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SOME RESULTS ON FIELDS OF VALUES OF A MATRIX*

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1. For a square matrix A the so-called field of values $G[A]$ is defined as the following set of complex numbers:[1]

$$G[A] := \{x^H A x \mid x^H x = 1\}. \tag{1.1}$$

It is well known that this set contains the spectrum $\Lambda(A) = \{\lambda_i(A)\}$ of A. Moreover, Toeplitz [11] proved in 1918 that $G[A]$ also contains the convex hull $\mathcal{H}(\Lambda(A))$ of $\Lambda(A)$. Hausdorff [5] generalized this result by proving that $G[A]$ is convex. The concept of a field of values was generalized by Bauer [1] in 1962. Starting with an arbitrary norm $\|\cdot\|$ in C^n (or R^n) and its *dual norm* (defined on the dual space of C^n (or R^n) of row vectors y^H)

$$\| y^H \|^D := \max_{x \neq 0} \frac{\operatorname{Re} y^H x}{\| x \|},$$

he defined the field of values $G[A, \|\cdot\|]$ with respect to the norm $\|\cdot\|$ as the set

$$G[A, \|\cdot\|] := \{y^H A x \mid \| y^H \|^D \| x \| = \operatorname{Re} y^H x = 1\}. \tag{1.2}$$

Vectors with the property

$$\operatorname{Re} y^H x = \| y^H \|^D \| x \| \neq 0$$

are called *dual vectors* and are denoted by

$$y^H \parallel x.$$

It is well known that a dual pair of vectors y^H and x is geometrically characterized by the fact that y is the normal to the (real) supporting hyperplane

$$H_y := \{z \mid \operatorname{Re} y^H z = \| y^H \|^D\}$$

to the convex body

$$B := \{z \mid \| z \| \leqq 1\}$$

passing through the boundary point $x/\| x \|$ of B.

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[1] We use := throughout to express equality by definition.

Clearly, the original definition (1.1) of $G(A)$ is obtained from (1.2) for the Euclidean norm $\| x \| := \sqrt{x^H x}$. Again, Bauer and Witzgall [1], [1a] have shown that $G[A, \| \cdot \|]$ contains $\Lambda(A)$. However, $G[A, \| \cdot \|]$ is not always convex, as can be shown by counterexamples (see Nirschl and Schneider [9]). In this situation, the question arises whether Toeplitz' result still holds for the generalized field of values (1.2). Sections 3 and 4 of this paper are devoted to this problem. A theorem of Stoer and Witzgall [10] (see Theorem 3.1) settles this question positively for diagonal matrices A and absolute norms, i.e., those norms with the property[2]

$$(1.3) \qquad \| x \| = \| \, | x | \, \|.$$

In order to answer this question for more general classes of norms (and matrices) this theorem is generalized.

In §2 some theorems on norms used later are listed.

Other questions arise in connection with the *orthogonal field of values* $O[A]$ relative to a norm $\| \cdot \|$. This set is defined [1] by

$$(1.4) \qquad O[A] := \{ y^H A x \mid y^H x = 0, \, \| y^H \|^D \, \| x \| = 1 \}$$

and was used implicitly (for normal matrices A and the Euclidean norm only) by Mirsky [8] in order to derive lower bounds for the *spread* $s(A)$ of A:

$$(1.5) \qquad s(A) := \max_{i,j} |(\lambda_i(A) - \lambda_j(A))|.$$

In the second part of this paper (§§5 and 6), $O[A]$ is investigated systematically. Here, Theorem 3.1 mentioned above again turns out to be useful.

2. In the sequel, we shall often use several results, partly known, on absolute and other norms, which are listed in the following theorems.

THEOREM 2.1 [2]. *The dual of an absolute norm is absolute.*

THEOREM 2.2 [2]. *Absolute norms in C^n (or R^n) are equivalently defined by each of the following properties*:

(i) $| x | \leqq | y |$ *implies* $\| x \| \leqq \| y \|$ *for all* x, y (*monotonicity of absolute norms*);

(ii) $\text{lub}\,(D) = \max_i | d_i |$ *holds for all diagonal matrices*

$$D = \text{diag}\,(d_1, \cdots, d_n).$$

Here, lub $(\cdot)$ means the *least upper bound norm* associated with $\| \cdot \|$. It is defined for all $n \times n$ matrices A by

$$(2.3) \qquad \text{lub}\,(A) := \max_{x \neq 0} \frac{\| Ax \|}{\| x \|}.$$

[2] If $x = (x_1, \cdots, x_n)^T$, then $| x | := (| x_1 |, \cdots, | x_n |)^T$.

For lub $(\cdot)$, we note the further property [1]

$$\text{(2.4)} \qquad \operatorname{lub}(A) = \max_{x,y\neq 0} \frac{\operatorname{Re} y^H A x}{\| y^H \|^D \| x \|}.$$

The following theorem is essentially due to Nirschl and Schneider [9].

THEOREM 2.5. *Let* $\|\cdot\|$ *be an absolute norm in* C^n *or* R^n *and* x *and* z *two vectors satisfying* $0 \leqq x \leqq z$ *and* $\| x \| = \| z \|$. *Then the* jth *component* y_j *of any* y *dual to* x: $y^H \parallel x$ *vanishes*, $y_j = 0$, *whenever* $x_j < z_j$.

Apart from absolute norms we now define more generally *orthant-monotonic* norms in R^n and C^n.

