A DISTRIBUTED PATH ALGORITHM
AND ITS CORRECTNESS PROOF*

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ABSTRACT

A distributed program is developed to allow a process in a network to determine a path from itself to any other process, assuming that the topology of the entire network is not known to any process and that each process knows only the names of the processes to which it is directly connected. The solution, written in CSP, is proved correct and deadlock-free.

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1. Introduction

Consider a network of processes connected by unidirectional communication links. Such a network can be modeled by a directed graph in which processes are represented by nodes and communication links by edges. Let the graph representing the network be strongly connected, and assume each process has a unique name that is known to all other processes in the network. Further, assume that the topology of the entire network is not known to any process, although each process does know the names of the processes to which it is directly connected. Desired is a distributed program that will allow a process to determine a path—a sequence of names of connected processes—from itself to any other process.

The program will be described in a variant of Hoare's Communicating Sequential Processes notation (CSP) [Hoar78], extended to include output guards. The solution is proved correct and deadlock-free using the proof method of Levin and Gries [LeGr81].

2. The Environment

Given are processes \( P_1, P_2, ..., P_N \). Associated with each process \( P_i \) are the successors of \( P_i \), a set \( T_i \) of processes to which \( P_i \) can transmit messages, and the predecessors of \( P_i \), a set \( R_i \) of processes from which \( P_i \) can receive messages. We require \( j \in R_i \iff i \in T_j \).

A path is a finite sequence of process names. A sequence consisting of \( a_0, a_1, ..., a_n \) in that order will be denoted by \(<a_0, a_1, ..., a_n>\), the null sequence by \(<>\). In order to construct and examine sequences, two functions will be useful:

\[
\text{first}(S) = \begin{cases} 
 a_0 & \text{if } S = <a_0, ..., a_n> \\
 \Phi & \text{if } S = <> 
\end{cases}
\]

\[
\text{last}(S) = \begin{cases} 
 a_n & \text{if } S = <a_0, ..., a_n> \\
 \Phi & \text{if } S = <> 
\end{cases}
\]

where \( \Phi \) is a name distinct from any \( P_i \). In addition, the sequence obtained by concatenating an element \( i \) to the end of a sequence \(<a_0, ..., a_n>\) will be denoted by \(<a_0, ..., a_n> \cdot i\).

If at termination, process \( P_i \) is to have a path from \( S \) to \( D \) in a variable \( p_i \), the desired postcondition of \( P_i \) must include the predicate \( \text{path}(p_i,S,D) \), where
\[\text{path}(p,x,y) \equiv [\text{first}(p)=x \land \text{last}(p)=y \land (p=<a_0, \ldots, a_n> \Rightarrow ((\forall i: 0 < i \leq n: a_i \in T_{i-1}) \land (\forall i: 0 \leq i < n: a_i \in R_{i+1})))\]

That is, \(p\) is a feasible path from \(x\) to \(y\).

3. The Algorithm

We now develop a program to solve the problem described above. The proof of the program is detailed in sections 4 and 5.

Execution of the program can be viewed as being divided into two phases. The first phase commences after some process \(P_S\) has selected another process, say \(P_D\), with which it wishes to communicate. \(P_S\) then sends information about a partial path and the desired destination (\(P_D\)) to each member of \(T_S\). These processes, upon receipt of such a message, concatenate their names to the partial paths they receive, and send the results to their successors, etc. A wave of partial paths emanates from \(P_S\) through the network. Thus, the first phase is characterized by no process possessing a complete path.

Since the graph representing the network is strongly connected, eventually the wave of messages will reach \(P_D\). Like any other process, \(P_D\) concatenates its name to a path. At this point, the system enters the second phase, which is characterized by at least one process having a complete path. \(P_D\) sends this complete path to all members of \(T_D\), which in turn send it to their successors, etc. This second wave spreading out from \(P_D\) will eventually reach \(P_S\), since the graph is strongly connected. \(P_D\) cannot simply send the complete path back along its own reverse, since the network edges are directed.

Once a process has sent the complete path to all of its successors, it can terminate. In CSP, as originally defined in [Hoar78], this would pose no problem, since a communication guard that names a terminated process is false. However, the proof system of [LeGr81] is for a variant of CSP that does not support this distributed termination facility. In this variant, a guard evaluates to false only when its Boolean component is false, so termination of a guarded command loop must be done based on local
information only. Since a process is sending a complete path to all of its successors, this suggests that it should also wait to receive a complete path from all of its predecessors. Once this happens, the process will be involved in no further communications and can terminate without causing deadlock. The program developed in this section does exactly this.

