, it follows

⁷This is the one argument that needs the assumption that the set of primitive propositions can be different in different structures in $\mathcal{M}(\Psi)$, and the fact that every $\Psi \setminus \Phi$ is countable. We have assumed for simplicity that the propositions $p_{i, \omega}$ are all in $\Psi \setminus \Phi$, and that they can be chosen in such a way so that $\Psi \setminus (\Phi \cup \{p_{i, \omega} : i \in \{1, \dots, n\}, \omega \in \Omega\})$ is countable.

that $(M, \omega, i) \models \varphi$, as desired. ■

The proof that (a) implies (c) shows that, starting from an arbitrary structure M , we can construct a common-interpretation structure M' that is equivalent to M in the sense that the same formulas hold in both models. Note that because the probability measures in the structure M' constructed in the proof of Proposition 4.1 have disjoint support, M' does not satisfy the CPA, even if the original structure M does. As the next result shows, this is not an accident.

Proposition 4.2 *For all formulas $\varphi \in \mathcal{L}_n^C(\Psi)$, if either $\mathcal{M}^{cpa}(\Psi) \models^{out} \varphi$, $\mathcal{M}^{cpa}(\Psi) \models^{in} \varphi$, $\mathcal{M}_c^{cpa, ai}(\Psi) \models^{out, ai} \varphi$, or $\mathcal{M}^{cpa, ai}(\Psi) \models^{in, ai} \varphi$, then $\mathcal{M}_c^{cpa}(\Psi) \models \varphi$. Moreover, if $\mathcal{M}_c^{cpa}(\Psi) \models \varphi$, then $\mathcal{M}^{cpa}(\Psi) \models^{out} \varphi$ and $\mathcal{M}^{cpa, ai}(\Psi) \models^{out, ai} \varphi$. However, in general, if $\mathcal{M}_c^{cpa}(\Psi) \models \varphi$, then it may not be the case that $\mathcal{M}^{cpa}(\Psi) \models^{in} \varphi$.*

Proof. All the implications are straightforward, with proofs along the same lines as that of Proposition 4.1. To prove the last claim, let $p \in \Psi$ be a primitive proposition. Aumann's agreeing to disagree result shows that $\mathcal{M}_c^{cpa}(\Psi) \models \neg CB_G(B_1 p \wedge B_2 \neg p)$, while Example 3.1 shows that $\mathcal{M}^{cpa}(\Psi) \not\models^{in} \neg CB_G(B_1 p \wedge B_2 \neg p)$. ■

Proposition 4.2 depends on the fact that we are considering belief and common belief rather than knowledge and common knowledge, where knowledge is defined in the usual way, as truth in all possible worlds:

$(M, \omega, i) \models^{in} K_i \varphi$ iff $(M, \omega', i) \models^{in} \varphi$ for all $\omega' \in \Pi_i(\omega)$, with K_i the knowledge operator for player i , and where we have assumed that $\omega \in \Pi_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$. Aumann's result holds if we consider common belief (as long as what we are agreeing about are judgments of probability and expectation). With knowledge, there are formulas that are valid with a common interpretation that are not valid under innermost-scope semantics when there is ambiguity. For example, $\mathcal{M}_c(\Psi) \models K_i \varphi \Rightarrow \varphi$, while it is easy to construct a structure M with ambiguity such that $(M, \omega, 1) \models^{in} K_2 p \wedge \neg p$. What is true is that $\mathcal{M}(\Psi) \models^{in} (K_1 \varphi \Rightarrow \varphi) \vee \dots \vee (K_n \varphi \Rightarrow \varphi)$. This is because we have $(M, \omega, i) \models^{in} K_i \varphi \Rightarrow \varphi$, so one of $K_1 \varphi \Rightarrow \varphi, \dots, K_n \varphi \Rightarrow \varphi$ must hold. As shown in (Halpern 2009), this axiom essentially characterizes knowledge if there is ambiguity.

As noted above, the proof of Proposition 4.1 demonstrates that, given a structure M with ambiguity and a common prior, we can construct an equivalent common-interpretation structure M' with heterogeneous priors, where M and M' are said to be *equivalent (under innermost scope)* if for every formula ψ , $M \models^{in} \psi$ if and only if $M' \models^{in} \psi$. The converse does not hold, as the next example illustrates: when formulas are interpreted using innermost scope, there is a common-interpretation structure with heterogeneous priors that cannot be converted into an equivalent structure with ambiguity that satisfies the CPA.