DEFINITION 2.6. A norm $\|\cdot\|$ in R^n is called *orthant-monotonic*, if $\| x \| \leqq \| y \|$ is true for all vectors $x = (x_1, \cdots, x_n)^T$, $y = (y_1, \cdots, y_n)^T$ satisfying

$$| x | \leqq | y | \quad \text{and} \quad x_i y_i \geqq 0 \quad \text{for} \quad i = 1, 2, \cdots, n.$$

This class of norms is investigated more systematically in [4]. Any norm $\|\cdot\|$ in C^n gives rise to a norm $\|\cdot\|_R$ in the underlying R^{2n} by defining

$$\| x_R \|_R := \| x \|,$$

where $x_R := \begin{pmatrix} x' \\ x'' \end{pmatrix} \in R^{2n}$, if $x = x' + i \cdot x''$, $x', x'' \in R^n$. Since $\operatorname{Re} y^H x = (y_R)^H x_R$, we have

$$\text{(2.7)} \qquad y^H \parallel x \quad \text{if and only if} \quad (y_R)^H \parallel_R x_R.$$

With the aid of $\|\cdot\|_R$ we give the following definition.

DEFINITION 2.8. A norm $\|\cdot\|$ in C^n is called *orthant-monotonic in* C^n, if $\|\cdot\|_R$ is orthant-monotonic in R^{2n}. Naturally, we can also state:

(2.9) *absolute norms are orthant-monotonic.*

Any norm $\|\cdot\|$ in C^n or R^n induces a norm $\|\cdot\|_L$ in any subspace L of C^n or R^n by setting

$$\| x \|_L := \| x \| \quad \text{for} \quad x \in L.$$

Orthant-monotonic norms have the property that for certain subspaces L, the coordinate-subspaces, the induced norm is again orthant-monotonic. By a coordinate-subspace $L = V_\eta$ of C^n or R^n we mean a subspace spanned by some subset $\{e_i, i \in \eta \subseteq N := \{1, 2, \cdots, n\}\}$ of the set of all axisvectors $e_1 := (1, 0, \cdots, 0)^T, \cdots, e_n := (0, \cdots, 0, 1)^T$. Then every $x \in R^n$ (or C^n) can be written in the form

$$x = x_\eta \oplus x_{\eta'} \quad \text{with} \quad x_\eta \in V_\eta, \qquad x_{\eta'} \in V_{\eta'}, \qquad \eta' := N \backslash \eta.$$

Hence we have the following lemma.

LEMMA 2.10. *If* $\| \cdot \|_\eta$ *is defined by* $\| x_\eta \|_\eta := \| x \|$ *for* $x = x_\eta \oplus O_{\eta'}$ *and* $\| \cdot \|$ *is orthant-monotonic, then* $\| \cdot \|_\eta$ *is orthant-monotonic in* V_η.

The following theorem shows that duality is preserved by the transition from $\| x_\eta \|_\eta$ to $\| x \|$.

THEOREM 2.11. *Let* $\| \cdot \|$ *be an orthant-monotonic norm in* R^n *or* C^n *and* $\eta \subseteq N = \{1, 2, \cdots, n\}$, $\eta' := N \setminus \eta$. *Then the vectors* x_η, y_η^H *are dual to each other with respect to* $\| \cdot \|_\eta$, $(y_\eta)^H \|_\eta x_\eta$, *if and only if the vectors* $x := x_\eta \oplus O_{\eta'}$, $y := y_\eta \oplus O_{\eta'}$ *are dual with respect to* $\| \cdot \|$: $y^H \| x$.

Proof. Naturally,

$$y^H x = (y_\eta)^H x_\eta, \qquad \| x_\eta \|_\eta = \| x \|.$$

Hence, the theorem is proved, if we show that

$$(\| y_\eta^H \|_\eta)^D := \max_{x_\eta \neq 0} \frac{\operatorname{Re} (y_\eta)^H x_\eta}{\| x_\eta \|_\eta} = (\| y_\eta^H \|^D)_\eta .$$

But, by definition,

$$(\| y_\eta^H \|^D)_\eta = \| y^H \|^D \quad \text{if} \quad y = y_\eta \oplus O_\eta .$$

Hence,

$$(\| y_\eta^H \|^D)_\eta = \max_{x \neq 0} \frac{\operatorname{Re} y^H x}{\| x \|} = \max_{x \neq 0} \frac{\operatorname{Re} (y_\eta)^H x_\eta}{\| x \|}$$

if $x = x_\eta \oplus x_{\eta'}$. But $\| \cdot \|$ is orthant-monotonic. Hence,

$$\| x \| \geqq \| x_\eta \oplus O_{\eta'} \| = \| x_\eta \|_\eta ,$$

which implies

$$(\| y_\eta^H \|^D)_\eta = \max_{x_\eta \neq 0} \frac{\operatorname{Re} (y_\eta)^H x_\eta}{\| x_\eta \|_\eta} = (\| y_\eta^H \|_\eta)^D.$$

3. As mentioned above, in [10] the following theorem was proved.

THEOREM 3.1. *Let* $\| \cdot \|$ *be an absolute norm in* R^n *or* C^n. *Then for any vectors* $u > 0$, $v > 0$, *there exists one (and up to positive multiples only one) diagonal matrix* D *with*

$$D \geqq O \quad \text{and} \quad v^H D \| D^{-1} u.$$

For $G[A, \| \cdot \|]$ as defined by (1.2), this theorem has the following consequences. If $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix, then the theorem implies that for absolute norms,

$$G[A, \| \cdot \|] \supseteq \mathcal{H}(\Lambda(A)).$$

Namely, for the elements

$$\mu = \sum_{i=1}^n \tau_i \lambda_i , \qquad \tau_i > 0, \qquad \sum_{i=1}^n \tau_i = 1,$$

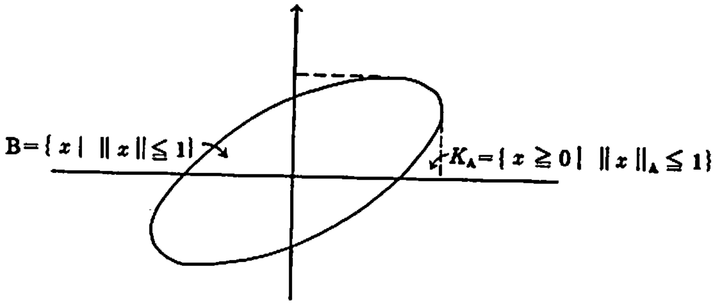

FIG. 1

of $\mathcal{H}(\Lambda(A))$, it implies that for

$$u := (u_1, \cdots, u_n)^T > 0, \qquad v := (1, 1, \cdots, 1)^T > 0,$$

there exist vectors

$$x := D^{-1}u = (x_1, \cdots, x_n)^T, \qquad y^H := (y_1, \cdots, y_n) = v^H D$$

such that $\bar{y}_i x_i = r_i$ and $y^H \parallel x$, hence $\mu = y^H A x \in G[A, \|\cdot\|]$. Since $G[A, \|\cdot\|]$ is a closed set, this implies the following.

