A Nonstandard Characterization of Sequential Equilibrium, Perfect Equilibrium, and Proper Equilibrium: Erratum

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As pointed out by [an anonymous referee/XXXX], Theorem 1.4 as stated in [Halpern 2009], which gives a characterization of perfect equilibrium in terms of nonstandard probability, is incorrect. Fortunately, the theorem is easily corrected. In this erratum, I briefly explain the problem and its solution.

Both sequential equilibrium and perfect equilibrium consider best responses at all information sets, even ones off the equilibrium path. The subtlety arises in making clear what it means to make a best response at an information set off the equilibrium path. In the standard definition of what it means for a behavioral strategy profile $\vec{\sigma}$ to be a sequential or perfect equilibrium, a sequence of *completely-mixed behavioral strategy profiles* that converges to $\vec{\sigma}$ is considered (where a behavioral strategy σ_i for player i is *completely mixed* if, at each information set I for player i, it assigns positive probability to all possible actions at I). In [Halpern 2009], this sequence of behavioral strategy profiles converging to $\vec{\sigma}$ is replaced by a single completely-mixed *nonstandard* strategy profile $\vec{\sigma}'$ (i.e., a strategy profile where each strategy can assign a nonstandard probability to each action) that differs infinitesimally from $\vec{\sigma}$. (The notion of being completely mixed remains the same: each action at an information set gets positive probability, but now that probability can be an infinitesimal.) The idea is that, by using a completely-mixed strategy $\vec{\sigma}'$, the notion of σ_i' being a best response to $\vec{\sigma}'_{-i}$ becomes completely unambiguous.

While the characterization of sequential equilibrium does involve σ_i' being a best response to $\vec{\sigma}'_{-i}$ at all information sets for player i, the characterization of perfect equilibrium requires that σ be a best response to $\vec{\sigma}'_{-i}$ at all information sets I for player i. The problem is that $(\sigma_i, \vec{\sigma}'_{-i})$ is not a completely mixed strategy, and may not reach an information set I at all. This leads to problems with the definition of best response (Definition 2.1) in [Halpern 2009].

To solve this problem, we should go back to the original motivation for these definitions. What we really want to say is that, after reaching I using $\vec{\sigma}'$, σ_i is the best continuation for player i from then on. To make this precise, we need to know i's beliefs regarding the relatively likelihood of histories in I; ths belief is determined by the completely-mixed strategy profile used to reach I. Recall the notation from [Halpern 2009]; given a behavioral-strategy profile $\vec{\sigma}$, $\Pr_{\vec{\sigma}}$ is the distribution on terminal histories induced by $\vec{\sigma}$. Since we can identify a partial history with the terminal histories that extend it, $\Pr_{\vec{\sigma}}(h)$ and $\Pr_{\vec{\sigma}}(I)$ are well defined for a partial history h and an information set I. As usual, we take a *belief system* μ to be a function that associates with each information set I a probability denoted μ_I on the histories in I. Given a behavioral strategy $\vec{\sigma}$ and a belief system μ in an extensive-form game Γ, let

$$\mathrm{EU}_i((\vec{\sigma},\mu) \mid I) = \sum_{h\in I}\sum_{z\in Z} \mu_I(h)\Pr_{\vec{\sigma}}(z \mid h)u_i(z),$$

where Z is the set of terminal histories in Γ. Finally, if $\vec{\sigma}$ is a completely-mixed behavioral strategy profile, let $\mu^{\vec{\sigma}}$ be the belief system determined by $\vec{\sigma}$ in the obvious way:

$$\mu_I^{\vec{\sigma}}(h) = \Pr_{\vec{\sigma}}(h \mid I).$$

We can now define best response:

Definition 2.1$'$: If $\epsilon \geq 0$ and I is an information set for player i that is reached with positive probability by $\vec{\sigma}'$, then σ_i is an *ϵ-best response to $\vec{\sigma}'_{-i}$ for i conditional on having reached I using $\vec{\sigma}'$* if, for every strategy τ for player i, we have $\mathrm{EU}_i(((\sigma_i,\vec{\sigma}'_{-i}),\mu_I^{\vec{\sigma}'}) \mid I) \geq \mathrm{EU}_i(((\tau_i,\vec{\sigma}'_{-i}),\mu_I^{\vec{\sigma}'}) \mid I)$. ∎

