A Propositional Modal Logic of Time Intervals*

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Abstract: In certain areas of artificial intelligence there is need to represent continuous change and to make statements that are interpreted with respect to time intervals rather than time points. To this end we develop a modal temporal logic based on time intervals, a logic which can be viewed as a generalization of point-based modal temporal logic. We discuss related logics, give an intuitive presentation of the new logic, and define its formal syntax and semantics. We make no assumption about the underlying nature of time, allowing it to be discrete (such as the natural numbers) or continuous (such as the rationals or the reals), linear or branching, complete (such as the reals) or not (such as the rationals). We show, however, that there are formulas in the logic that allow us to distinguish all these situations. We also give a translation of our logic into first-order logic, which allows us to apply some results on first-order logic to our modal one. Finally, we consider the difficulty of validity problems for the logic. This turns out to depend critically, and in surprising ways, on our assumptions about time. For example, if we take our underlying temporal structure to be the rationals, we can show that the validity problem is r.e.-complete, if it is the reals then we can show that validity is Π^1_1-hard, and if it is the natural numbers, then validity is Π^1_1-complete.

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1 Introduction

In at least two areas of Artificial Intelligence, known as "qualitative physics" and "automatic planning", there is a need for reasoning about continuous processes (such as water filling a slightly leaky container) and having assertions refer to time intervals rather than time points. For example, "the liquid level increased by three inches," "the robot carried out the task" and "I solved the problem while jogging to the ocean and back" may be true at certain intervals but at no time instant. We are interested in a logic in which such statements can be made, and whose formal semantics mirror the intuitive meaning of these statements.

The machinery developed in theoretical computer science so far is inadequate for our purposes. Modal temporal logic, developed in formal philosophy (e.g., [24]) and first applied to reasoning about programs in [21], interprets formulas over time *points*. Also, in most formulations, time is assumed to be discrete (the one exception of which we are aware is [4]). A similar comment applies to dynamic logic [22]: formulas are interpreted over time points, and furthermore there is no way to state what happens *during* the execution of a program.

There have been several extensions of these logics that deal with time *intervals* rather than just time points. Again, the initial idea of dealing with intervals goes back to the philosophers (e.g., [9], and more recently [12, 26, 5, 31]). In computer science, there has recently been work on *process logic* [23, 19, 11], where intervals (or "paths") represent pieces of computation, and even more recently work on *interval temporal logic* [17, 8]. We review these logics and others in more detail in a later section.

While these interval logics and process logics come closer to satisfying our goals than point-based temporal logics do, they are still not adequate for our needs: they either view time as being discrete, rather than continuous, or they do not provide an adequate set of modal operators for describing the situations we have in mind, or both.

Our logic can be viewed as an extension of point-based modal temporal logic, where we simply replace the notion of satisfaction by a state ($s \models$) by the notion of satisfaction by an interval, or an ordered pair of states ($\langle s,t\rangle \models$). Intuitively, we think of the ordered pair $\langle s,t\rangle$ as the closed interval consisting of all points between s and t. When dealing with only time points, a single accessibility relation and modality are sufficient, since the only relation between non-identical points is "after" (and its inverse "before"). The situation for intervals is slightly more complex, since in addition to "after" we have "immediately after", "during", "beginning", "ending" "overlapping", etc. It turns out that we can express all twelve relations between two distinct intervals (see [1]) by six modal operators: $\langle \mathtt{B}\rangle$, $\langle \mathtt{E}\rangle$, and $\langle \mathtt{A}\rangle$ (for "begin", "end" and "after"), and their "transposes" $\langle \overline{\mathtt{B}}\rangle$, $\langle \overline{\mathtt{E}}\rangle$, and $\langle \overline{\mathtt{A}}\rangle$. (In fact, as pointed out by Venema [33], we can express these twelve relations using just $\langle \mathtt{B}\rangle$, $\langle \mathtt{E}\rangle$, and their transposes.) The semantics of these operators is quite natural and simple. For example, $\langle \mathtt{B}\rangle\varphi$ is true of an interval $\langle s,t\rangle$ exactly if φ is true of some beginning interval $\langle s,t'\rangle$ with $t' < t$. Similarly, $\langle \overline{\mathtt{B}}\rangle\varphi$ is true of an interval $\langle s,t\rangle$ exactly if φ is true of some interval of which $\langle s,t\rangle$ is a beginning, that is an interval $\langle s,t'\rangle$ with $t' > t$. The definitions of the other two pairs of modal operators is similar.

Although for most of our applications we want to view time as having the structure of the reals, for the sake of generality we give the semantics of these operators with respect to an arbitrary temporal structure. Thus, time could be discrete (such as the natural numbers) or continuous (such as the rationals or the reals), linear or branching, complete (such as the reals) or not (such as the rationals). Our simple logic has enough expressive power to distinguish these situations.

We also consider the complexity of the validity problem for our logic. When we speak of validity, we do so with respect to a class of temporal structures (where the class could be a singleton consisting just of the reals, the rationals, or the natural numbers). It turns out that the validity problem is surprisingly sensitive to assumptions we make on the structure of time. For example, we can show that in any class of temporal structures that includes at least one with an infinite ascending sequence (i.e., a sequence $t_1 < t_2 < \ldots$), the validity problem is at least r.e.-hard. If all the structures in the class are in addition *complete* (so that every sequence with an upper bound has a *least* upper bound), then the validity problem becomes Π^1_1-hard. As corollaries to these results, we obtain that the validity problem for the rationals is r.e.-hard, while that for the reals or the natural numbers is Π^1_1-hard. (The notion of Π^1_1 is defined formally in Section 8.) We also give some upper bounds for the validity problem, showing, for example, that the validity problem is r.e.-complete for the rationals, and Π^1_1-complete for the natural numbers.

The rest of the paper is organized as follows: In the next section, we list a few choices that need to be made when constructing an interval-based logic, and decisions we have made on these issues with regard to our logic. In Section 3 we give the informal syntax and semantics of our logic; we make things formal in Section 4. In Section 5 we show how the logic can be used to capture situations of interest in qualitative physics and automatic planning. In Section 6 we show how formulas of the logic can distinguish various temporal structures. In Section 7 we show that our logic can be translated into a first-order one, allowing us to apply techniques of first-order logic to our logic. In Section 8 we present our results on the difficulty of the validity problem. In Section 9 we review related work on interval logics, both in philosophy and in computer science. We conclude in Section 10 with some interesting open problems.

2 Making initial choices

We mentioned in the introduction previous logics of time intervals. In philosophy we have the logics discussed by Hamblin [9], Humberstone [12], Roper [26] and Burgess [5]. In computer science we also have several interval-based logics. *Process logic* [23, 11] is a generalization of *dynamic logic*. *Interval temporal logic* [17, 8] is a generalization of point-based temporal logic. For other related work see [19, 18, 27].

In Section 9 we discuss these logics and their relation to ours in more detail. Here let us just point out issues that distinguish between the different logics, both in philosophy and in computer science.

1. Ontology. Are intervals primitive objects in the logic, or are they defined in terms of points, which are the only primitive objects? In philosophy one finds logic of both kinds. In computer science almost all interval-based logics construct intervals out of points (the

only exception of which we are aware is Allen's logic [2]). We will join the majority, and construct intervals out of points.

2. Commitment to a particular underlying temporal structure. With no exception, all interval-based temporal logics in computer science have been committed to the discrete and linear view of time. This has not been the case in philosophy. Our logic will be most general in this respect: we will only assume that the set of time points that lie between any two points is totally ordered. This will allow branching and linear time, dense and discrete time, bounded and unbounded time, and so on. Of course, further restrictions that we place on the nature of time induces special properties on the logic, as will become clear when we discuss the complexity of the validity problem for our logic.

3. Choice of tense operators. In computer science, the strong commitment to a discrete and linear order dictated fairly standard modal operators. In philosophy there has been less uniformity. Our logic will be very general in this respect too. We will introduce three very natural pairs of modal operators, which are sufficient to represent all twelve possible relations between two distinct intervals.

4. The relation between the truth-value of a formula over an interval to its truth value over parts of that interval. In computer science, the issue that arises is whether or not *locality* is assumed; a logic is local if an primitive proposition is true over an interval iff it is true over its starting point. In philosophy, the assumption that is sometimes made is of *homogeneity.* A logic is homogeneous when, roughly speaking, a proposition is true over an interval iff it is true over all of its subintervals. In our logic we do not assume homogeneity, or any other connection between the truth value of a proposition over an interval to its truth value over any part of that interval.

3 Informal syntax and semantics

Implicit in point-based modal temporal logic is the notion of *now*, the current instant of time. By way of contrast, in our logic, the key notion is *the current interval.* Formulas are interpreted over intervals, and we have modal operators that let us refer to other intervals besides the current one. As we mentioned above, we view an interval as an ordered set of points; however, our modal operators will allow us to talk about individual points (or, more accurately, point intervals) as well.

Our logic has more modal operators than one usually encounters in point-based modal temporal logics. This is because there are only two possible relationships between two comparable and distinct points t and t' (namely, that t precedes t' or that t' precedes t), while two comparable and distinct intervals can stand in one of twelve different relationships. Specifically, well-formed formulas in our logic will be those of propositional calculus, augmented by the modal operators $\langle \mathtt{A} \rangle$, $\langle \mathtt{B} \rangle$, $\langle \mathtt{E} \rangle$, $\langle \overline{\mathtt{A}} \rangle$, $\langle \overline{\mathtt{B}} \rangle$, and $\langle \overline{\mathtt{E}} \rangle$. Their informal meaning is as follows:

$\langle \mathtt{A} \rangle\varphi$: φ holds at some interval beginning immediately after the end of the current one

```
Current interval:               |---------------|
⟨A⟩:                           .               |------|
⟨B⟩:                           |-------|       .
⟨E⟩:                           .  |------------|
⟨Ā⟩:                  |--------|               .
⟨B̄⟩:                           |------------------------|
⟨Ē⟩:                    |----------------------|
```

Figure 1: The six basic modal operators

$\langle \mathtt{B} \rangle \varphi$: φ holds at some interval ending during the current one, beginning when the current one begins

$\langle \mathtt{E} \rangle \varphi$: φ holds at some interval beginning during the current one, ending when the current one ends

$\langle \overline{\mathtt{A}} \rangle \varphi$: φ holds at some interval ending immediately before the beginning of the current one

$\langle \overline{\mathtt{B}} \rangle \varphi$: φ holds at some interval of which the current one is a beginning

$\langle \overline{\mathtt{E}} \rangle \varphi$: φ holds at some interval of which the current one is an end.

Pictorially, the modal operators pick out intervals as shown in Figure 1.

Using these operators we can define some more complex ones:

$\langle \mathtt{L} \rangle \varphi =_{def} \langle \mathtt{A} \rangle \langle \mathtt{A} \rangle \varphi$: φ holds at some later interval

$\langle \mathtt{D} \rangle \varphi =_{def} \langle \mathtt{B} \rangle \langle \mathtt{E} \rangle \varphi \equiv \langle \mathtt{E} \rangle \langle \mathtt{B} \rangle \varphi$: φ holds at some interval during the current one (i.e. φ holds at some proper subinterval of the current interval)

$\langle \mathtt{O} \rangle \varphi =_{def} \langle \mathtt{E} \rangle \langle \overline{\mathtt{B}} \rangle \varphi$: φ holds at some "future overlapping" interval.

$\langle \overline{\mathtt{L}} \rangle$, $\langle \overline{\mathtt{D}} \rangle$ and $\langle \overline{\mathtt{O}} \rangle$ are similarly defined. Pictorially, these operators select intervals as shown in Figure 2. In fact, these modal operators (`A, B, E, L, D, O` and their "transposes") exactly define the twelve possible relations between two distinct intervals (see [1]).