THEOREM 3.2. *If A is a diagonal matrix and $\|\cdot\|$ an absolute norm, then* $G[A, \|\cdot\|] \supseteq \mathcal{H}(\Lambda(A))$.

(See [4] for a generalization: absolute norms in C^n are characterized by the property $G[A, \|\cdot\|] = \mathcal{H}(\Lambda(A))$ for diagonal A.)

It is remarkable that in contrast with the results for norms in C^n (see the counterexamples at the beginning of §4) the assertion of Theorem 3.1 is also true for every norm in R^n.

THEOREM 3.3. *Let $\|\cdot\|$ be any norm in R^n. Then to any pair of vectors $u > 0$, $v > 0$ of R^n there exists one, and up to positive multiples only one, nonsingular diagonal matrix $D \geqq 0$ with*

$$v^H D \parallel D^{-1}u.$$

Proof. The proof given in [10] for the uniqueness of D in the case of absolute norms is also valid for all norms in R^n. We do not repeat it here. The proof of the existence of D may be outlined as follows: Given $\|\cdot\|$ we define the largest absolute norm $\|\cdot\|_A$ with the property $\|x\| \geqq \|x\|_A$ for $x \geqq 0$ (see Fig. 1). Then, by Theorem 3.1, there exists a diagonal matrix $D \geqq 0$ with the property

$$v^H D \parallel_A D^{-1}u.$$

We then show that $D^{-1}u/\|D^{-1}u\|_A$ must lie on the common boundary of $K := \{x \geqq 0 \mid \|x\| \leqq 1\} = B \cap \{x \geqq 0\}$ and $K_A := \{x \geqq 0 \mid \|x\|_A \leqq 1\}$. This will yield the desired result $v^H D \parallel D^{-1}u$.

For the formal proof, given a norm $\| \cdot \|$ in R^n and the associated convex bodies B and K,

$$B := \{x \mid \| x \| \leqq 1\}, \qquad K := B \cap \{x \geqq 0\},$$

we define the norm $\| \cdot \|_A^D$:

$$\| y^H \|_A^D := \max_{\substack{x \geqq 0 \\ x \neq 0}} \frac{| y |^H x}{\| x \|} = \max_{x \in K} | y |^H x.$$

It is easy to verify that $\| \cdot \|_A^D$ is a norm. Moreover, it is absolute, since $\| y^H \|_A^D$ depends only on $| y |^H$. Therefore, the dual norm $\| \cdot \|_A$,

$$\| x \|_A := \max_{y \neq 0} \frac{y^H x}{\| y^H \|_A^D},$$

with the corresponding convex bodies B_A and K_A ,

$$(3.4) \qquad \begin{aligned} B_A &:= \{x \mid y^H x \leqq \| y^H \|_A^D = \max_{z \in K} | y |^H z \text{ for all } y \neq 0\}, \\ K_A &:= B_A \cap \{x \geqq 0\}, \end{aligned}$$

is absolute (see Theorem 2.1). Obviously,

$$K_A = \{x \mid x \geqq 0\} \bigcap_{\substack{y \geqq 0 \\ y \neq 0}} H_y^- \quad \text{with} \quad H_y^- := \{x \mid y^H x \leqq \max_{z \in K} y^H z\}.$$

Also, by definition of $\| \cdot \|_A^D$,

$$(3.5) \qquad K \subset K_A .$$

By Theorem 3.1, when applied to the absolute norm $\| \cdot \|_A$, there exists a diagonal matrix $D \geqq O$ such that

$$y^H := v^H D \|_A D^{-1} u =: x \quad \text{with} \quad \| x \|_A = 1.$$

Since D is nonsingular and nonnegative, we have $y^H > 0, x > 0$. Hence, x is a boundary point of B_A and K_A . Suppose for the moment that x is also a boundary point of B and K. Then one can show that y^H and x form also a dual pair with respect to the original norm $\| \cdot \|$: $y^H \| x$, which proves the theorem. Indeed, by definition of y^H,

$$H_y := \{z \mid y^H z = \| y^H \|_A^D\}$$

is a supporting plane of B_A and therefore also of K_A , and from (3.5), also of K. Suppose H_y is not a supporting plane of B. Then there exists a point $x_1 \ngeq 0$ such that

$$x_1 \in H_y^+ := \{z \mid y^H z > \| y^H \|_A^D\}, \qquad x_1 \in B.$$

Since B is convex, every point

$$x_2 \in [x_1, x) := \{z \mid z = \lambda x_1 + (1 - \lambda)x, 0 < \lambda \leqq 1\}$$

is also in $B \cap H_y^+$. Moreover, $x > 0$ implies that there exists a point $x_2 > 0$, $x_2 \in [x_1, x)$, hence $x_2 \in K \cap H_y^+$. Because H_y is a supporting plane of K, $H_y^+ \cap K = \emptyset$, this contradicts $x_2 \in K \cap H_y^+$. Hence, H_y is a supporting plane of B and $y^H \parallel x$.