To develop a program from the above, we first make several observations:

1. A process $P_i$ is involved in at most $|R_i| + 2|T_i| + 1$ communications. It could send a partial and a complete path to each member of $T_i$, receive a complete path from each member of $R_i$, and receive a partial path from some member of $R_i$. This suggests using a loop that will iterate at most $|R_i| + 2|T_i| + 1$ times.

2. A process need receive at most one partial path. Indeed, allowing it to receive more than one could result in a program that does not terminate, with two processes eternally exchanging and extending partial paths.

3. A complete path is a partial path that has the destination as its last element. Each process $P_i$ will use two different variables: $pp_i$ to hold a partial path and $p_i$ to hold a complete path. Given this, $P_i$ will be in the first phase if $pp_i \neq <>$ and in the second if $p_i \neq <>$.

For a process to terminate, it must send a complete path to all of its successors and receive one from all of its predecessors. We can use a set of Boolean variables $\{s_i^j, j \in T_i\}$, that have value true iff a complete path has been sent by $P_i$ to $P_j$, and another set $\{r_i^j, j \in R_i\}$, that have value true iff a complete path has been received by $P_i$ from $P_j$. The strengthened postcondition for process $P_i$ is:

$$Q_i: \text{path}(p_i, S, D) \land (\forall j \in T_i: s_i^j) \land (\forall j \in R_i: r_i^j)$$

We can now write a program:
\[ P :: \text{for } i := 1 \text{ to } n \\
\quad \text{for } j \in T_i: s_i^j := \text{false} \text{ rof } \\
\quad \text{for } j \in R_i: r_i^j := \text{false} \text{ rof } \\
\quad \text{rof } \\
\quad [ P_1 \ | \ \cdots \ | \ P_N ] \]

where

\[ P_i :: \text{if } i = S \rightarrow p_i, pp_i, d_i := \langle\rangle, \langle S\rangle, D \\
\quad \text{if } i \neq S \rightarrow p_i, pp_i, d_i := \langle\rangle, \langle\rangle, \Phi \\
\quad \text{f1 } \\
\quad \text{do } \text{f1 } \\
\quad \ldots \\
\quad \text{f1 } \\
\quad \text{skip } \\
\quad \text{f1 } \\
\quad \text{f1 } \]

4. The Weak Correctness Proof

In the logic of [LeGr81], a proof of a CSP program consists of a weak correctness proof, which shows correctness in the absence of deadlock, and a proof that the program is deadlock-free. A weak correctness proof is itself divided into a sequential proof, a satisfaction proof, and a non-interference proof.

A sequential proof is an annotation of the program using axioms and rules of inference, in which any assertion may appear as the postcondition of a communication statement. A satisfaction proof shows that such assertions are valid. A non-interference proof is used to show that assertions involving shared auxiliary variables are valid in light of concurrent execution.\(^1\) Our proof of \(P_i\) uses a set of auxiliary variables \(\{a_i^j, j \in T_i\}\), which are initially false. Variable \(a_i^j\) will be set to true iff a path (partial or complete) is sent from \(P_i\) to \(P_j\). That is,

\(^1\)Although shared program variables are not allowed in a CSP program, shared auxiliary variables are allowed.
\[ a^l_i \Rightarrow pp_j \not\in \mathbb{L} \lor p_j \not\in \mathbb{L} \]

### 4.1. The Sequential Proof

Here is the annotated program \( P_i \) with auxiliary variables added.