We can define the duals of all these operators as usual: $[\mathtt{X}]\varphi \equiv \neg \langle \mathtt{X} \rangle \neg \varphi$ (where `X` is `A, B, E,` etc.). While $\langle \mathtt{B} \rangle \varphi$ intuitively says that φ is true at some beginning interval, $[\mathtt{B}]\varphi$ says that φ is true at all beginning intervals.

We define both the `B` and `E` operators so that they refer to *strict* subintervals. In particular, $[\mathtt{B}]\varphi$ is vacuously true of point intervals of the form $\langle s, s \rangle$, since they have no strict beginning intervals. Thus, if we define the formula `false` to be $\mathtt{q} \wedge \neg \mathtt{q}$ for some primitive proposition `q`, we have that the formula $[\mathtt{B}]$`false` holds precisely of point intervals.

We can use this observation to define a "beginning point" modal operator `[[BP]]`, where `[[BP]]`φ says that φ holds at the beginning point of the interval:

```
Current interval:            |---------------|
⟨L⟩:                           .                          |------|
⟨D⟩:                              |----|       .
⟨O⟩:                           .    |----------------|
⟨L̄⟩:             |--------|                    .
⟨D̄⟩:                       |---------------------------|
⟨Ō⟩:                   |---------------|       .
```

Figure 2: Derived modal operators

$[\![\mathtt{BP}]\!]\varphi \equiv ((\varphi\wedge [\mathtt{B}]\mathtt{false}) \vee \langle\mathtt{B}\rangle(\varphi\wedge [\mathtt{B}]\mathtt{false}))$

By analogy, it is easy to define the "end point" modal operator:

$[\![\mathtt{EP}]\!]\varphi \equiv ((\varphi\wedge [\mathtt{B}]\mathtt{false}) \vee \langle\mathtt{E}\rangle(\varphi\wedge [\mathtt{B}]\mathtt{false}))$

Notice that both "point" operators are their own duals: $[\![\mathtt{BP}]\!]\varphi \equiv \neg[\![\mathtt{BP}]\!]\neg\varphi$, and similarly for $[\![\mathtt{EP}]\!]$. To emphasize this fact, we have chosen the single $[\![\,]\!]$ notation over the double $[\,]$ and $\langle\rangle$ notation.

As observed by Venema [33], the $\langle\mathtt{A}\rangle$ and $\langle\overline{\mathtt{A}}\rangle$ operators are actually definable in terms of the other modal operators. $\langle\mathtt{A}\rangle$ is definable by $\langle\mathtt{A}\rangle\varphi =_{def} [\![\mathtt{EP}]\!](\langle\overline{\mathtt{B}}\rangle\varphi)$. $\langle\overline{\mathtt{A}}\rangle$ is similarly definable in terms of $\langle\mathtt{B}\rangle$ and $\langle\overline{\mathtt{E}}\rangle$.

4 Formal syntax and semantics

Syntax. Given a set Φ_0 of primitive propositions, we form the set of all formulas by closing off under conjunction, negation, and the modal operators discussed above. Thus, if φ and ψ are formulas, then so are $\neg\varphi_1$, $\varphi_1\wedge\varphi_2$, $\langle\mathtt{A}\rangle\varphi_1$, $\langle\mathtt{B}\rangle\varphi_1$, $\langle\mathtt{E}\rangle\varphi_1$, $\langle\overline{\mathtt{A}}\rangle\varphi_1$, $\langle\overline{\mathtt{B}}\rangle\varphi_1$ and $\langle\overline{\mathtt{E}}\rangle\varphi_1$. We use the standard abbreviations: $\vee$, $\supset$, and so on.

Semantics. An *interpretation* is a pair $\langle S, V\rangle$. S is a *temporal structure* $\langle T, \leq\rangle$, where T is a set of time points and $\leq$ is a partial order on T. V is a function which assigns meaning to the primitive propositions by associating each primitive proposition with the set of intervals where it is true. Thus, $V : \Phi_0 \to 2^I$, where $I = \{\langle t_1, t_2\rangle : t_1 \leq t_2)\}$. The only assumptions we will make about $\leq$ is that it has "linear intervals," which means that for any two points t_1 and t_2 such that $t_1 \leq t_2$, the set of points $\{t : t_1 \leq t \leq t_2\}$ is totally ordered. In other words, if $t_1 \leq t_3$, $t_1 \leq t_4$, $t_3 \leq t_2$ and $t_4 \leq t_2$, then either $t_3 \leq t_4$ or $t_4 \leq t_3$. Note that given this assumption, the set of points induce a forest-like structure with respect to $\leq$ (a forest is a collection of trees). Actually, no part of the discussion in this paper depends on the assumption. We make it simply because it fits our intuition about the nature of time. In particular, given the assumption about the linearity of intervals, one can intuitively think of the pair $\langle t_1, t_2\rangle$ as the closed interval of points between t_1 and t_2. To investigate the logic in its full generality, we have not imposed any further assumptions on the nature of time, such as linearity or continuity. Of course, we can easily do so. In fact, as we shall show in the next section, some of these assumptions can essentially be expressed by formulas in the logic.

We interpret formulas over pairs $\langle t_1, t_2\rangle$ such that $t_1, t_2 \in T$ and $t_1 \leq t_2$. Given an interpretation $\mathbf{M}$ and an interval $\langle t_1, t_2\rangle$, a formula φ is either true in the interval (written $\mathbf{M},\langle t_1, t_2\rangle \models \varphi$) or false (written $\mathbf{M},\langle t_1, t_2\rangle \not\models \varphi$). When clear from the context, the interpretation $\mathbf{M}$ may be omitted, and so in those cases we write simply $\langle t_1, t_2\rangle \models \varphi$ and $\langle t_1, t_2\rangle \not\models \varphi$.

The truth value of formulas is determined by the semantic rules given below. For convenience, we define the strict (i.e., irreflexive) version of $\leq$:

$$t_1 < t_2 =_{def} t_1 \leq t_2 \wedge \neg(t_2 \leq t_1).$$

1. For all $\varphi \in \Phi_0$, we have $\langle t_1, t_2\rangle \models \varphi$ iff $\langle t_1, t_2\rangle \in V(\varphi)$.
2. $\langle t_1, t_2\rangle \models \neg\varphi$ iff $\langle t_1, t_2\rangle \not\models \varphi$
3. $\langle t_1, t_2\rangle \models \varphi_1 \wedge \varphi_2$ iff $\langle t_1, t_2\rangle \models \varphi_1$ and $\langle t_1, t_2\rangle \models \varphi_2$
4. $\langle t_1, t_2\rangle \models \langle \mathtt{A}\rangle\varphi$ iff there exists t_3 such that $t_2 < t_3$ and $\langle t_2, t_3\rangle \models \varphi$
5. $\langle t_1, t_2\rangle \models \langle \mathtt{B}\rangle\varphi$ iff there exists t_3 such that $t_1 \leq t_3$, $t_3 < t_2$ and $\langle t_1, t_3\rangle \models \varphi$
6. $\langle t_1, t_2\rangle \models \langle \mathtt{E}\rangle\varphi$ iff there exists t_3 such that $t_1 < t_3$, $t_3 \leq t_2$ and $\langle t_3, t_2\rangle \models \varphi$
7. $\langle t_1, t_2\rangle \models \langle \overline{\mathtt{A}}\rangle\varphi$ iff there exists t_3 such that $t_3 < t_1$ and $\langle t_3, t_1\rangle \models \varphi$
8. $\langle t_1, t_2\rangle \models \langle \overline{\mathtt{B}}\rangle\varphi$ iff there exists t_3 such that $t_2 < t_3$ and $\langle t_1, t_3\rangle \models \varphi$
9. $\langle t_1, t_2\rangle \models \langle \overline{\mathtt{E}}\rangle\varphi$ iff there exists t_3 such that $t_3 < t_1$ and $\langle t_3, t_2\rangle \models \varphi$

These definitions induce a meaning on the derived modal operators as well. The reader may verify that $\langle t_1, t_2\rangle \models \langle \mathtt{D}\rangle\varphi$ iff there exist t_3, t_4 such that $t_1 < t_3 < t_4 < t_2$ and $\langle t_3, t_4\rangle \models \varphi$, that $\langle t_1, t_2\rangle \models [[\mathtt{BP}]]\,\varphi$ iff $\langle t_1, t_1\rangle \models \varphi$, and so on.

A formula φ is said to be *satisfiable* with respect to a class of temporal structures $\mathcal{A}$ if, in some interpretation $\langle\langle T, <\rangle, V\rangle$ such that $\langle T, <\rangle \in \mathcal{A}$, we have $\langle t_1, t_2\rangle \models \varphi$ for some $t_1 \in T$ and $t_2 \in T$ with $t_1 \leq t_2$. A formula is satisfiable in a given temporal structure if it is satisfiable with respect to the singleton consisting of that structure. φ is *valid* with respect to $\mathcal{A}$ if $\neg\varphi$ is not satisfiable with respect to $\mathcal{A}$.

Of particular interest to us will be three particular temporal structures, namely, the natural numbers, the rationals, and reals, endowed with the usual ordering relation. We denote these three structures $\mathcal{N}$, $\mathcal{Q}$, and $\mathcal{R}$ respectively.

5 Expressing assertions in the logic

The area of Artificial Intelligence known as qualitative physics is concerned with reasoning about relatively simple physical situations, using only rough and qualitative information, in much the same way as people do in everyday life. Typical problems are predicting the outcome of placing a kettle on a burner, reasoning about liquids flowing between containers, and reasoning about collisions between moving objects. Although reasoning about time is clearly central to qualitative physics, the actual work that has been done makes little use of explicit temporal formalisms (see, e.g., [6]).

To illustrate the fact that our logic lends itself nicely to this research domain, consider representing the sentence "if you open the tap then, unless someone punctures the canteen, the canteen will eventually be filled." In our logic this assertion is represented by the formula

$\texttt{open-tap} \supset \langle \texttt{A} \rangle (\neg \langle \texttt{D} \rangle \texttt{puncture} \supset [[\texttt{EP}]]\texttt{filled})$

Another area of Artificial Intelligence heavily involved in temporal reasoning is automatic planning, where a (usually simulated) robot must reason about carrying out outstanding tasks, managing available resources, meeting various deadlines and interacting with other agents. Here there has been some use of temporal formalisms, most notably by McDermott [16] and Allen [2]. Our logic is in fact related to their logics; its translation into first-order logic, either as described in a later section or as described in [29], results in a logic that is not unlike those of McDermott and Allen. In [28] we argue, however, that our logic (and its translation into first-order logic) has the advantages of very clear semantics, greater simplicity, and improved flexibility.

To see how our logic lends itself easily to the planning domain, consider the assertion "if the robot executes the `charge-battery` routine then at the beginning of the *following* execution of the `navigate` routine its batteries will be fully charged." This example is somewhat more complex than the qualitative physics one. We choose it not only because it is typical of statements that come up naturally in the process of planning, but because it illustrates another property of our logic. Our logic is intended as a very basic and fundamental vehicle for representing temporal information. If AI has taught us anything, it is that intelligent information processing relies on a detailed and finely-structured knowledge representation. In this example we demonstrate how more complex definitions can be built on top of our "assembly language" logic.