In order to complete the proof of the theorem, it remains to be shown that $x > 0$ is also a boundary point of B and K. Suppose this is not true, i.e.,

$$1 = \| x \|_A < \| x \|.$$

Then we show that there exists a point x_1 with the properties

$$\text{(3.6)} \qquad x_1 \geqq x > 0 \quad \text{and} \quad \| x_1 \| = 1.$$

If this were not true, then the closed convex set

$$T := \{z \mid z \geqq x\}$$

would satisfy $T \cap K = \emptyset$. Since K is a compact convex set there would exist a hyperplane

$$H_{\bar{y}} = \{z \mid \bar{y}^H z = t\}, \qquad \| \bar{y}^H \|^D = 1,$$

strictly separating K from T (compare, e.g., [3]):

$$\text{(3.7)} \qquad K \subset \{z \mid \bar{y}^H z < t\} =: H_{\bar{y}}^-, \qquad T \subseteq \{z \mid \bar{y}^H z > t\} =: H_{\bar{y}}^+.$$

Since $0 \in K$, it follows $t > 0$; by definition of T it follows also $\bar{y}^H \geqq 0$. From $K \subseteq H_{\bar{y}}^-$ we have

$$t > \max_{z \in K} \bar{y}^H z,$$

but this implies also $K_A \subset H_{\bar{y}}^-$ in view of $\bar{y}^H \geqq 0$ and the definition of K_A. Since $x \in K_A$ and also $x \in T$, we have a contradiction to (3.7). Hence there exists an x_1 for which (3.6) holds. Since, by assumption $\| x \| > 1$, we have

$$x_1 \geqq x > 0, \qquad x_1 \neq x,$$

and $K \subseteq K_A$, and the monotonicity of the absolute norm $\| \cdot \|_A$ (Theorem 2.2) implies at once

$$1 = \| x_1 \| \geqq \| x_1 \|_A \geqq \| x \|_A = 1,$$

which shows $\| x_1 \|_A = 1$. Because of $x_1 \geqq x$, $x_1 \neq x$, from Theorem 2.5 it then follows that at least one component of y^H vanishes, in contradiction to $y^H > 0$. This completes the proof of Theorem 3.3.

Now, let A be an arbitrary complex $n \times n$ matrix with real *eigenvectors*, which is similar to a diagonal matrix Λ :

$$T^{-1}AT = \Lambda, \quad T \text{ a } \textit{real} \text{ nonsingular matrix.}$$

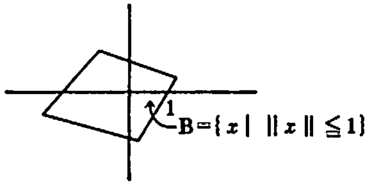

FIG. 2

If $\| \cdot \|$ is an arbitrary norm in R^n (not necessarily strictly homogeneous, i.e., $\| \alpha x \| = | \alpha | \| x \|$ for all (real) α), then by a result of Nirschl and Schneider [9],

$$G[A, \| \cdot \|] = G[T^{-1}AT, \| \cdot \|_T],$$

where $\| x \|_T := \| Tx \|$ for $x \in R^n$. Since T is real, $\| \cdot \|_T$ is another norm on R^n, and Theorem 3.3 when applied to $\| \cdot \|_T$ gives the following theorem.

THEOREM 3.8. *If the complex $n \times n$ matrix A has n real linear independent eigenvectors and $\| \cdot \|$ is any norm in R^n, then*

$$G[A, \| \cdot \|] \supseteq \mathcal{K}(\Lambda(A)).$$

4. The following example shows that the basic Theorem 3.1 is not true for every norm in C^n. Define in C^1 a norm by the convex body in Fig. 2 and take $u := v := 1$. Then it is easily seen that there is no $d > 0$ such that $v^H d \parallel d^{-1} u$. It is an open question, whether the assertion of Theorem 3.1 holds for arbitrary norms in C^n, if it is not required that $D \geqq O$. However, the theorem remains true as it stands for orthant-monotonic norms in C^n. To prove this, we first note a slight generalization of the theorem for orthant-monotonic norms in R^n.

THEOREM 4.1. *Let $\| \cdot \|$ be an orthant-monotonic norm in R^n and $u = (u_1, \cdots, u_n)^T \geqq 0$, $v = (v_1, \cdots, v_n)^T \geqq 0$ be nonzero vectors with*

$$u_i = 0 \quad \textit{if and only if} \quad v_i = 0.$$

Then there exists a diagonal matrix $D \geqq O$ such that

$$v^H D \parallel D^{-1} u.$$

Proof. Define

$$\eta := \{i \mid u_i \neq 0\} \subset N = \{1, 2, \cdots, n\}.$$

Then, $u_\eta > 0$, $v_\eta > 0$. By Theorem 3.3, when applied to the induced norm $\| \cdot \|_\eta$ (see Lemma 2.10), there exists a diagonal matrix $D_\eta \geqq O$ such that

$$(v_\eta)^H D_\eta \parallel_\eta D_\eta u_\eta .$$

Hence, by (2.11), the vectors

$$y := (D_\eta v_\eta) \oplus O_{\eta'}, \qquad x := (D_\eta^{-1} u_\eta) \oplus O_{\eta'}$$

are dual with respect to $\|\cdot\|$: $y^H \| x$. Clearly, the matrix D_η can be extended to a diagonal matrix $D := D_\eta \oplus D_{\eta'}$ such that D is nonsingular and

$$D \geqq O, \qquad y^H = v^H D = ((v_\eta)^H D_\eta) \oplus ((O_{\eta'})^H D_{\eta'}),$$
$$x = D^{-1}u = (D_\eta^{-1}u_\eta) \oplus (D_{\eta'}^{-1}O_{\eta'}).$$

This proves the theorem. (Note that $D \geqq O$ is no longer uniquely defined because of the arbitrary choice of $D_{\eta'}$.) This theorem can be used to prove a similar result in C^n.