\[
P_i ::
\begin{align*}
\{DL_i \land (\forall j \in T_i: \neg a^l_j) \land (\forall j \in R_i: \neg r^l_j \land \neg a^l_j)\} \\
\text{if } i = S \rightarrow p_i, pp_i, d_i := \langle\rangle, \langle\rangle, S, D \\
\text{fi} \\
\{INV_i \land DL_i\} \\
\text{do} \quad \begin{array}{l}
P_j((pp_i, d_i, a^l_j) \rightarrow \{INV_i \land INV_2 \land DL_i \land d_i = D \land \text{path}(pp_i, S, j) \land p_i = \langle\rangle \land a^l_j\}) \\
\text{if } d_i = i \rightarrow p_i, pp_i := pp_i \cdot i, \langle\rangle \\
\text{fi} \\
\text{fi} \\
\{INV_i \land DL_i\} \\
\text{do} \quad \begin{array}{l}
P_j((pp_i, d_i, true) \rightarrow \text{skip} \quad \{INV_i \land DL_i\}) \\
\text{fi} \\
\{INV_i \land DL_i\} \\
\text{do} \quad \begin{array}{l}
P_j((pp_i, r_j, a_j) \rightarrow \{INV_2 \land INV_3 \land DL_i \land \text{path}(p_i, S, D)\}) \\
pp_i := \langle\rangle \quad \{INV_i \land DL_i\} \\
\text{fi} \\
\{INV_i \land DL_i\} \\
\text{do} \quad \begin{array}{l}
P_j((pp_i, true, true) \rightarrow \{INV_i \land DL_2 \land DL_3 \land r^l_i \land p_i \not\in \mathbb{L} \land \langle\rangle \land (\forall k \in T_i: k \not\in j: \neg a^l_k \Rightarrow \neg r^l_k)\}) \\
sp_i := true \quad \{INV_i \land DL_i\} \\
\text{fi} \\
\{INV_i \land DL_i \land (p_i \not\in \mathbb{L} \lor pp_i \not\in \mathbb{L}) \land pp_i = \langle\rangle \land (\forall j \in R_i: r^l_j) \land (\forall j \in T_i: a^l_j) \land (\forall j \in R_i: (r^l_j)) \land (\forall j \in T_i: (a^l_j))\}
\end{array}
\end{array}
\end{align*}
\]
where

\[ INV_i \equiv INV_{i1} : (p_i = <> \lor pp_i = <>) \land \\
INV_{i2} : (p_i = <> \lor path(p_i,S,D)) \land \\
INV_{i3} : (pp_i = <> \lor (path(pp_i,S,i) \land d_i = D)) \]

\[ DL_i \equiv DL_{i1} : (\forall j \in T_i : a_j^i = r_j^i) \land \\
DL_{i2} : (p_i = <> \Rightarrow (\forall j \in R_i : r_j^i) \land (\forall j \in T_i : \neg r_j^i)) \land \\
DL_{i3} : (\forall k \in R_i : a_k^i \Rightarrow (p_i \neq <> \lor pp_i \neq <>)) \]

4.2. The Satisfaction Proof

A satisfaction proof shows the validity of the postconditions of communication actions. Suppose that process \( A \) contains \( B!(\overline{x}) \) and process \( B \) contains \( A?\overline{x} \), where \( \overline{x} \) is a vector of expressions and \( \overline{\bar{x}} \) is a vector of variables. Suppose also that \( \overline{x} \) and \( \overline{\bar{x}} \) are of the same length and that their corresponding elements are of the same type. In this case, \( B!(\overline{x}) \) and \( A?\overline{x} \) form a matching pair. To show that communication establishes the desired postconditions, the following rule is used:

If \( \{ P \} B!(\overline{x}) \{ Q \} \) and \( \{ R \} A?\overline{x} \{ V \} \) form a matching pair, then \((P \land R) \Rightarrow (Q \land V)\).

The program has only two matching pairs: the send/receive of a partial path and the send/receive of a complete path. We consider each below for process \( P_i \) sending to process \( P_j \).

Sending a partial path. The above rule requires us to show

\[ (INV_i \land DL_i \land pp_i \neq <> \land INV_j \land DL_j \land p_j = <> \land pp_j = <>) \Rightarrow \\
(INV_i \land DL_i \land INV_{i1} \land INV_{i2} \land DL_j \land d_j = D \land path(pp_j,S,i) \land p_j = <> \land a_j^i_{pp_j,i} \land true) \]

From the antecedent, we can deduce

\[ INV_i \land DL_i \land INV_j \land DL_j \land d_i = D \land p_i = <> \land p_j = <> \land pp_j = <> \land path(pp_j,S,i), \]