In [29] we categorize proposition types, showing how we can arrive at coherent definitions of, and distinctions between, what are usually called *events, facts, properties, processes*, and the like. For example, we define *liquid propositions* to be those that hold over an intervals iff they hold over all its subintervals (that is, propositions for which the philosophers' assumption of *homogeneity*, mentioned in Section 2, holds). For our present purposes it is enough to define the notion of *solid propositions.* A proposition is said to be solid if no two distinct overlapping intervals ever satisfy it. For example, 'The robot executed the `navigate` routine' is a solid proposition. It is easy to define the notion in the logic:

$$\texttt{solid}(\varphi) =_{def} \varphi \supset \neg\langle \texttt{B} \rangle \varphi \wedge \neg\langle \texttt{E} \rangle \varphi \wedge \neg\langle \texttt{D} \rangle \varphi \wedge \neg\langle \texttt{O} \rangle \varphi$$

It is easy to check that φ is a solid proposition in a given temporal structure if $\texttt{solid}(\varphi)$ holds for every interval in that structure. (Although the formula $\texttt{solid}(\varphi)$ can be true of an interval if φ is true of a preceding overlapping interval, this cannot happen if solid(φ) holds for every interval.)

Assertions of the form "the next time that" are very common, and so it will be useful to define a new binary modal operator. For a solid proposition φ, [[NTT]](φ,ϑ) will mean that ϑ holds in the first interval which satisfies φ and which begins after the current interval:

$$[[\mathtt{NTT}]](\varphi,\vartheta) =_{def} [\mathtt{A}]([\mathtt{D}]\neg\varphi \supset [\mathtt{A}](\varphi \supset \vartheta))$$

Given this definition, the assertion about the robot is simply

$$\texttt{charge-battery} \supset [[\mathtt{NTT}]](\texttt{navigate},[[\mathtt{BP}]]\texttt{battery-full})$$

6 Distinguishing among temporal structures

We have so far deliberately refrained from imposing all but the most elementary constraints on the underlying structure of time. For most applications we will indeed want to add further constraints, such as discreteness, linearity or unboundedness. Interestingly, our logic is sufficiently expressive to capture several such constraints in the logic itself: there are formulas that restrict the class of structures exactly to those satisfying the appropriate constraint. In this section we give several examples of such formulas.

Discreteness. A point is *discrete* in a temporal structure if, along any path in the structure which includes that point, the point has a "closest point" on each side (unless it has *no* points on that side). Formally, we say that a point r is discrete in a temporal structure S if for all points $t \in S$, if $r < t$ ($t < r$) then there exists a point $s \in S$ such that $r < s \leq t$ (resp. $t \leq s < r$) and such that there does not exist a point $s' \in S$ with $r < s' < s$ (resp. $s < s' < r$).

A temporal structure is discrete if all points in it are discrete.

Now consider the following formulas:

$$\begin{aligned}
&\texttt{length0} =_{def} [\mathtt{B}]\texttt{false}\\
&\texttt{length1} =_{def} \langle\mathtt{B}\rangle\texttt{true} \wedge [\mathtt{B}][\mathtt{B}]\texttt{false}\\
&\texttt{discrete} =_{def} \texttt{length0} \vee \texttt{length1} \vee (\langle\mathtt{B}\rangle\texttt{length1} \wedge \langle\mathtt{E}\rangle\texttt{length1})
\end{aligned}$$

It was noted earlier that [B]false, and therefore also length0, are true exactly of point intervals. Similarly, length1 is true of $\langle s, t\rangle$ exactly if $s < t$ and there are no points between s and t. It is easily seen that a temporal structure is discrete exactly if the formula discrete is valid in that structure. Thus, discrete is valid in $\mathcal{N}$, but is not even satisfiable by any nonpoint interval in either $\mathcal{Q}$ or $\mathcal{R}$.

Density. A temporal structure is *dense* if between any two comparable points there is a third point, i.e., if $r < t$ entails that there exists an s such that $r < s < t$. Consider the following formula

$$\texttt{dense} =_{def} \neg\texttt{length1}$$

Clearly, the formula `dense` is valid in a structure S iff S is dense. In particular, `dense` is valid in $\mathcal{R}$ and $\mathcal{Q}$, but not in $\mathcal{N}$.

Unboundedness. A temporal structure is *unbounded* if for any point s there exist points r and t such that $r < s < t$. The following definition:

$$\texttt{unbounded} =_{def} \langle\texttt{A}\rangle\texttt{true} \wedge \langle\overline{\texttt{A}}\rangle\texttt{true}$$

guarantees that `unbounded` is valid exactly for the class of unbounded structures.

Linearity. A temporal structure is *linear* if any two points that are comparable under the symmetric and transitive closure of '$\leq$' are also comparable under '$\leq$' ; i.e., if there is no branching in the forest induced by the structure (notice that this does not preclude having many "parallel" time lines). Thus, in a linear temporal structure, if two distinct intervals start at the same point, then one must be a prefix of another. Similarly, if two distinct intervals *end* at the same point, then one must be a suffix of another. Consider the following definition:

$$\begin{aligned}&\texttt{linear-time} =_{def}\\ &\quad(\langle\texttt{A}\rangle\texttt{p} \supset [\texttt{A}](\texttt{p} \vee \langle\texttt{B}\rangle\texttt{p} \vee \langle\overline{\texttt{B}}\rangle\texttt{p})) \wedge\\ &\quad(\langle\overline{\texttt{A}}\rangle\texttt{p} \supset [\overline{\texttt{A}}](\texttt{p} \vee \langle\texttt{E}\rangle\texttt{p} \vee \langle\overline{\texttt{E}}\rangle\texttt{p})),\end{aligned}$$

where `p` is a primitive proposition. It is easy to check that `linear-time` captures the notion of a linear temporal structure, in that `linear-time` is valid with respect to linear temporal structures, while for any non-linear temporal structure S, there is a valuation V such that `linear-time` is not valid in the interpretation $\langle S, V\rangle$.

Completeness. It is a standard result of first-order logic that any two dense, linear and unbounded structures are *elementarily equivalent*: they cannot be distinguished by formulas in the first-order logic whose only relation symbols are '$=$' and '$<$' (see, e.g., [7]). In particular, it follows that $\mathcal{Q}$ and $\mathcal{R}$ are elementarily equivalent. However, as we are about to show, $\mathcal{Q}$ and $\mathcal{R}$ *are* distinguishable in our logic (although, as we shall show in section 7, it is the case that all formulas satisfiable in $\mathcal{R}$ are also satisfiable in $\mathcal{Q}$).

The crucial property that distinguishes $\mathcal{R}$ from $\mathcal{Q}$ is that $\mathcal{R}$ is *complete*: all sequences with an upper bound have a *least* upper bound. $\mathcal{Q}$ is not complete. For example, an increasing sequence of rationals converging to $\sqrt{2}$ will not have a least upper bound in $\mathcal{Q}$. We now give a formula in our logic that distinguishes complete from incomplete temporal structures.

Anticipating some of the constructs that we will need in the next section, let `p` and `#` be two primitive propositions and define

$$\texttt{cell} =_{def} \ \texttt{[[BP]]\#} \wedge \texttt{[[EP]]\#} \wedge \texttt{[D]p} \wedge \langle\texttt{D}\rangle\texttt{p}.$$

Thus `cell` is satisfied by an interval exactly if both its begin point and its end point satisfy `#` (intuitively, the cell "delimiters"), and if all interior intervals satisfy `p` (intuitively, the cell "content"), and there is some interior interval satisfying `p`.

Now consider the following formula:

$$\texttt{telescoping} =_{def} \ \langle\texttt{B}\rangle\texttt{cell} \wedge \texttt{[[EP]]}\neg\texttt{\#} \wedge \texttt{[E]([[BP]]\#} \supset \langle\texttt{B}\rangle\texttt{cell).}$$

If $\langle s, t\rangle \models \texttt{telescoping}$ then it is easy to show by induction on k that there exists a sequence $s_0, s_1, \ldots, s_k, \ldots$ such that $s = s_0$, $s_0 < s_1 < \ldots < t$, and $\langle s_i, s_{i+1}\rangle \models \texttt{cell}$. Thus, we have the following picture:

```
s0      s1      s2      ...          t
|-------|-----|----|--|-|...|||      |
   cell   cell   cell   ...         ¬#
```

In a complete temporal structure, the sequence $s_0, s_1, \ldots$ has a least upper bound, say s'. It is easy to see that $\langle s, s'\rangle \models$ [E]($\neg$length0 $\supset$ $\langle$D$\rangle$cell). Define

$$\texttt{complete} =_{def} \ \texttt{telescoping} \supset \langle\texttt{B}\rangle\texttt{([E](}\neg\texttt{length0} \supset \langle\texttt{D}\rangle\texttt{cell))}$$

The discussion above shows that `complete` is valid in complete temporal structures. In particular, it is valid for $\mathcal{R}$. Moreover, $\neg$`complete` is satisfiable in any structure which is not complete. For example, in $\mathcal{Q}$, consider an infinite sequence of rational numbers $s_0, s_1, \ldots$ converging to $\sqrt{2}$, with $s_0 = 1$. Define a valuation V such that $\langle s_i, s_{i+1}\rangle \models \texttt{cell}$ and for every point $t > \sqrt{2}$, $\langle t, t\rangle \models \neg\#$. It is now easy to check that with respect to this valuation, we have $\langle 1, 2\rangle \models \neg\texttt{complete}$.

7 Translation into first-order logic

Suppose we have a modal logic with Kripke semantics and a formula φ in that logic. It is well known that we can find a first-order formula φ_t which is satisfiable if φ is satisfiable. In fact, if $(M, s) \models \varphi$ for some structure M and world s, then we can construct a first-order structure M_t such that $M_t \models \varphi_t$, where the possible worlds in M are the objects in the domain of M_t and the accessibility relation is made into a binary first-order relation. Although the satisfiability of φ implies the satisfiability of φ_t, then converse is not true in general. It will hold if the accessibility relations are characterizable in first-order logic. While in many cases of interest the accessibility relation is so characterizable, it is not always the case [31].

We consider a similar of our logic translation into first-order logic. Such a translation was already discussed in [29]; the translation we use here is slightly different, and is a a variant of a translation suggested to us by J. van Benthem. The advantage of using this translation is that it allows us to reduce problems of interest to us here to well-known results in first-order logic. In the previous section we showed how various classes of temporal structures can be distinguished by formulas in the logic. In particular, we showed that there are formulas valid in $\mathcal{R}$ but not in $\mathcal{Q}$, and therefore that there are formulas satisfiable in $\mathcal{Q}$ but not in $\mathcal{R}$. Using the translation into first-order logic, in this section we show that the converse does not hold: every formula satisfiable in $\mathcal{R}$ is also satisfiable in $\mathcal{Q}$. (We will make use of this translation also in the next section, when we discuss upper bounds for the validity problem.)

The translation is straightforward. We use the variable symbols $\mathtt{t}_1$, $\mathtt{t}_2$, …, which will designate time points. The target language is the first-order logic with '$=$' and '$\prec$', and binary predicate symbols $\mathtt{p}_1$, $\mathtt{p}_2$, … corresponding to the primitive propositions in the modal logic. Intuitively, where in the modal logic we would say that a proposition $\mathtt{p}$ was satisfied by the interval $\langle t_1, t_2\rangle$ in a certain interpretation, in the first-order logic we will say that the formula $\mathtt{p(t_1,t_2)}$ is true under the appropriate interpretation.