Example 4.3 We construct a structure M with heterogeneous priors for which there is no equivalent ambiguous structure that satisfies the CPA. The structure M has three

players, one primitive proposition p , and two states, ω_1 and ω_2 . In ω_1 , p is true according to all players; in ω_2 , the proposition is false. Player 1 knows the state: his information partition is $\Pi_1 = \{\{\omega_1\}, \{\omega_2\}\}$. The other players have no information on the state, that is, $\Pi_i = \{\{\omega_1, \omega_2\}\}$ for $i = 2, 3$. Player 2 assigns probability $\frac{2}{3}$ to ω_1 , and player 3 assigns probability $\frac{3}{4}$ to ω_1 . Hence, M is a common-interpretation structure with heterogeneous priors. We claim that there is no equivalent structure M' that satisfies the CPA.

To see this, suppose that M' is an equivalent structure that satisfies the CPA, with a common prior ν and a state space Ω' . As $M' \models^{in} pr_2(p) = \frac{2}{3}$ and $M' \models^{in} pr_3(p) = \frac{3}{4}$, we must have

$$\begin{aligned}\nu(\{\omega' \in \Omega' : (M', \omega', 2) \models^{in} p\}) &= \frac{2}{3}, \\ \nu(\{\omega' \in \Omega' : (M', \omega', 3) \models^{in} p\}) &= \frac{3}{4}.\end{aligned}$$

Observe that $M \models^{in} B_2(p \Leftrightarrow B_1p) \wedge B_3(p \Leftrightarrow B_1p)$. Thus, we must have $M' \models^{in} B_2(p \Leftrightarrow B_1p) \wedge B_3(p \Leftrightarrow B_1p)$. Let $E = \{\omega' \in \Omega' : (M', \omega', 1) \models^{in} B_1p\}$. It follows that we must have $\nu(E) = 2/3$ and $\nu(E) = 3/4$, a contradiction. ■

Example 4.3 demonstrates that there is no structure M' that is equivalent to the structure M (defined in Example 4.3) that satisfies the CPA. In fact, as we show now, an even stronger result holds: In any structure M' that is equivalent to M , whether it satisfies the CPA or not, players have a common interpretation.

Proposition 4.4 *If a structure $M' \in \mathcal{M}(\Psi)$ is equivalent under innermost scope to the structure M defined in Example 4.3, then $M' \in \mathcal{M}_c$.*

Proof. Note that $M \models p \Leftrightarrow (pr_1(p) = 1)$. Hence, if a structure M' is equivalent to M , we must have that $M' \models^{in} p \Leftrightarrow (pr_1(p) = 1)$, that is, for all $\omega \in \Omega$ and $i \in N$, $(M', \omega, i) \models^{in} p \Leftrightarrow (pr_1(p) = 1)$. By a similar argument, we obtain that for every $\omega \in \Omega$ and $i \in N$, it must be the case that $(M', \omega, i) \models^{in} \neg p \Leftrightarrow (pr_1(\neg p) = 1)$. Since the truth of a probability formula does not depend on the player under the innermost-scope semantics, it follows that for each $i, j \in N$, the interpretations π'_i and π'_j in M' coincide. In other words, M' is a common-interpretation structure. ■

5 Discussion

We have defined a logic for reasoning about ambiguity, and considered the tradeoff between having a common prior (so that everyone starts out with the same belief) and having a common interpretation (so that everyone interprets all formulas the same way). We showed that, in a precise sense, allowing different beliefs is more general than allowing multiple interpretations. But we view that as a feature, not a weakness, of considering ambiguity. Ambiguity can be viewed as a reason for differences of beliefs; as such, it provides some structure to these differences.

We have not discussed axiomatizations of the logic. From Proposition 4.1 it follows that for formulas in $\mathcal{L}_n^C(\Psi)$, we can get the same axiomatization with respect to structures in $\mathcal{M}(\Psi)$ for both the \models^{out} and \models^{in} semantics; moreover,

this axiomatization is the same as that for the common-interpretation case. An axiomatization for this case is already given in (Fagin and Halpern 1994). Things get more interesting if we consider $\mathcal{M}^{cpa}(\Psi)$, the structures that satisfy the CPA. Halpern 2002 provides an axiom that says that it cannot be common knowledge that players disagree in expectation, and shows that it can be used to obtain a sound and complete characterization of common-interpretation structures with a common prior. (The axiomatization is actually given for common knowledge rather than common belief, but a similar result holds with common belief.) By Proposition 4.2, the axiomatization remains sound for outermost-scope semantics if we assume the CPA. However, using Example 4.3, we can show that this is no longer the case for the innermost-scope semantics. The set of formulas valid for innermost-scope semantics in the class of structures satisfying the CPA is strictly between the set of formulas valid in all structures and the set of formulas valid for outermost-scope semantics in the class of structures satisfying the CPA. Finding an elegant complete axiomatization remains an open problem.

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