By way of contrast, here is Definition 2.1 in [Halpern 2009]:

Definition 2.1: If $\epsilon \geq 0$ and I is an information set for player i, σ_i is an *ϵ-best response to $\vec{\sigma}'_{-i}$ for i conditional on having reached I* if, for every strategy τ for player i that agrees with σ_i except possibly at I and information sets preceded by I, we have $\mathrm{EU}_i(\sigma_i,\vec{\sigma}'_{-i}) \geq \mathrm{EU}_i(\tau,\vec{\sigma}'_{-i}) - \epsilon$. ∎

Note the differences here. Definition 2.1$'$ enforces the assumption that I is reached using $\vec{\sigma}'$ by using the belief system $\mu^{\vec{\sigma}'}$. In Definition 2.1, σ_i' plays no role; rather, σ_i and τ have to agree up to I, so the implicit assumption is that I is reached using $(\sigma_i,\vec{\sigma}'_{-i})$, not $\vec{\sigma}'$. It is this difference that causes problems.[1] To understand

[1]There is actually a second problem in Definition 2.1: it uses the *ex ante* probability, rather than the probability conditional on reaching I. This problem is also corrected in Definition 2.1$'$. However, it is the former problem that is the deeper conceptual problem.

why, consider the statement of Theorem 1.4 taken from [Halpern 2009] (where a best response is taken to be a 0-best response, according to Definition 2.1), which is intended to be a characterization of perfect equilibrium:

Theorem 1.4: *If* Γ *is an extensive-form game with perfect recall, then* $\vec{\sigma}$ *is a perfect equilibrium in* Γ *iff there exists a nonstandard completely-mixed strategy profile* $\vec{\sigma}'$ *that differs infinitesimally from* $\vec{\sigma}$ *such that, for each player* i *and each information set* I *for player* i*,* σ_i *is a best response to* $\vec{\sigma}'_{-i}$*, conditional on having reached* I*.*

This statement is problematic because, in evaluating whether σ_i is a best response to σ'_{-i} conditional on having reached I, according to Definition 2.1, we implicitly assume that I is reached using $(\sigma_i, \vec{\sigma}'_{-i})$. But σ_i is not completely mixed and, indeed, might be such that I is not reached at all! Thus, the theorem is incorrect. A correct version of the theorem can be obtained by simply adding the words "using $\vec{\sigma}'$" at the end of the theorem (i.e., I is reached using $\vec{\sigma}'$), where "best response to $\vec{\sigma}'_{-i}$, conditional on having reached I using $\vec{\sigma}'$ is defined by Definition 2.1$'$. The proof of Theorem 1.4 sketched in [Halpern 2009] actually proves the corrected version, not Theorem 1.4, so no further changes are required here.

A similar change must be made in Definition 2.2, which is intended to be equivalent to the standard definition of perfect equilibrium. Specifically, it should say that $\vec{\sigma}$ is a perfect equilibrium in an extensive-form game Γ iff there exists a sequence $\vec{\sigma}^n$ of completely-mixed behavior strategies such that $\vec{\sigma}^n \to \vec{\sigma}$ and, for all n and each information set I of player i, σ_i is a best response to $\vec{\sigma}^n_{-i}$ conditional on having reached I *using* $\vec{\sigma}^n$. Again, the phrase "using $\vec{\sigma}^n$" needs to be added to the definition given in [Halpern 2009]. Similarly, "using $\vec{\sigma}^n$" needs to be added to the characterization of sequential equilibrium (Theorem 1.2).[2]

References

Halpern, J. Y. (2009). A nonstandard characterization of sequential equilibrium, perfect equilibrium, and proper equilibrium. *International Journal of Game Theory 38*(1), 37–50.

[2]The issue of how the information set is reached is actually not significant in Theorem 1.2. However, it is still critical to consider probabilities conditional on reaching I as defined in Definition 2.1$'$, rather than the ex ante probability as in Definition 2.1. Again, if we use Definition 2.1$'$ to define best response rather than Definition 2.1, then Theorem 1.2 is correct (and the proof in [Halpern 2009] actually proves the corrected statement).