Each modal formula φ is translated into a first-order formula φ_t with two free variables: $\mathtt{t}_1$ and $\mathtt{t}_2$. Although the first-order formula will not say this, the reader should think of $\mathtt{t}_1$ and $\mathtt{t}_2$ as satisfying $\mathtt{t}_1 \preceq \mathtt{t}_2$ (where $\mathtt{t}_1 \preceq \mathtt{t}_2$ abbreviates $\mathtt{t}_1 \prec \mathtt{t}_2 \vee (\mathtt{t}_2 = \mathtt{t}_1)$):

1. If $\mathtt{p}$ is a primitive proposition then $\mathtt{p}_t = \mathtt{p(t_1,t_2)}$
2. $(\neg\varphi)_t = \neg(\varphi_t)$
3. $(\varphi \wedge \varphi')_t = \varphi_t \wedge \varphi'_t$
4. $(\langle\mathtt{B}\rangle\varphi)_t = \exists\mathtt{t}_3(\mathtt{t}_1 \preceq \mathtt{t}_3 \wedge \mathtt{t}_3 \prec \mathtt{t}_2 \wedge \varphi_t[\mathtt{t}_3/\mathtt{t}_2])$. By $\varphi_t[\mathtt{t}_3/\mathtt{t}_2]$ we mean φ_t, with all free occurrences of $\mathtt{t}_2$ replaced by $\mathtt{t}_3$.
5. $(\langle\mathtt{E}\rangle\varphi)_t = \exists\mathtt{t}_3(\mathtt{t}_1 \prec \mathtt{t}_3 \wedge \mathtt{t}_3 \preceq \mathtt{t}_2 \wedge \varphi_t[\mathtt{t}_3/\mathtt{t}_1])$.
6. $(\langle\mathtt{A}\rangle\varphi)_t = \exists\mathtt{t}_3(\mathtt{t}_2 \prec \mathtt{t}_3 \wedge \varphi_t[\mathtt{t}_2/\mathtt{t}_1, \mathtt{t}_3/\mathtt{t}_2])$.
7. …and similarly for the other modal operators.

In order to make precise the connection between a modal formula and its first-order counterpart, we define the notion of a *faithful* first-order interpretation.

Definition 7.1: Let $M = \langle\langle T, <\rangle, V\rangle$ be a modal interpretation, and $M_t = \langle T_t, V_t\rangle$ a first-order interpretation (V_t is the meaning function that determines the denotation of constant symbols, predicate symbols, and the relation symbol $\prec$). We will say that M_t is *faithful* to M if it has the following three properties.

1. $T = T_t$.
2. $< \,=\, V_t(\prec)$.
3. For any primitive proposition $\mathtt{p}$, we have $V(\mathtt{p}) = V_t(\mathtt{p}_t) \cap \{\langle t_1, t_2\rangle : t_1 \leq t_2\}$. Note that we place no constraint on V_t as far as "reversed intervals" go; if $t_2 < t_1$ then we do not care whether $\langle t_1, t_2\rangle \in V_t(\mathtt{p}_t)$. ∎

Note that all modal interpretations have faithful first-order ones. Conversely, all first-order interpretations in which '$\preceq$' has the right properties (i.e., it is a partial order with linear intervals) are faithful to some modal interpretation.

Lemma 7.2: *Let $M = \langle\langle T, <\rangle, V\rangle$ be a modal interpretation, $M_t = \langle T, V_t\rangle$ a first-order interpretation that is faithful to M, φ a modal formula, $s_1, s_2 \in T$ with $s_1 \leq s_2$, and v a valuation mapping variables to time points such that $v(\mathtt{t}_1)$=s_1 and $v(\mathtt{t}_2)$=s_2. Then $M, \langle s_1, s_2\rangle \models \varphi$ iff $M_t, v \models \varphi_t$.*

Proof: By a straightforward induction on the structure of φ. We consider the case that φ is of the form $\langle \mathtt{B}\rangle\varphi'$ here. Observe that $M, \langle s_1, s_2\rangle \models \langle \mathtt{B}\rangle\varphi'$ iff $M, \langle s_1, s_3\rangle \models \varphi'$ for some s_3 with $s_1 \leq s_3 < s_2$ iff (by the inductive hypothesis) $M_t, v' \models \varphi'$, where $v'(\mathtt{t}_1)$=s_1 and $v'(\mathtt{t}_2)$=s_3, iff $M_t, v \models \exists \mathtt{t}_3(\mathtt{t}_1 \preceq \mathtt{t}_3 \wedge \mathtt{t}_3 \prec \mathtt{t}_2 \wedge \varphi'_t[\mathtt{t}_3/\mathtt{t}_2])$, where $v(\mathtt{t}_1)$=s_1 and $v(\mathtt{t}_2)$=s_2, iff $M_t, v \models (\langle \mathtt{B}\rangle\varphi')_t$. This completes the proof in this case; we leave the remaining cases to the reader. ∎

Clearly, there are first-order interpretations that are not faithful to *any* modal interpretation. For example, a first-order interpretation need not associate a partial order with "$\prec$". However, we can exclude such uninteresting interpretations by expressing the appropriate properties of time in a first-order formula. Let `po` be the first-order formula saying that '$\preceq$' denotes a partial order, and `li` the first-order formula saying that intervals are linear (that is, the set of points that lie between any two points is totally ordered). Recall that these were the only assumptions we made about temporal structures.

$$\mathtt{po} =_{def} \forall \mathtt{t}_1,\mathtt{t}_2,\mathtt{t}_3(\mathtt{t}_1 \preceq \mathtt{t}_1 \wedge ((\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \mathtt{t}_2 \preceq \mathtt{t}_1) \supset \mathtt{t}_1 = \mathtt{t}_2) \wedge ((\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \mathtt{t}_2 \preceq \mathtt{t}_3) \supset \mathtt{t}_1 \preceq \mathtt{t}_3)$$

$$\mathtt{li} =_{def} \forall \mathtt{t}_1,\mathtt{t}_2,\mathtt{t}_3,\mathtt{t}_4((\mathtt{t}_1 \preceq \mathtt{t}_3 \preceq \mathtt{t}_2 \wedge \mathtt{t}_1 \preceq \mathtt{t}_4 \preceq \mathtt{t}_2) \supset (\mathtt{t}_3 \preceq \mathtt{t}_4 \vee \mathtt{t}_4 \preceq \mathtt{t}_3))$$

$$\mathtt{ok} =_{def} \mathtt{po} \wedge \mathtt{li}$$

These definitions immediately give us:

Lemma 7.3: *If M_t is a first-order interpretation such that $M_t \models$ `ok`, then there is a modal interpretation M such that M_t is faithful to M.*

Proposition 7.4: *Let φ be a modal formula, $\mathcal{A}$ a class of temporal structures, and $\mathtt{time}_\mathcal{A}$ a first-order sentence (i.e. a formula with no free variables) whose only relation symbols are '$\prec$' and '$=$' (that is, it expresses some property of time) and whose class of models is exactly $\mathcal{A}$. Moreover, suppose that $\mathtt{time}_\mathcal{A} \supset \mathtt{ok}$ is valid. Then for any first-order interpretation M_t, if $M_t \models \mathtt{time}_\mathcal{A} \wedge \exists \mathtt{t}_1,\mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$, then there exists a modal interpretation $M = \langle S, V\rangle$ such that M_t is faithful to M, $S \in \mathcal{A}$, and for some time points s_1, s_2 with $s_1 \leq s_2$, we have $M, \langle s_1, s_2\rangle \models \varphi$. Conversely, if $M = \langle S, V\rangle$ is a modal interpretation such that $S \in \mathcal{A}$, and for some time points s_1, s_2 with $s_1 \leq s_2$, we have $M, \langle s_1, s_2\rangle \models \varphi$, and M_t is a first-order interpretation faithful to M, then $M_t \models \mathtt{time}_\mathcal{A} \wedge \exists \mathtt{t}_1,\mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$.*

Proof: Suppose $M_t \models \mathtt{time}_\mathcal{A} \wedge \exists\ \mathtt{t}_1,\mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$. Since $\mathtt{time}_\mathcal{A} \supset \mathtt{ok}$ is valid, it follows from Lemma 7.3 that there is a modal interpretation M such that M_t is faithful to M. By

Lemma 7.2, it now easily follows that there exist time points s_1, s_2 in M with $s_1 \leq s_2$ such that that there is a modal interpretation M and time points s_1, s_2 in M such that $s_1 \leq s_2$ and $M, \langle s_1, s_2 \rangle \models \varphi$. The proof of the converse is similar and left to the reader. ∎

Corollary 7.5: *Let φ be a modal formula, $\mathcal{A}$ a class of temporal structures, and* $\mathtt{time}_{\mathcal{A}}$ *a first-order sentence whose only relation symbols are '$\prec$' and '$=$' and whose class of models is exactly $\mathcal{A}$. Then*

1. *the first-order formula* $\mathtt{time}_{\mathcal{A}} \wedge \exists \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$ *is satisfiable iff the modal formula φ is satisfiable with respect to $\mathcal{A}$,*

2. *the first-order formula* $\mathtt{time}_{\mathcal{A}} \supset \forall \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \supset \varphi_t)$ *is valid iff the modal formula φ is valid with respect to $\mathcal{A}$.*

Proof: Again, we just briefly give the idea of the proof. For part 1, suppose $\mathtt{time}_{\mathcal{A}} \wedge \exists \mathtt{t}_1, \mathtt{t}_2 (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$ is satisfiable say in the first-order structure M_t. Since $\mathtt{time}_{\mathcal{A}}$ is a sentence, it follows that $M_t \models \mathtt{time}_{\mathcal{A}} \wedge \exists \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$.[1] The result now follows from Proposition 7.4. The proof of the converse is similar, as is the proof of part 2. ∎

We are now ready to apply some known results on first-order logic. First we recall the well-known Löwenheim-Skolem theorem (see, e.g., [7]):

Theorem 7.6: *(Löwenheim-Skolem) If a first-order formula is satisfiable, then it is satisfiable in a countable interpretation.*

Corollary 7.7: *If a formula in our modal logic is satisfiable in some temporal structure, then it is satisfiable in a countable structure.*

Proof: Let φ be a modal formula. If φ is satisfiable in some modal interpretation, then by Corollary 7.5, $\mathtt{ok} \wedge \exists \mathtt{t}_1, \mathtt{t}_2 (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$ is satisfiable in some first-order interpretation. By Theorem 7.6, this last formula is satisfiable also in some countable interpretation M_t. By Proposition 7.4 there exists a modal structure M and points t_1, t_2 such that $t_1 \leq t_2$ and $M, \langle t_1, t_2 \rangle \models \varphi$, with M_t faithful to M. Finally, we note that by definition, since M_t is countable and faithful to M, M too is countable. ∎

The next theorem is originally due to Cantor, and is proved by the well-known zig-zag argument [7]:

Theorem 7.8: *Any two countable, linear, dense and unbounded structures are isomorphic with respect to '$<$' (i.e., there exists a bijection f between the two structures, such that $t_1 < t_2$ iff $f(t_1) < f(t_2)$).*

---

[1] Note this would not necessarily be the case if $\mathtt{time}_{\mathcal{A}}$ were an open formula, say with free variable $\mathtt{x}_1, \ldots, \mathtt{x}_k$, for then the satisfiability of $\mathtt{time}_{\mathcal{A}} \wedge \exists \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$ in M_t would amount to $M_t \models \exists \mathtt{x}_1, \ldots, \mathtt{x}_k (\mathtt{time}_{\mathcal{A}}) \wedge \exists \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$, while $M_t \models \mathtt{time}_{\mathcal{A}} \wedge \exists \mathtt{t}_1, \mathtt{t}_2 (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$ corresponds to $M_t \models \forall \mathtt{x}_1, \ldots, \mathtt{x}_k (\mathtt{time}_{\mathcal{A}}) \wedge \exists \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$.