THEOREM 4.2. *Let $\|\cdot\|$ be an orthant-monotonic norm in C^n and $u > 0$, $v > 0$ be two positive vectors of C^n. Then, there exists (up to positive multiples) exactly one diagonal matrix $D \geqq O$ such that*

$$v^H D \| D^{-1}u.$$

Proof. The norm $\|\cdot\|_R$ in R^{2n} (see Definition 2.8) is orthant-monotonic. The vectors u_R, v_R in R^{2n} satisfy together with $\|\cdot\|_R$ the hypotheses of Theorem 4.1. Hence, there exists a real diagonal $2n \times 2n$ matrix

$$D = \begin{pmatrix} D_1 & O \\ O & D_2 \end{pmatrix} \geqq O$$

such that

$$(v^H D_1, 0^H) = (v^H, 0^H)D \|_R D^{-1}\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} D_1^{-1}u \\ 0 \end{pmatrix}.$$

By (2.7), we then have $v^H D_1 \| D_1^{-1}u$, which was to be proved. (The uniqueness follows from the proof given in [10].)

Clearly, the above theorem can be sharpened along the lines indicated by Theorem 4.1; it is not necessary to require strict positivity of u and v.

THEOREM 4.3. *The result of Theorem 4.1 is true also for orthant-monotonic norms in C^n.*

Again, the last theorem gives the following result for fields of values of diagonal matrices (compare Theorem 3.2).

COROLLARY 4.4. *If $\|\cdot\|$ is any orthant-monotonic norm in C^n, then $G[D, \|\cdot\|] \supseteq \mathcal{H}(\Lambda(D))$ for all diagonal matrices D.*

5. In this section, we derive several properties of the orthogonal field of values $O[A]$ of A associated with a norm in C^n, $n \geqq 2$,

$$(5.1) \qquad O[A] := \{y^H A x \mid y^H x = 0, \ \|y^H\|^D \|x\| = 1\}$$

(for $n = 1$, $O[A]$ is empty). Moreover, we shall consider only strictly homogeneous norms.

From the definition follows [1] the translation invariance of $O[A]$:

$$(5.2) \qquad O[A + \tau I] = O[A].$$

$O[A]$ is associated with the function [1]

$$\text{ort}\,(A) := \max_{\gamma \in O[A]} \text{Re}\,\gamma. \tag{5.3}$$

$O[A]$ has a simple shape, as given in the following theorem.

THEOREM 5.4. *If* $\|\cdot\|$ *is a strictly homogeneous norm in* C^n, $n \geqq 2$, *then*

$$O[A] = \{\mu \mid |\mu| \leqq \text{ort}\,(A)\}$$

is a circle with center 0 *and radius* ort (A).

Proof. We show that each real number μ with $0 \leqq \mu \leqq \text{ort}\,(A)$ belongs to $O[A]$. It then follows by the strict homogeneity of $\|\cdot\|$ that $e^{i\phi}\mu \in O[A]$ for all real ϕ.

Now, let x_0 be an eigenvector of A and choose $y_0 \neq 0$ such that $y_0^H x_0 = 0$. Then,

$$0 = \frac{y_0^H A x_0}{\|y_0^H\|^D \|x_0\|} \in O[A].$$

By definition of ort (A), there are vectors x_1, y_1 such that $\|y_1^H\|^D = \|x_1\| = 1$, $y_1^H x_1 = 0$ and

$$\text{ort}\,(A) = \max_{\substack{y_1 x \neq 0 \\ y^H x = 0}} \frac{\text{Re}\, y^H A x}{\|y^H\|^D \|x\|} = y_1^H A x_1 \in O[A],$$

since $\|\cdot\|$ is strictly homogeneous.

Now, if $n \geqq 3$, then there exists a $y_2 \neq 0$ such that

$$y_2^H A x_1 = y_2^H x_1 = 0.$$

Since $\|\cdot\|$ is continuous, there exists to each $0 < \mu \leqq \text{ort}\,(A) = y_1^H A x_1$ a number $k \geqq 0$ with

$$\mu = \frac{y_1^H A x_1}{\|(y_1 + k y_2)^H\|^D \|x_1\|} = \frac{(y_1 + k y_2)^H A x_1}{\|(y_1 + k y_2)^H\|^D \|x_1\|} \in O[A].$$

If $n = 2$, then the function

$$f(a, b) := \left| \frac{(b, a) A \begin{pmatrix} a \\ -b \end{pmatrix}}{\|(b, a)\|^D \left\| \begin{pmatrix} a \\ -b \end{pmatrix} \right\|} \right|$$

is continuous for all complex $(a, b) \neq (0, 0)$. Since

$$O[A] \supset \{f(a, b) \mid (a, b) \neq (0, 0), a, b \in C\}$$

and f assumes the values 0 and $y_1^H A x_1$, it also assumes every real value between 0 and ort $(A) = y_1^H A x_1$, since the set $\{(a, b) \mid a, b \in C, (a,b) \neq (0, 0)\}$ is connected. This completes the proof of Theorem 5.4.

A further property of ort (A) is gained by (2.4):

$$\begin{aligned}\text{ort}(A) = \text{ort}(A + \tau I) &= \max_{\substack{y,x\neq 0\\ y^H x = 0}} \frac{\text{Re}\, y^H(A + \tau I)x}{\|y^H\|^D \|x\|}\\ &\leq \max_{y,x\neq 0} \frac{\text{Re}\, y^H(A + I)x}{\|y^H\|^D \|x\|}\\ &= \text{lub}(A + \tau I),\end{aligned}$$

and therefore,

$$\text{ort}(A) \leq \min_{\tau} \text{lub}(A + \tau I). \tag{5.5}$$

Matrices A with lub $(A) = \min_\tau$ lub $(A + \tau I)$ are called *centered*.