Theorem 7.9: *A formula in our modal logic is valid with respect to the class of dense, linear and unbounded structures iff it is valid in the rationals, $\mathcal{Q}$.*

Proof: Clearly if a formula is valid with respect to all dense, linear, and unbounded structures, it is valid in $\mathcal{Q}$. For the converse, it suffices to show that if a formula is satisfiable in some dense, linear and unbounded structure, then it is also satisfiable in $\mathcal{Q}$. Let φ be a formula and S a dense, linear and unbounded structure, such that for some modal interpretation $M = \langle S, V\rangle$ and time points t_1, t_2, we have $M, \langle t_1, t_2\rangle \models \varphi$. Let φ_t be the first-order counterpart of φ. Also, let `dense1`, `linear1`, and `unbounded1` be the three first-order formulas asserting that time is (respectively) dense, linear and unbounded (their definition is straightforward, and is omitted). Let $\mathtt{dlu} =_{def} \mathtt{dense1} \wedge \mathtt{linear1} \wedge \mathtt{unbounded1}$. Note that $\mathtt{dlu} \supset \mathtt{ok}$. Let $M_t = \langle S, V_t\rangle$ be a first-order interpretation that is faithful to M. By Proposition 7.4, we have $M_t \models \mathtt{dlu} \wedge \exists \mathtt{t}_1, \mathtt{t}_2 (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$. By Theorem 7.6, there is a countable interpretation M'_t such that $M'_t \models \mathtt{dlu} \wedge \exists \mathtt{t}_1, \mathtt{t}_2 (\mathtt{t}_1 \preceq \mathtt{t}_2 \wedge \varphi_t)$. Taking $M' = \langle S', V'\rangle$ to be the modal interpretation to which M'_t is faithful, by Proposition 7.4 we have that $M', \langle t_3, t_4\rangle \models \varphi$ for some time points t_3 and t_4. By Theorem 7.8, we have that $\mathcal{Q}$ and S' are isomorphic with respect to '$<$'. Let V'' be the valuation that corresponds to V' under the isomorphism; thus, for example, if $f : S \rightarrow \mathcal{Q}$ is the isomorphism, we $\langle t_1, t_2\rangle \in V'(p_t)$ iff $\langle f(t_1), f(t_2)\rangle \in V''(p_t)$. It is easy to check that $\langle \mathcal{Q}, V''\rangle \models \varphi$. ∎

Corollary 7.10: *If a formula in our modal logic is satisfiable in $\mathcal{R}$ then it is also satisfiable in $\mathcal{Q}$.*

8 The complexity of the validity problem

We now turn our attention to the complexity of the validity problem for the logic. We begin with a brief review of the notions of Π^1_1 and its dual Σ^1_1. Further details can be found in [25] or any other standard textbook of recursive function theory.

Formulas of *second-order arithmetic with set variables* consist of formulas of first-order arithmetic (that is, in the language with constant symbols **0** and **1**, together with the function symbols $+$ and $\times$) augmented with expressions of the form $x \in X$, where x is a number variable and X is a set variable, together with quantification over set variables and number variables. A *sentence* is a formula with no free variables. Second-order arithmetic with set variables is a very powerful language. For example, the following (true) sentence of the language expresses the law of mathematical induction over the natural numbers:

$$\forall X(0 \in X \wedge \forall x((x \in X \supset x+1 \in X) \supset \forall x(x \in X)))$$

A Π^1_1 sentence (resp. Σ^1_1 sentence) of second-order arithmetic with set variables is one of the form $\forall X_1 \ldots \forall X_n \varphi$ (resp. $\exists X_1 \ldots \exists X_n \varphi$), where φ is a formula of second-order arithmetic with set variables whose free set variables are among $X_1, \ldots, X_n$ that has no quantification over set variables. A set A of natural numbers is in Π^1_1 (resp. Σ^1_1) if there is a Π^1_1 sentence (resp. Σ^1_1

sentence) $\psi(x)$ with one free number variable x and no free set variables such that $a \in A$ iff $\psi(a)$ holds. Π^1_1 hardness and completeness are defined in the obvious way (the reduction is via one-one recursive functions). It is well-known that Π^1_1-hard sets are not recursively enumerable (see [25]). In particular, if the validity problem for a class of temporal structures is Π^1_1-hard, it follows that there can be no complete (recursive) axiomatization for the formulas that are valid with respect to that class of structures.

Later in the paper we will also briefly consider the notion of Π^2_1 and its dual Σ^2_1. In order to define these notions, we need to go to *third-order arithmetic*, which is the result of taking second-order arithmetic with set variables, and further augmenting to allow expressions of the form $X \in \mathcal{X}$, where X is a set variable and $\mathcal{X}$ is a set of sets variable, together with quantification over set of sets variables (as well as set variables and number variables). Again we take a sentence to be a formula with no free variables. A Π^2_1 sentence (resp. Σ^2_1 sentence) of third-order arithmetic is one of the form $\forall \mathcal{X}_1 \ldots \forall \mathcal{X}_n \varphi$ (resp. $\exists \mathcal{X}_1 \ldots \exists \mathcal{X}_n \varphi$), where φ is a formula of third-order arithmetic that has no quantification over set variables or set of sets variables. The definition of A being in Π^2_1- or Π^2_1-hard is analogous to that for Π^1_1.

The degree to which the complexity of our logic depends on the underlying temporal structure is striking; depending on the class of temporal structures being considered, the validity problem ranges from being decidable to being Π^1_1-hard (correspondingly, the satisfiability problem ranges from being decidable to being Σ^1_1-hard.) Actually, we show that for most interesting classes of temporal structures validity and satisfiability are undecidable. One gets decidability only in very restricted cases, such as when the set of temporal models considered is a finite collection of structures, each consisting of a finite set of natural numbers (since in this case one can simply perform an exhaustive check on all structures). The various hardness properties hold even if we weaken the logic by restricting it to the `B`, `E` and `A` operators. We also discuss upper bounds for these problems.

8.1 Lower bounds

To make our results precise, we need a few brief definitions. A temporal structure is said to contain an *infinitely ascending sequence* if it contains an infinite sequence of points $t_0, t_1, t_2, \ldots$ such that $t_i < t_{i+1}$. Note that any unbounded structure contains an infinite ascending sequence. A class of temporal structures contains an infinitely ascending sequence if *at least one* of the structures in it does. We have already defined *complete* temporal structures; those in which any sequence with an upper bound has a least upper bound. A class of temporal structures is said to be complete if *all* structures in the class are complete. A class $\mathcal{A}$ of structures is said to have *unboundedly ascending sequences* if for any natural number n there is a structure $T \in \mathcal{A}$ which contains a sequence $t_1, t_2, \ldots, t_n$ such that $t_i < t_{i+1}$, $0 < i < n$.

We now state all our lower bound results, and then prove them in detail.

Theorem 8.1: *The validity problem for any class of temporal structures that contains an infinitely ascending sequence is r.e.-hard.*

Corollary 8.2: *The validity problem for $\mathcal{N}$, $\mathcal{Q}$, and $\mathcal{R}$ is r.e.-hard.*

In fact, Theorem 8.1 tells us that the validity problem for almost any interesting class of temporal structures will be r.e.-hard. For example, we have:

Corollary 8.3: *The validity problem for each of the following classes of temporal structures is r.e.-hard:*

1. *The class of all temporal structures.*
2. *The class of all linear temporal structures.*
3. *The class of all discrete temporal structures.*
4. *The class of all dense temporal structures.*
5. *The class of all dense, linear, unbounded temporal structures.*

In the case of classes that are complete as well as containing an infinitely ascending sequence, we can show that the validity problem is even harder.

Theorem 8.4: *The validity problem for complete classes of temporal structures which contain an infinitely ascending sequence is* Π_1^1*-hard.*

Corollary 8.5: *The validity problem for* $\mathcal{R}$ *and for* $\mathcal{N}$ *is* Π_1^1*-hard.*

Even for classes of structures which contain no infinite ascending sequence we can often get undecidability results:

Theorem 8.6: *The validity problem for any complete class of temporal structures which has unboundedly ascending sequences is co-r.e.-hard.*

Let $\mathcal{K}$ be the set of temporal structures consisting of the initial segments of the natural numbers, with the usual ordering:

$$\mathcal{K} \;=\; \{\langle\ [0..n]\ ,\leq\rangle :\ n=0,1,2,\ldots\},$$

$\mathcal{K}$ is useful, for example, when reasoning about possible computations of a program, knowing that the computation is finite but having no bound on its length.

Corollary 8.7: *The validity problem for* $\mathcal{K}$ *is co-r.e.-hard.*

8.2 Proofs of the lower bounds

The proofs for all these results are quite similar. The idea is to construct formulas that essentially encode the computation of a Turing machine. For Theorem 8.1, we construct a formula that is satisfiable iff the TM started on a blank tape never halts. Since the non-halting problem is co-r.e.-hard, this makes satisfiability co-r.e.-hard, and thus validity r.e.-hard. For Theorem 8.4, we construct a formula that is satisfiable iff there is a computation of the TM that enters the start state infinitely often. For nondeterministic TM's, this problem is known to be Σ_1^1-hard [10], so this gives us that satisfiability is Σ_1^1-hard, and thus that validity is Π_1^1-hard. Finally, for Theorem 8.6, we construct a formula that is satisfiable iff the TM halts.

We proceed as follows. Fix a TM M (the construction we are about to describe is independent of whether M is deterministic). We assume without loss of generality that M only writes the symbols 0 and 1. Let Q be the set of M's states, with q_0 the unique start state and q_f the unique halting state. We will assume that our language contains all the primitive propositions in the set $L = \{0,1,^*,\#,(q,0),(q,1),(q,B)\text{: } q{\in}Q\}$, as well as the proposition `corr` which we discuss later.

The computation of M started on a blank tape in state q_0 is encoded as a sequence of IDs separated by pairs of asterisks: `* ID1 ** ID2 ** ID3 **` .... Each ID consists of a sequence of cells. Just as in Section 6, a cell is an interval whose first and last points satisfy $\#$, and whose interior satisfies the "content" of cell, which is one of the elements of L. (Thus, we allow for contents other than `p`, which was the only content considered in Section 6.) As usual, 0 means that the content of the cell is 0 and that the head is not pointing at the cell, and likewise for 1. Similarly, (q,0) means that the content of the cell is 0, that the head is pointing at the cell, and that M is in state q. A similar statement holds of (q,1) and (q,B). B represents the "blank" tape symbol. Thus we have the following slight modification of the definitions of Section 6:

$$\begin{aligned}\texttt{cell}(l) &=_{def} [[\texttt{BP}]]\texttt{\#} \wedge [[\texttt{EP}]]\texttt{\#} \wedge [\texttt{D}]l \wedge \langle\texttt{D}\rangle l\\ \texttt{cell} &=_{def} \textstyle\bigvee_{l\in\ L,\ l\neq\#}\ \texttt{cell}(l)\end{aligned}$$

(Note that since the Turing machine is finite then so is L, and hence so is the above disjunction.)

An ID is simply an interval delimited by `*`-cells, with at least one non-* cell in its interior.

$$\texttt{ID} =_{def} \langle\texttt{B}\rangle\texttt{cell(*)} \wedge \langle\texttt{E}\rangle\texttt{cell(*)} \wedge \langle\texttt{D}\rangle\texttt{cell} \wedge \neg\langle\texttt{D}\rangle\texttt{cell(*)}$$

A final (resp. start) ID is an ID such that one of its cells has the head in the final (resp. start) state:

$$\begin{aligned}\texttt{final-ID} &=_{def} \texttt{ID} \wedge \langle\texttt{D}\rangle(\texttt{cell}((q_f,\texttt{0})) \vee \texttt{cell}((q_f,1)) \vee \texttt{cell}((q_f,\texttt{B})))\\ \texttt{start-ID} &=_{def} \texttt{ID} \wedge \langle\texttt{D}\rangle(\texttt{cell}((q_0,0)) \vee \texttt{cell}((q_0,1)) \vee \texttt{cell}((q_0,\texttt{B})))\end{aligned}$$

For convenience, we define a new modal operator `F`.

$$\begin{aligned}\langle\texttt{F}\rangle\varphi &=_{def} \langle\texttt{A}\rangle\varphi \vee \langle\texttt{L}\rangle\varphi\\ [\texttt{F}]\varphi &=_{def} \neg\langle\texttt{F}\rangle\neg\varphi\end{aligned}$$

Intuitively, $[\mathtt{F}]\varphi$ says that φ holds of all future intervals.

We want to force there to be an infinite sequence of IDs or a finite one ending with a final-ID. This is the job of the following formula:

$$\texttt{ID-sequence} =_{def} [\mathtt{F}]((\texttt{ID} \wedge \neg\texttt{final-ID}) \supset \langle\mathtt{A}\rangle\texttt{ID})$$

Definition 8.8: We say that there is *a computation starting from* s_0 if either there exists a finite sequence $s_0, s_1, \ldots, s_k$, $k \geq 1$, such that $s = s_0$, $s_0 < s_1 < \ldots < s_k$, $\langle s_i, s_{i+1}\rangle \models \texttt{ID}$ for $i < k$ and $\langle s_{k-1}, s_k\rangle \models \texttt{final-ID}$, or there exists an infinite sequence $s_0 < s_1 < s_2 < \ldots$ and $\langle s_i, s_{i+1}\rangle \models \texttt{ID}$. ∎

The following is immediate from the definition:

Lemma 8.9: $M, \langle s_0, s_0\rangle \models \texttt{ID-sequence}$ *iff there is a computation starting from* s_0.

We next want to write formulas which force the sequence of IDs to encode the computation of the TM M starting on a blank tape. We first need to make sure that the contents of each cell are unique.

$$\texttt{unique-val} =_{def} [\mathtt{F}](\textstyle\bigwedge_{l,l' \in L, l \neq l'} l \supset \neg l')$$

We next want to make sure that the computation starts right and continues right. In order to do this, we need a few preliminary formulas. The formula `2-cell(x,y)` holds of an interval in case it consists of two consecutive cells, with respective contents `x` and `y`. Similarly, `3-cell(x,y,z)` holds of an interval just in case the interval consists of three consecutive cells, with respective contents `x`, `y` and `z`.

$$\begin{aligned}&\texttt{2-cell(x,y)} =_{def}\\ &\quad\langle\mathtt{B}\rangle\texttt{cell(x)} \wedge \langle\mathtt{E}\rangle\texttt{cell(y)} \wedge [\mathtt{D}]([[\mathtt{BP}]]\texttt{\#} \wedge [[\mathtt{EP}]]\texttt{\#} \supset \texttt{length0})\end{aligned}$$

$$\begin{aligned}\texttt{3-cell(x,y,z)} =_{def}\ &\langle\mathtt{B}\rangle\texttt{cell(x)} \wedge \langle\mathtt{E}\rangle\texttt{cell(z)} \wedge \langle\mathtt{D}\rangle\texttt{cell(y)} \wedge\\ &[\mathtt{D}]([[\mathtt{BP}]]\texttt{\#} \wedge [[\mathtt{EP}]]\texttt{\#} \supset (\texttt{length0} \vee \texttt{cell(y)})\end{aligned}$$

The following formula now guarantees that the first ID encodes a blank tape with M in the initial state

$$\texttt{init-ID} =_{def} [\mathtt{A}](\texttt{ID} \supset \texttt{3-cell(*,(}\, q_0, \mathrm{B}\texttt{),*)})$$

Finally, we want to ensure that consecutive IDs obey the rules of the transition function. To that end we will use the proposition `corr` (read: "corresponds"), which will be true of an interval iff the interval starts and ends with a cell, and these cells are corresponding cells in consecutive IDs. In the following diagram, each dashed segment represents an interval for which `corr` holds:

```
ID:          n                  n+1               n+2
   #*# # #  ...  #*#*# # #   ...     #*#*# #   ...
   |-----------------|
     |-----------------|
       |-----------------|
            .....
                    |---------------------|
                      |---------------------|
                               ......
```

The formulas described below guarantee this property of `corr`. The first formula guarantees that an interval for which `corr` is true starts and ends with a cell:

$$\texttt{cell-rule} =_{def} \texttt{corr} \supset (\langle \texttt{B}\rangle\texttt{cell} \wedge \langle \texttt{E}\rangle\texttt{cell})$$

Also, an interval that starts with an ID and ends with the * that starts the next ID or ends with an ID and starts with the * that ends the previous ID satisfies `corr`:

$$\begin{aligned}\texttt{ID-rule} =_{def} \; & ((\langle \texttt{B}\rangle\texttt{ID} \wedge \langle \texttt{E}\rangle(\texttt{2-cell(*,*)}) \wedge \neg\langle \texttt{D}\rangle\texttt{ID}) \supset \texttt{corr}) \wedge \\ & ((\langle \texttt{E}\rangle\texttt{ID} \wedge \langle \texttt{B}\rangle(\texttt{2-cell(*,*)}) \wedge \neg\langle \texttt{D}\rangle\texttt{ID}) \supset \texttt{corr})\end{aligned}$$

Next, we stipulate that one `corr` interval may not properly contain another:

$$\texttt{not-contains-corr} =_{def} \texttt{corr} \supset (\neg\langle \texttt{B}\rangle\texttt{corr} \wedge \neg\langle \texttt{D}\rangle\texttt{corr} \wedge \neg\langle \texttt{E}\rangle\texttt{corr})$$

The next formula states that if an interval starts with a cell and ends with an interval satisfying `corr`, then it starts with an interval satisfying `corr`:

$$\texttt{corr-starts} =_{def} (\langle \texttt{B}\rangle\texttt{cell} \wedge \langle \texttt{E}\rangle\texttt{corr}) \supset \langle \texttt{B}\rangle\texttt{corr}$$

Similarly, if an interval ends with a cell which is not the last cell of an ID, and starts with an interval satisfying `corr`, then it ends with an interval satisfying `corr`. Note that we do not require that the last cell of an ID end a `corr` interval since it may not correspond to a cell in the previous ID. This can happen if the head was in the last cell of the previous ID and moved right.

$$\texttt{corr-ends} =_{def} (\langle \texttt{E}\rangle\texttt{cell} \wedge \langle \texttt{A}\rangle(\texttt{cell} \wedge \neg\texttt{cell(*)}) \wedge \langle \texttt{B}\rangle\texttt{corr}) \supset \langle \texttt{E}\rangle\texttt{corr}$$

Finally, the formula `corr-properties` says that the properties of `corr` described above hold of every future interval:

$$\begin{aligned}&\texttt{corr-properties} =_{def} \\ &\quad [\texttt{F}](\texttt{cell-rule} \wedge \texttt{ID-rule} \wedge \texttt{not-contains-corr} \wedge \texttt{corr-starts} \wedge \texttt{corr-ends})\end{aligned}$$

Definition 8.10: We say an interval $\langle s,t\rangle$ can be *subdivided into cells* if there exist points $s_0, s_1, \ldots, s_k$ such that $s_0 = s$, $s_k = t$, $s_0 < \ldots < s_k$, and $\langle s_i, s_{i+1}\rangle \models cell$. Note that provided `unique-val` holds, there is at most one way of subdividing an interval into cells. ∎

Our next formula guarantees that an interval which starts and ends with a cell can either be subdivided into a finite number of cells or contains an infinite sequence of cells:

$$\texttt{subdivide} =_{def} (\langle \texttt{B}\rangle\texttt{cell} \wedge \langle \texttt{E}\rangle\texttt{cell}) \supset [\texttt{E}]([[\texttt{BP}]]\texttt{\#} \supset (\texttt{cell} \vee \langle \texttt{B}\rangle\texttt{cell}))$$

Lemma 8.11: *If $\langle s,t\rangle \models$ `subdivide` then either $\langle s,t\rangle$ can be subdivided into a finite number of cells, or else there is a sequence $s_0, s_1, \ldots$ and a point t' such that $s_0 = s$, $s_0 < s_1 < \ldots t' < t$, and $\langle s_i, s_{i+1}\rangle \models$ `cell`.*

Proof: Since $\langle s,t\rangle \models \langle$B$\rangle$`cell` $\wedge \langle$E$\rangle$`cell`, there must be some s_1, t' with $s < s_1 \leq t' < t$ such that $\langle s, s_1\rangle \models$ `cell` and $\langle t, t'\rangle \models$ `cell`. Now we show by induction on k, using `subdivide`, that if we have found $s_0, \ldots, s_k$ with $s_0 < \ldots < s_k \leq t'$ and $\langle s_i, s_{i+1}\rangle \models$ `cell` for $i < k$, then either $s_k = t'$ or there exists s_{k+1} such that $s_k < s_{k+1}$ and $\langle s_k, s_{k+1}\rangle \models$ `cell`. ∎

Lemma 8.12: *Suppose $\langle s_0, s_0\rangle \models$ `corr-properties` $\wedge$ `unique-val` $\wedge$ `init-ID` $\wedge$ [F]`subdivide`, and suppose that there is a (finite or infinite) computation $s_0, s_1, s_2, \ldots$ starting from s_0. Then for each $i \geq 0$, the interval $\langle s_{i+1}, s_{i+2}\rangle$ can be subdivided into a finite number of cells which is the same or one more than the interval $\langle s_i, s_{i+1}\rangle$. Moreover, the intervals which satisfy* `corr` *and which either start in $\langle s_i, s_{i+1}\rangle$ or end in $\langle s_{i+1}, s_{i+2}\rangle$ are exactly those which start with a cell in $\langle s_i, s_{i+1}\rangle$ and end with the corresponding cell in $\langle s_{i+1}, s_{i+2}\rangle$.*

Proof: The formula `init-ID` guarantees that the first interval $\langle s_0, s_1\rangle$ can be subdivided into exactly three cells. Now suppose that the interval $\langle s_i, s_{i+1}\rangle$ can be subdivided into a finite number of cells. That is, there exist points $t_0, \ldots, t_k$ such that $s_i = t_0 < \ldots < t_k = s_{i+1}$ such that $\langle t_j, t_{j+1}\rangle \models$ `cell` for $j < k$.

Now look at the interval $\langle s_{i+1}, s_{i+2}\rangle$. Since it starts with a *-cell and ends with one, from `subdivide` we have that it starts with a sequence of cells.

To prove the lemma we show

1. this sequence must be at least k cells long,
2. among all the intervals that either start with a cell in $\langle s_i, s_{i+1}\rangle$ or end with one of the first $k-1$ cells in $\langle s_{i+1}, s_{i+2}\rangle$, `corr` holds of exactly those that start and end with corresponding cells in the two intervals, and
3. this sequence is at most $k+1$ cells long.

Note that since we have $\langle s_i, s_{i+1}\rangle \models$ `ID` and $\langle s_{i+1}, s_{i+2}\rangle \models$ `ID`, it follows that there exists t and u with $s_i < t < s_{i+1} < u < s_{i+2}$ such that $\langle t, s_{i+1}\rangle \models$ `cell(*)` and $\langle s_{i+1}, u\rangle \models$ `cell(*)`. By `ID-rule`, it follows that $\langle s_i, u\rangle \models$ `corr` and $\langle t, s_{i+2}\rangle \models$ `corr`; i.e., the interval $\langle s_i, s_{i+2}\rangle$ can be decomposed into two consecutive subintervals, each satisfying `corr`. From `corr-starts`, it follows that each of the k cells in $\langle s_i, s_{i+1}\rangle$ starts an interval satisfying `corr`. These intervals must all end between u and s_{i+2}, for if they did not, we could obtain a contradiction using `not-contains-corr` and the fact that both $\langle s_i, u\rangle$ and $\langle t, s_{i+2}\rangle$ satisfy `corr`. Moreover, from `not-contains-corr` we have that no two distinct intervals satisfying `corr` can end with the same cell. Therefore there must be at least k distinct cells in the interval $\langle s_{i+1}, s_{i+2}\rangle$. From Lemma 8.11, it follows that this interval must actually start with at least k cells. This proves (1).