The following example shows that equality does not always hold in (5.5). Take in C^2,

$$\|x\| := \max(|x_1|, |x_2|), \qquad \|y^H\|^D := |y_1| + |y_2|,$$

$$A := \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \qquad \text{lub}(A) = \max_i \sum_k |a_{ik}| = 3.$$

It is easy to see that A is centered. Any vectors x, y, with $\|x\| = \|y^H\|^D = 1$ and Re $y^H Ax =$ lub $(A) = 3$ must satisfy

$$\|Ax\| = 3\|x\| \quad \text{and} \quad y \parallel Ax.$$

Hence, we have for x and y only two possibilities:

$$x = e^{i\phi}\begin{pmatrix}1\\1\end{pmatrix}, \qquad Ax = e^{i\phi}\begin{pmatrix}3\\1\end{pmatrix}, \qquad y = e^{i\phi}\begin{pmatrix}1\\0\end{pmatrix}$$

or

$$x = e^{i\phi}\begin{pmatrix}1\\-1\end{pmatrix}, \qquad Ax = e^{i\phi}\begin{pmatrix}-1\\3\end{pmatrix}, \qquad y = e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}.$$

In either case, $y^H x = 0$ is not true. However, we show later that equality holds in (5.5) for the Euclidean norm (cf. Theorems 6.2 and 5.9).

The following theorem can be regarded as a substitute for the missing equality sign in (5.5).

THEOREM 5.6. *Let* $\|\cdot\|$ *be a strictly homogeneous norm in* C^n *and* A *a centered* $n \times n$ *matrix. Then*

$$0 \in \mathcal{H}(M),$$

where $M := \{y^H x \mid \|y^H\|^D\|x\| = 1,\ y^H Ax = \text{lub}(A)\}$. (Note that if $0 \in M$, then equality holds in (5.5).)

Proof. We show first that for any angle ϕ, $0 \leqq \phi \leqq 2\pi$, there exist vectors y^H and x such that

(5.7) $\quad \operatorname{Re} e^{i\phi} y^H x \geqq 0 \quad \text{and} \quad y^H A x = \operatorname{lub}(A), \qquad \| y^H \|^D = \| x \| = 1.$

Indeed, for every $k > 0$, there exist vectors y_k, x_k such that

$$\operatorname{lub}\left(A + \frac{e^{i\phi}}{k} I\right) = \operatorname{Re} y_k A x_k + \operatorname{Re} \frac{e^{i\phi}}{k} y_k^H x_k, \qquad \| y_k^H \|^D = \| x_k \| = 1.$$

Since A is centered, we have

$$\operatorname{lub}\left(A + \frac{e^{i\phi}}{k} I\right) \geqq \operatorname{lub}(A) \geqq \operatorname{Re} y_k^H A x_k,$$

proving $\operatorname{Re} e^{i\phi} y_k^H x_k \geqq 0$. Since

$$\lim_{k\to\infty} \operatorname{lub}\left(A + \frac{e^{i\phi}}{k} I\right) = \operatorname{lub}(A)$$

and y_k, x_k belong to the compact sets $\{y \mid \| y^H \|^D = 1\}$ and $\{x \mid \| x \| = 1\}$, respectively, there are subsequences of the y_k^H, x_k converging to vectors y^H, x with the properties (5.7).

Now (5.7) means that in each of the half-planes

$$H_\phi^- := \{\alpha \mid \operatorname{Re} e^{i\phi}\alpha \leqq 0\} \quad \text{and} \quad H_\phi^+ := \{\alpha \mid \operatorname{Re} e^{i\phi}\alpha \geqq 0\}$$

there is an element of M:

(5.8) $\qquad H_\phi^- \cap M \neq \varnothing, \qquad H_\phi^+ \cap M \neq \varnothing.$

Hence, if 0 were not in the (compact) convex hull $\mathcal{H}(M)$ of the compact set M, there would exist a line

$$H_\phi := \{\alpha \mid \operatorname{Re} e^{i\phi}\alpha = 0\}$$

separating 0 strictly from $\mathcal{H}(M)$ in contradiction to (5.8).

It is known that the spectral radius $\rho(A) = \max_i |\lambda_i(A)|$ of a matrix A is less or equal to lub (A):

$$\rho(A) \leqq \operatorname{lub}(A).$$

In connection with inequality (5.5) this gives rise to the question whether it is true that

$$\operatorname{ort}(A) \geqq \min_\tau \rho(A + \tau I),$$

that is, whether the spectrum $\Lambda(A)$ of A can be translated into $O[A]$. A partial answer is given in the following theorem.

THEOREM 5.9. *Let* $\| \cdot \|$ *be an absolute norm in* C^n *and* $T = (t_{ik})$ *be an* *upper (lower) triangular* $n \times n$ *matrix. Then*

$$\text{ort}\,(T) \geqq \min_{\tau} \rho(T + \tau I) = \min_{\tau} \max_{j} | t_{jj} + \tau |.$$

Proof. Without loss of generality we assume that T is upper triangular. We may also assume (because of (5.2)) that

$$\rho(T) = \min_{\tau} \rho(T + \tau I),$$

that is,

$$0 \in \mathcal{K}(Q), \qquad Q := \{t_{ii} \mid | t_{ii} | = \max_{j} | t_{jj} |\}.$$

Hence, by a theorem of Carathéodory (see, e.g., [3]) one can find at most $k \leqq 3$ indices i_j and numbers μ_j such that

$$0 = \sum_{j=1}^{k} \mu_j t_{i_j i_j}, \qquad t_{i_j, i_j} \in Q, \qquad \mu_j > 0, \qquad \sum_{j=1}^{k} \mu_j = 1.$$

If $k = 1$, then $t_{ii} = 0$ for all i and we are finished. We treat here only the case $k = 3$; the case $k = 2$ can be similarly proved. Moreover, we suppose without loss of generality $i_j = j, j = 1, 2, 3$. We then have

$$0 = \sum_{j=1}^{3} \mu_j t_{jj} = | t_{11} | \sum_{j=1}^{3} \mu_j e^{i\phi_j}, \qquad \mu_j > 0, \qquad \sum_{j=1}^{3} \mu_j = 1,$$

where ϕ_j is such that

$$t_{jj} = | t_{jj} | e^{i\phi_j}.$$

Define

$$w := \sum_{j=1}^{3} \mu_j e^{i\phi_j} e_j, \qquad u := \sum_{j=1}^{3} e_j, \qquad v := | w |,$$

e_j being the jth axisvector. Hence,

$$w^H u = \sum_{j=1}^{3} \mu_j e^{-i\phi_j} = \overline{\sum_{j=1}^{3} \mu_j e^{i\phi_j}} = 0,$$

$$v^H u = \sum_{j=1}^{3} \mu_j = 1,$$

$$u \geqq 0, \qquad v \geqq 0, \qquad u_i = 0 \text{ if and only if } v_i = 0.$$

By Theorem 4.3 there exists a diagonal matrix $D \geqq O$, $D = \text{diag}\,(d_i)$ with $v^H D \parallel D^{-1} u$. Since $\| \cdot \|$ is absolute, we have for every phase matrix $D_\Theta := \text{diag}\,(e^{i\theta_1}, \cdots, e^{i\theta_n})$, $| D_\Theta | = I$,

$$\| w^H D D_\Theta \|^D \| D_\Theta^{-1} D^{-1} u \| = \| v^H D \|^D \| D^{-1} u \| = v^H u = 1.$$

Then

$$(w^H DD_\Theta)T(D_\Theta^{-1}D^{-1}u) = |t_{11}|\sum_{j=1}^{3}\mu_j + \sum_{j<k\leqq 3} P_{jk}e^{i(\theta_j-\theta_k)},$$

with

$$P_{jk} := \frac{d_j}{d_k}\mu_j t_{jk} e^{-i\phi_j}.$$

It is clear that we can choose the angles θ_j, $j = 1, 2, 3$, in such a way that

$$\mathrm{Re} \sum_{j<k\leqq 3} P_{jk}e^{i(\theta_j-\theta_k)} \geqq 0.$$

We have then (if $\theta_j = 0$ for $j > 3$)

$$\mathrm{Re}\,(w^H DD_\Theta TD_\Theta^{-1}D^{-1}u) \geqq |t_{11}|\sum_{j=1}^{3}\mu_j = |t_{11}|$$

and simultaneously

$$w^H DD_\Theta D_\Theta^{-1}D^{-1}u = w^H u = 0,$$
$$\| w^H DD_\Theta \|^D \| D_\Theta^{-1}D^{-1}u \| = 1.$$

This proves Theorem 5.9, and we note the following corollary.

COROLLARY 5.10. *If* $\|\cdot\|$ *is an absolute norm in* C^n *and* $D = \mathrm{diag}\,(d_1, \cdots, d_n)$ *is a diagonal matrix, then*

$$\mathrm{ort}\,(D) = \min_\tau \mathrm{lub}\,(D + \tau I) = \min_\tau \max_j |d_j + \tau|.$$

This is a simple consequence of Theorem 2.2.

6. For the Euclidean norm $\| x \| := \sqrt{x^H x}$ many results of §5 can be sharpened. In this case $O[A]$ is defined by

$$O[A] := \{y^H Ax \mid y^H y = x^H x = 1, y^H x = 0\}.$$

Also, it is known that lub (A) is given by

$$\mathrm{lub}\,(A) = \sqrt{\rho(A^H A)},$$

and any normed vectors y, x, $\| y \| = \| x \| = 1$ with $y^H Ax = \mathrm{lub}\,(A)$ must satisfy $A^H Ax = \mathrm{lub}^2\,(A)x$, $\| Ax \| = \mathrm{lub}\,(A)\| x \|$ and $y^H \parallel Ax$, that is, since $\|\cdot\|$ is self-dual,

$$y^H = \frac{x^H A^H}{\| Ax \|}.$$

This shows that for centered A the set

$$M = \{y^H x \mid y^H Ax = \mathrm{lub}\,(A),\ y^H y = x^H x = 1\}$$

(see Theorem 5.6) is equivalently described by

$$(6.1)\quad \begin{aligned} M &= \left\{\frac{x^H A^H x}{\|Ax\|\,\|x\|} \mid A^H Ax = \operatorname{lub}^2(A)x\right\} \\ &= \frac{1}{\operatorname{lub}(A)} \cdot \left\{\frac{x^H A^H x}{x^H x} \mid A^H Ax = \operatorname{lub}^2(A)x\right\}. \end{aligned}$$

Now, the set

$$L := \{x \mid A^H Ax = \operatorname{lub}^2(A)x\}$$

is a certain linear subspace of C^n, say of dimension $k(\geqq 1)$. Introduce in L an orthonormal basis of k vectors $v_1, \cdots, v_k$, and form the $n \times k$ matrix $V := (v_1, \cdots, v_k)$. Then $V^H V = I_k$, and L is also given by

$$L = \{x \mid x = Vy \text{ for some } y \in C^k\}.$$

Hence, it follows from (6.1) that

$$\begin{aligned} M &= \frac{1}{\operatorname{lub}(A)} \cdot \left\{\frac{y^H V^H A^H V y}{y^H V^H V y} \mid 0 \neq y \in C^k\right\} \\ &= \frac{1}{\operatorname{lub}(A)} \cdot \left\{\frac{y^H Q y}{y^H y} \mid 0 \neq y \in C^k\right\} \end{aligned}$$

with the $k \times k$ matrix $Q := V^H A^H V$. Hence, for the Euclidean norm, M is equal to $(1/\operatorname{lub}(A))G[Q]$, namely, the multiple of the ordinary G-field of values for the matrix Q. Now, by Hausdorff [5], $G[Q]$ is convex. Together with Theorem 5.6 this implies the following.