For (2), we show by induction on j that for $j < k$, the formula `corr` holds for an interval starting with the j^{th} cell in $\langle s_i, s_{i+1}\rangle$ and ending with the j^{th} cell in $\langle s_i, s_{i+1}\rangle$. For $j = 1$, we have already shown that `corr` holds of the appropriate interval. If $j > 1$, we know that the j^{th} cell in $\langle s_i, s_{i+1}\rangle$ starts a `corr` interval. It cannot end before the j^{th} cell in the second ID without violating `not-contains-corr`. If it ends in a cell after the j^{th} cell, then by `corr-ends` we know that some `corr` interval ends with the j^{th} cell in the second ID. But now `not-contains-corr` and our inductive hypothesis tell us that this interval can start neither before nor after the j^{th} cell in $\langle s_i, s_{i+1}\rangle$. Thus `corr` holds for the required intervals. Another application of `not-contains-corr` shows that there can be no other intervals satisfying `corr` that either start in $\langle s_i, s_{i+1}\rangle$ or end with one of the first $k-1$ cells in $\langle s_{i+1}, s_{i+2}\rangle$.

For (3), observe that if the sequence is not either k or $k+1$ cells long, then from Lemma 8.11 it starts with at least $k+2$ cells, and the $(k+1)^{\text{st}}$ cell does not have contents $*$. Thus, by `corr-ends`, there is a an interval satisfying `corr` ending with the $(k+1)^{\text{st}}$ cell. Another application of `not-contains-corr` quickly leads to a contradiction. ∎

We are now in a position to ensure that the computation proceeds according to the transition function of M. Notice that the contents of any three consecutive cells determines the contents of the cell in the next ID which corresponds to the middle cell. Suppose the function δ describes this transition, so that if three consecutive cells in an ID are `i`, `j` and `k`, then δ`(i,j,k)` describes the contents of the cell in the next ID that corresponds to `j`. (Note that for a nondeterministic TM this function is really a relation). The following formula guarantees that the transitions of M are obeyed at all intervals in the future:

$$\texttt{obeys-}\delta =_{def}$$
$$[\texttt{F}](\textstyle\bigwedge_{i,j,k\in L}\ ((\texttt{corr} \land \langle\texttt{B}\rangle\texttt{3-cell(i,j,k)}) \supset [\texttt{A}](\texttt{cell} \supset \texttt{cell}(\delta\texttt{(i,j,k)}))))$$

Notice that this is the only formula where the details of the particular TM M play a role.

We can now finally define the formula `computation`:

$$\texttt{computation} =_{def}$$
$$\texttt{length0} \land \texttt{unique-val} \land \texttt{ID-sequence} \land \texttt{init-ID} \land$$
$$[\texttt{F}]\texttt{subdivide} \land [\texttt{F}]\texttt{corr-properties} \land \texttt{obeys-}\delta$$

Definition 8.13: Suppose $s_0, s_1, \ldots$ is a computation starting with s_0. This sequence *encodes a computation of M* if each interval $\langle s_i, s_{i+1}\rangle$ can be subdivided into a finite number of cells, and there is a complete computation **comp** of M (either infinite or ending with M in a final state) such that the j^{th} cell of the interval $\langle s_i, s_{i+1}\rangle$ is the same as the j^{th} cell in the i^{th} ID of **comp**. ∎

Lemma 8.14: *Let* $\langle s,t\rangle \models$ `computation`*. Then there is a computation starting with s. Moreover, any computation starting with s encodes a computation of M.*

Proof: Since $\langle s,t\rangle \models$ `computation`, in particular $\langle s,t\rangle \models$ `length0`, so that $s = t$. By Lemma 8.9 there is a sequence starting with s which encodes an infinite sequence of IDs. Suppose that $s_0, s_1, \ldots$ encodes an infinite sequence of IDs, with $s = s_0$. We must show that it encodes a computation of M. By Lemma 8.12 each interval encoding an ID can be subdivided into a finite number of cells. Lemma 8.12 also tells us that intervals starting and ending with corresponding cells in consecutive IDs satisfy `corr`. The formula `obeys-`δ is easily seen to guarantee that corresponding cells in consecutive IDs match up right, so that we really are encoding a prefix of a computation of M in any finite sequence of IDs, and a complete legal computation of M in any infinite sequence of IDs. ∎

All of the above constitutes the part common to all proofs. The proofs diverge on the punchline. We prove Theorem 8.1 by encoding the nonhalting problem, using the definition of `final-ID` given earlier. If M is deterministic, then the formula

`computation` $\wedge\ \neg\langle$`F`$\rangle$`final-ID`

is satisfiable in a class of temporal structures containing an infinitely ascending sequence exactly when M does not halt on a blank tape. This proves Theorem 8.1.

At first it might appear that we could strengthen the result by encoding the halting problem (rather than the nonhalting problem), by considering instead the formula

`computation` $\wedge\ \langle$`F`$\rangle$`final-ID`

Unfortunately, in general this alone will not suffice. Depending on the particular set of temporal structures that are being considered, this formula can be satisfied by "nonstandard" (or, at least, unintended) computations of the TM. For example, if we are considering any dense structure (e.g., $\mathcal{Q}$ or $\mathcal{R}$), there is nothing to exclude a structure which satisfies the conjuncts described thus far and which has the following form:

```
|-----------|-------|-----|---|...|||     ...      |-|
    ID1         ID2    ID3  .....                final-ID
```

In other words, we have not precluded models in which the computation is captured by an infinite sequence of intervals which "telescope" to the right, followed by an interval satisfying `final-ID`. Although this structure would satisfy the formula `computation` $\wedge\ \langle$`F`$\rangle$`final-ID`, it would not tell us that the Turing machine necessarily halted.

We are able to exclude such nonstandard models of computation in complete structures.

First we add a conjunct that in complete structures eliminates the possibility of an infinite number of cells:

$$\texttt{no-telescope} =_{def} \neg\langle \texttt{B}\rangle[\texttt{E}]\langle\texttt{D}\rangle\texttt{cell}$$

Lemma 8.15: *If $M = \langle S, V\rangle$, S is a complete structure, and $M, \langle s,t\rangle \models$ `no-telescope`, then there can be no sequence $s_0, s_1, \ldots$ and point t' such that $s = s_0$, $s_0 < s_1 < \ldots t' < t$ and $M, \langle s_i, s_{i+1}\rangle \models$ `cell`.*

Proof: Suppose there were such a sequence. Let s' be the least upper bound of $s_0, s_1, \ldots$. (Such as s' exists since S is complete.) Then it is easy to check that $M, \langle s, s'\rangle \models [\texttt{E}]\langle\texttt{D}\rangle\texttt{cell}$, contradicting our assumption that $M, \langle s,t\rangle \models$ `no-telescope`. ∎

Thus, the conjunction `subdivide` $\wedge$ `no-telescope` guarantees that any interval (in a complete structure) that starts and ends with a cell can be subdivided into a finite number of cells. We need one more formula which guarantees that if an interval starts and ends with an ID, then it can be subdivided into a finite number of IDs. Define

$$\texttt{subdivide-ID} =_{def} (\texttt{ID} \supset [\texttt{A}](\texttt{cell} \supset \texttt{cell(*)})) \wedge \neg\texttt{3-cell(*,*,*)}$$

Finally, let

$$\texttt{standard} =_{def} [\texttt{F}](\texttt{subdivide} \wedge \texttt{no-telescope} \wedge \texttt{subdivide-ID})$$

We leave it to the reader to check the following lemma

Lemma 8.16: *If $M = \langle S, V\rangle$, S is a complete structure, and $M, \langle s,t\rangle \models \texttt{standard} \wedge \langle\texttt{B}\rangle\texttt{ID} \wedge \langle\texttt{E}\rangle\texttt{ID}$, then $\langle s,t\rangle$ can be subdivided into a finite number of IDs.*

It is now easy to check that in a complete structure, if the formula

$$\texttt{computation} \wedge \texttt{standard} \wedge \langle\texttt{B}\rangle\texttt{start-ID} \wedge \langle\texttt{E}\rangle\texttt{final-ID}$$

is satisfiable then there is a halting computation of the Turing machine M. Moreover, if there is a halting computation of the Turing machine M, this formula is satisfiable in any complete class of temporal structures which has unboundedly ascending sequences. This proves Theorem 8.6.

Finally, we encode the question of whether the TM returns infinitely often to its start state. Observe that if the formula

$$\texttt{computation} \wedge \texttt{standard} \wedge [\texttt{F}](\texttt{start-ID} \supset \langle\texttt{L}\rangle\texttt{start-ID})$$

is satisfiable in a complete structure, then there is a computation where M returns infinitely often to its start state. Moreover, if M does return infinitely often to its start state, this formula is satisfiable in any complete class of temporal structures which contains an infinite ascending sequence. This, combined with the result mentioned above that the problem of deciding if a nondeterministic TM returns to its start state infinitely often is Σ_1^1 hard, proves Theorem 8.4.

This concludes the proof of our lower bound results.

Corollary 8.17: *All our hardness results hold even when we weaken the logic to include only the* `B`, `E` *and* `A` *operators.*

Proof: In our constructions we only used these three operators, and ones defined in terms of them. ∎

8.3 Upper bounds

We end by briefly discussing upper bounds for the complexity problems. We have the following results.

Theorem 8.18:

1. *The validity problem for each of the following classes of temporal structures is r.e.-complete:*
   - (a) *The class of all temporal structures.*
   - (b) *The class of all linear temporal structures.*
   - (c) *The class of all discrete temporal structures.*
   - (d) *The class of all dense temporal structures.*
   - (e) *The class of all dense, linear, unbounded temporal structures.*
2. *The validity problem for $\mathcal{Q}$ in r.e.-complete.*
3. *The validity problem for $\mathcal{N}$ is Π_1^1-complete.*
4. *The validity problem for $\mathcal{R}$ is in Π_1^2.*
5. *The validity problem for $\mathcal{K}$ is co-r.e.-complete.*

Proof: We have already proved all the lower bounds, so it only remains to show the upper bounds.

For (1), we prove the upper bound for dense, linear, unbounded structures. The other proofs are similar. By Corollary 7.5, φ is valid for the class of dense, linear, unbounded structures iff the first-order formula $\mathtt{dlu} \supset \forall \mathtt{t}_1, \mathtt{t}_2\ (\mathtt{t}_1 \preceq \mathtt{t}_2 \supset \varphi_t)$ is valid (where `dlu` is the first-order formula discussed in the proof of Theorem 7.9 which which characterizes dense, linear, unbounded structures). Since validity for first-order logic is well-known to be r.e., the result follows. Note that the proof actually shows that validity with respect to any first-order definable class of structures is r.e.

For (2), note that by Theorem 7.9 validity for $\mathcal{Q}$ is equivalent to validity for dense, linear, unbounded structures, so the result follows immediately from part (1).