THEOREM 6.2. *Let* $\|\cdot\|$ *be the Euclidean norm. Then for all matrices* A,

$$\operatorname{ort}(A) = \min_\tau \operatorname{lub}(A + \tau I).$$

COROLLARY 6.3. *For the Euclidean norm,*

$$\operatorname{ort}(A) \geqq \min_\tau \rho(A + \tau I)$$

holds for all matrices A.

For normal matrices A one has

$$\operatorname{lub}(A) = \sqrt{\rho(A^H A)} = \rho(A).$$

Hence we have a further corollary.

COROLLARY 6.4. *For the Euclidean norm,*

$$\operatorname{ort}(A) = \min_\tau \rho(A + \tau I)$$

holds for all normal matrices A.

Therefore,

$$\begin{aligned}\text{ort}\,(A) &= \max\,\{\text{Re}\, y^H A x \mid y^H y = x^H x = 1,\, y^H x = 0\} \\ &= \max\,\{| y^H A x | \mid y^H y = x^H x = 1,\, y^H x = 0\}\end{aligned}$$

gives the radius of the smallest circle containing all eigenvalues of a normal matrix A. By a result of Jung [7], this radius gives a lower bound of the spread $s(A)$ of A:

$$s(A) := \max_{i,j} | (\lambda_i(A) - \lambda_j(A)) | \geqq 3^{1/2}\, \text{ort}\,(A),$$

an estimate first proved by Mirsky [8] for normal matrices A.

Now, we shall derive a lower bound for the quantities

$$\bar{\rho}(A) := \min_{\tau} \rho(A + \tau I),\ s(A),$$

also for nonnormal matrices A. The basic technique is the same as that used by Henrici [6] in order to obtain upper bounds for the departure of $G[A]$ from the convex hull $\mathcal{H}(\Lambda(A))$ of the spectrum of A.

To any matrix A there is at least one unitary matrix U such that

$$U^H A U = T = D + M \quad \text{with} \quad D = \text{diag}\,(t_{11}, \cdots, t_{nn}),$$

where $T = (t_{ik})$ is an upper triangular matrix. If $\nu(A)$ is any matrix norm, then the *ν-departure of normality* $\Delta_\nu(A)$ of the matrix A is defined by (see [6])

$$\Delta_\nu(A) := \min_{U: U^H A U \text{ triangular}} \nu(M).$$

For the matrix norms

$$\alpha(A) := \sum_{i,j} | a_{ij} |,$$

$$\epsilon(A) := (\text{tr}\, A^H A)^{1/2},$$

Henrici [6] obtained the following computable estimate:

$$\Delta_\epsilon(A) \leqq \sqrt[4]{\frac{n^3 - n}{12}}\, \sqrt{\epsilon(A^H A - A A^H)}\,.$$

We now prove the following theorem.

THEOREM 6.5. *If A is any matrix and $\|\cdot\|$ the Euclidean norm, then*

$$\bar{\rho}(A) \geqq \text{ort}\,(A) - \Delta_\epsilon(A) \geqq \text{ort}\,(A) - \Delta_\alpha(A).$$

Proof. Given a point ξ of $O[A]$,

$$\xi = y^H A x, \qquad y^H y = x^H x = 1, \qquad y^H x = 0,$$

we have to find a point $\eta \in \{\mu \mid |\mu| \leqq \tilde{\rho}(A)\}$ such that

$$|\xi - \eta| \leqq \Delta_\epsilon(A) \leqq \Delta_\alpha(A).$$

Let $U^H A U = T = D + M$ be a triangular matrix, U unitary and $D = \text{diag}(t_{11}, \cdots, t_{nn})$. Define

$$u := Ux, \qquad v^H := y^H U^H.$$

Then $u^H u = v^H v = 1$, $v^H u = 0$ and

$$\xi = v^H T u = v^H D u + v^H M u = \eta + v^H M u$$

with $\eta := v^H D u \in \{\mu \mid |\mu| \leqq \tilde{\rho}(A)\}$, since $v^H u = 0$. Cauchy's inequality then yields

$$|\xi - \eta|^2 = |v^H M u|^2 = \Big|\sum_{i<j} m_{ij}\bar{v}_i u_j\Big|^2 \leqq \sum_{i<j} |m_{ij}|^2 \sum_{i<j} |v_i u_j|^2$$
$$\leqq \epsilon(M^2) \sum_{i<j} |v_i u_j| \leqq \epsilon(M)^2 (v^H v\, u^H u)^{1/2} = \epsilon(M)^2.$$

This proves $|\xi - \eta| \leqq \Delta_\epsilon(A)$ and $\tilde{\rho}(A) \geqq \text{ort}(A) - \Delta_\epsilon(A)$. The second part follows from the inequality $\epsilon(A) \leqq \alpha(A)$ being true for all matrices A.

Both inequalities in Theorem 6.5 are sharp. There are nonnormal matrices A with

$$\tilde{\rho}(A) = \text{ort}(A) - \Delta_\epsilon(A) = \text{ort}(A) - \Delta_\alpha(A). \tag{6.6}$$

Take, for example, $A := I + M$ with

$$m_{ij} := \begin{cases} 1 & \text{for } i = n-1, j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{\rho}(A) = 0$, $\epsilon(M) = \alpha(M) = 1$. The vectors $y := e_{n-1}$, $x := e_n$ satisfy $\| y^H \|^D = 1 = \| x \|$, $y^H x = 0$, and yield $y^H A x = 1 \in O[A]$, proving that (6.6) can hold.

In terms of the spread $s(A)$ of A, the result of Theorem 6.5 reads as follows: $3^{1/2}(\text{ort}(A) - \Delta_\epsilon(A)) \leqq s(A)$. It should be noted, however, that the above estimates can be rather pessimistic. It is possible that ort $(A) - \Delta_\epsilon(A) \leqq 0$.

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