For (3), we show that the satisfiability problem for $\mathcal{N}$ is in Σ^1_1. We do this by showing that given a modal formula φ, we can construct a Σ^1_1 sentence ψ_φ such that ψ_φ is true iff φ is satisfiable in $\mathcal{N}$. Suppose φ has k subformulas $\varphi_1, \ldots, \varphi_k$, where $\varphi = \varphi_k$. (A subformula of φ is simply a substring of φ which is also a formula.) A pair $\langle m, n \rangle$ of natural numbers representing an interval can be encoded by a single number using the pairing function $f(m,n) = 1/2((m+n)^2 + 3m + n)$. It is well known [25] that f is a one-one onto map from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$. We use the sets $X_1, \ldots, X_k$ to encode the intervals where the formulas $\varphi_1, \ldots, \varphi_k$ are true. Thus the formula ψ_φ is of the form $\exists X_1 \ldots X_k \psi'$, where ψ' is a conjunction encoding some conditions that the sets X_i, $i = i, \ldots, k$ must satisfy. For example, if φ_j is of the form $\varphi_{j_1} \wedge \varphi_{j_2}$, then one of the conjuncts in ψ' is:

$$x \in X_j \equiv (x \in X_{j_1} \wedge x \in X_{j_2}).$$

Similarly, if φ_j is of the form $\langle B \rangle \varphi_{j'}$, then then (using some obvious abbreviations) we have a conjunct in ψ' of the form:

$$x \in X_j \equiv \exists x_1, x_2, x_3, y((x_1 \leq x_2) \wedge (x_2 \leq x_3) \wedge (x = f(x_1, x_2)) \wedge (y = f(x_1, x_3)) \wedge (y \in X_{j'})).$$

Finally, we have a conjunction if ψ' stating that X_k is nonempty: $\exists x(x \in X_k)$. We leave it to the reader to check that φ is satisfiable iff ψ_φ is satisfiable.

The proof that the satisfiability problem for $\mathcal{R}$ is in Σ^2_1 proceeds along very similar lines. The only difference is that we can no longer represent an interval of reals $\langle t_1, t_2 \rangle$ by a single number. Rather, we have to represent it by a set of natural numbers, which encodes two Cauchy sequences of rational numbers (where a rational number q is represented by a triple $\langle m_1, m_2, m_3 \rangle$, where m_1/m_2 encodes the absolute value of q, and m_3, which is either 0 or 1, encodes its sign). Given a modal formula φ, we can construct a Σ^2_1 sentence σ_φ such that σ_φ is true iff φ is satisfiable in $\mathcal{R}$. The sentence σ_φ is similar in structure to ψ_φ, except that all the number variables (which were previously used to encode intervals of natural numbers) are replaced by set variables (which encode intervals of reals, as described above), and all the set variables are replaced by set of sets variables. In addition, we need clauses to guarantee that all the set variables used do encode intervals of reals as described above (i.e. to guarantee that they encode a pair of Cauchy sequences). We omit the details here.

Finally, for $\mathcal{K}$, note that it is obviously decidable if a modal formula φ is satisfiable in a particular finite initial segment of the natural numbers. Since φ is satisfiable in $\mathcal{K}$ iff it is satisfiable in some initial segment, the satisfiability problem for $\mathcal{K}$ is clearly r.e. (We just check the initial segments one by one, and report that φ is satisfiable if it is satisfiable in some initial segment.) Thus, the validity problem for $\mathcal{K}$ is co-r.e. ∎

9 Related work: interval-based modal logics of time

There is a very rich literature on the algebra of intervals and on interval logics, spanning philosophy and computer science. In AI in particular there have been several important developments in this areas over the past few years. However, none of these have to do with *modal* interval logics, which are the topic of this comparison section. We will therefore not discuss recent work by Allen and Hayes [3], Ladkin [14], Ladkin and Maddux [15], or Kowalski and Sergot [13]. An concise overview of this and other literature is offered in van Benthem's [32].

Although most of the work on modal temporal logic has been point-based, recent years have seen a growing interest in interval-based modal logics. As usual, the initial idea of dealing with intervals goes back to the philosophers (e.g., [9], and more recently [12, 26, 5, 31]). In computer science, there has recently been work on *process logic* [23, 19, 11], where intervals (or "paths") represent pieces of computation, and even more recently work on *interval temporal logic* [17, 8]. The one property all the interval-based logics have in common is that they interpret propositions over intervals of time. They differ among themselves, however, on several counts.

The first distinction is the ontological one mentioned in Section 2: are intervals primitive objects, or is it points that are taken as primitive, with intervals defined in terms of points. In philosophy one finds logics of both kinds. For example, in the logic of Burgess [5], intervals are defined by their end points, whereas in the logics of Humberstone [12] and Roper [26] intervals are primitive objects related by the $\subset$ (*subinterval*) and $<$ (*completely before*) relations. In computer science all interval-based modal logics construct intervals out of points.

Another distinction between various logics was mentioned earlier and stems from the commitment to a particular underlying temporal structure. With no exception, all interval-based modal temporal logics in computer science have been committed to the discrete linear view of time (see details below). This has not been the case in philosophy. Burgess explicitly assumes a dense linear order. Roper assumes linearity, but apparently nothing beyond that.

Another source of difference between the modal logics is the choice of tense operators. In computer science, the strong commitment to a discrete linear order dictated fairly standard modal operators (see details below). In philosophy there has been less uniformity. For example, the only operator discussed by Humberstone is F, standing for "in some interval after the current interval". The one other operator mentioned by Humberstone as a subject for future research is R, standing for "in some interval *immediately* after the current interval." Roper uses two other modal operators, which are also adopted by Burgess: G (for "in all intervals beginning during the current interval"), and H (for "in all intervals ending in the current interval"). Clearly all these operators are easily definable in our logic.

For a more detailed discussion of temporal logics in philosophy, both point-based and interval-based, see [31]. We now discuss particular interval-based logics in computer science. SOAPL is a fairly complex logic introduced by Parikh [19]. It has two kinds of formulas: those interpreted over states and those interpreted over "paths", or sequences of states. Parikh proved that validity in the logic is (nonelementarily) decidable, but did not provide a complete deductive system. Nishimura's logic [18] is an attempt to merge temporal logic with dynamic logic. He showed his logic to be as expressive as SOAPL. His logic too is rather complex, and maintains the distinction between "state" formulas and "path" formulas.

At roughly the same time Pratt introduced *process logic*, in which formulas are interpreted over paths [23]. Later Harel, Kozen and Parikh refined the formulation [11]. They introduced two new modal operators (in addition to the ones introduced by dynamic logic): `f` (*first*) and `suf` (roughly, *until*). More precisely,

$\langle s_1, \ldots, s_n \rangle \models \mathtt{f}\varphi$ iff $\langle s_1 \rangle \models \varphi$, and

$\langle s_1, \ldots, s_n \rangle \models \varphi\, \mathtt{suf}\, \psi$ iff, for some j, $\langle s_j, \ldots, s_n \rangle \models \psi$, and for all i such that $1 \leq i < j$, $\langle s_i, \ldots, s_n \rangle \models \psi$.

Harel et al. show that satisfiability in the resulting logic is decidable (although not necessarily elementarily so), and give a complete axiomatization for it. Those results depend on assuming that a primitive proposition is true over an interval iff it is true at the first time point of that interval; that is, assuming locality. This property is captured in process logic by the axiom schema $\mathtt{p} \equiv \mathtt{fp}$, for a primitive proposition `p`. The assumption of locality is really at odds with the reason for our interest in a logic of intervals, since, as Harel et al. themselves put it, "every path formula ultimately expresses properties of *states*." They mention the fact that without the axiom $\mathtt{p} \equiv \mathtt{fp}$ there are path properties that cannot be expressed, and leave open the question of decidability in the absence of this axiom of locality. Later Streett settled this question by showing that global propositional process logic is Π^1_1-complete [30].

Interval temporal logic [8, 17] is also an extension of temporal logic in which formulas are interpreted over paths. The two modal operators considered there are $\bigcirc$ (*next*) and ; (*chop*). The meaning of these operators is given by:

$\langle s_1, \ldots, s_n \rangle \models \varphi;\psi$ iff $\langle s_1, \ldots, s_i \rangle \models \varphi$ and $\langle s_i, \ldots, s_n \rangle \models \psi$ for some i

$\langle s_1, \ldots, s_n \rangle \models \bigcirc\varphi$ iff $\langle s_2, \ldots, s_n \rangle \models \varphi$.

Thus, the $\bigcirc$ operator strongly commits ITL to the discrete view of time. In [8] it is shown that satisfiability for ITL is undecidable, and that if locality is assumed then satisfiability is decidable but nonelementary. The ITL extension of temporal logic is different from ours in two ways. First, in our logic we are not committed to viewing time as discrete. Second, even if we assume discreteness of time in our logic, the two logics are not comparable in their expressive power. On the one hand the *chop* operator of ITL is not definable in our logic (this is proved formally by Venema in [33]), and on the other hand we provide means of referring to intervals *outside* the reference interval, which ITL does not.

Schwartz, Melliar-Smith, and Vogt [27] offer another interval-based temporal logic. They augment modal temporal logic by constructs referring to intervals explicitly: if φ is a formula then so is $\mathtt{[I]}\varphi$, where `I` is an interval designator. For example, the formula $\mathtt{[(x{=}y)} \Rightarrow \mathtt{(y{=}16)]}\Box\mathtt{(x{>}z)}$ is intended to mean that `x` is greater than `z` throughout the interval beginning at the first time `x` equals `y` and ending at the first time after that when `y` equals 16. Intervals are assumed to consist of linearly ordered and discrete time points, and again locality is assumed. In [20] it is shown that satisfiability for this logic is nonelementarily decidable.

Finally, Venema has recently obtained further results regarding our logic [33]. Besides the expressibility results mentioned above, he proves that all the classes of structures which were shown in Theorem 8.18 to have a validity problem that is r.e. complete, actually have a relatively elegant complete axiomatization. In particular, he shows that there is a complete axiomatization for $\mathcal{Q}$.

10 Conclusions

We presented a new interval logic which generalizes point-based temporal logic. The syntax and semantics are very simple, and the logic allows one to express naturally statements that refer to intervals of time and to continuous processes.

We showed that the logic is expressive enough to identify several classes of temporal structures, such as the classes of dense structures, linear structures, and complete structures. At the same time, we showed that some classes of structures *cannot* be distinguished in the logic, one example being the class of dense, linear and unbounded structures and the singleton consisting of the rationals $\mathcal{Q}$.

We gave several results on the complexity of the validity problem for the logic. For all but the simplest classes of temporal structures, validity is undecidable. For classes of structures with infinitely ascending sequences, such as the rationals $\mathcal{Q}$, validity is r.e-hard. For classes of structures which contain unboundedly ascending sequences, validity is co-r.e.-hard. For complete classes of structures with infinitely ascending sequences, such as $\mathcal{N}$ or $\mathcal{R}$, satisfiability is Π_1^1-hard. Notice that the Π_1^1-hardness and co-r.e.-hardness results imply nonaxiomatizability.

Finally, we gave several upper bounds for the validity problem. For $\mathcal{N}$, $\mathcal{Q}$ and $\mathcal{K}$, we showed that the upper bounds match the lower ones. For $\mathcal{R}$ we gave a less tight upper-bound.

It is surprising that such a natural logic of time has never been explored before. Many fascinating open problems still remain, and they include the following:

1. Can we find matching upper and lower bounds for the validity problem with respect to $\mathcal{R}$?

2. What results can we get for other natural classes of temporal structures?

3. What happens to the complexity of the validity problem if we slightly modify the logic? We have already remarked that our lower bounds hold even if we restrict the logic to the `B`, `E` and `A` operators, but we do not know what happens for weaker or incomparable combinations of modal operators, for example the set $\{\mathtt{D},\overline{\mathtt{D}}\}$ or the set $\{\mathtt{B},\mathtt{E}\}$.

4. The motivation for our logic was the need to reason about situations of interest in Artificial Intelligence. Are the hardness results for the validity problem a sign of failure? We think not. Our logic is very natural, and the meaning of the various operators is quite intuitive. The fact that an efficient general-purpose theorem prover for the logic is unattainable will hardly come as a shock to anyone in AI. What we need to do, now that we have a natural and expressive logic, is to identify classes of formulas about which reasoning is easier than in the general case.

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