which implies the result of the textual substitution

\[ INV_i \land DL_i \land (p_j = <> \lor pp_j = <> ) \land INV_{i2} \land DL_{i1} \land DL_{i2} \land \\
(\forall k \in R_i : a_k^i \Rightarrow (p_j \neq <> \lor pp_j \neq <>))_{true} \land d_j = D \land path(pp_j,S,i) \land p_j = <> \land true. \]
Sending a complete path. The rule requires us to show
\[
(INV_i \land DL_i \land \neg s_i \land p_i \not\iff \land INV_j \land DL_j \land \neg r_j) 
\Rightarrow \\
(INV_i \land DL_0 \land DL_i \land r_i \land p_i \not\iff \land (\forall k \in T_i : k \neq j : \neg s_k \Rightarrow \neg r_k) 
\land \quad INV_{\bar{j}} \land INV_{\bar{s}} \land DL_j \land path(p_j, S, D)_{j, true, true}.
\]
From the antecedent, we can deduce
\[
INV_i \land DL_i \land \neg s_i \land path(p_i, S, D) \land INV_j \land DL_j \land \neg r_j
\]
which implies the result of the textual substitution
\[
INV_i \land DL_0 \land DL_i \land true \land p_i \not\iff \land (\forall k \in T_i : k \neq j : \neg s_k \Rightarrow \neg r_k) \land \\
INV_{\bar{j}} \land INV_{\bar{s}} \land DL_j \land (DL_0)_{j, true, true} \land (DL_i)_{j, true, true} \land path(p_j, S, D).
\]

4.3. The Non-Interference Proof

Constructing a non-interference proof can be a formidable task, since it requires showing that no assignment statement or communication action in any process can invalidate any assertion in any other process. Fortunately, this task is greatly simplified if shared variables are synchronously altered. A variable v is synchronously altered in process P_i if its value is only changed by assignments to v in P_i, input commands in P_i, or input commands in other processes that match output commands in P_i. Under these restrictions, no process can asynchronously change the value of such a variable. Thus, synchronously altered variables are not subject to interference.

All variables referenced in assertions in P_i are synchronously altered, so the proof of non-interference is immediate.

4.4. Proving Termination

To prove total correctness, we must show that the iterative construct in each P_i terminates. The termination function of P_i is:
\[
Term_i : (N j \in T_i : \neg s_j) + (N j \in T_i : \neg a_j) + (N j \in R_i : \neg r_j) + \beta_i
\]
where

\[ N \text{ denotes "number of," and} \]

\[ \beta_i \equiv \text{if } p_i = p_i \neq <> \rightarrow 1 \quad \text{[} pp_i \neq <> \lor p_i \neq <> \rightarrow 0 \text{]}. \]

To show that this function has the desired properties, we must prove

(1) \( Term_i = 0 \) implies all guards in the loop of \( P_i \) are false, and

(2) each iteration of \( P_i \) decreases the value of \( Term_i \) by at least 1

For the first point, note that

\[ Term_i = 0 \Rightarrow (\forall j \in T_i : s_j) \land (\forall j \in R_i : r'_j) \land (pp_i \neq <> \lor p_i \neq <>). \]

so all but the second guard of the iterative construct will be false. In addition, from \( DL S_i \) and \( Term_i = 0 \), we can deduce \( p_i \neq <> \), which implies \( pp_i = <> \) by \( INV_i \). Hence the second guard is also false and the loop will terminate.

It is clear that sending a complete path will decrease \( Term_i \) by at least 1, as the path can only be sent if \( \neg s'_j \), and \( s'_j \) is set to true immediately thereafter. A complete path can only be received if \( \neg r'_j \), and receiving the path makes \( r'_j \) true, so this will also decrease \( Term_i \). A partial path can only be received if \( p_i = <> \land pp_i = <> \) (i.e. \( \beta_i = 1 \)), and receiving it will set \( \beta_i \) to 0. Finally, a partial path can only be sent from \( P_i \) to \( P_j \) if \( pp_j = <> \land p_j = <> \). From \( DL S_j \) we can then infer \( \neg s'_j \), and since the send will establish \( s'_j, \ Term_i \) will decrease by 1.

**Note:** No process name appears more than once in any \( p_i \) at any time. Informally, we can see this because a process concatenates its name to a path only within the first guard of the iterative construct. This will establish \( (pp_i \neq <> \lor p_i \neq <>). \) Once this has been established, no later action of the program will falsify it, so the concatenation can take place only once. However, since acyclicity of \( p_i \) is not essential, we have left the observation informal. **End of Note.**

5. The Deadlock-Freedom Proof

A formal proof of strong correctness consists of a weak correctness proof and a proof that progress is possible in any possible configuration (that is, in any program state other than complete termination).
The latter proof is carried out by showing that the assumption of deadlock implies a contradiction.

In a CSP program, processes can only be blocked at communication commands, at the end of parallel constructs, and at alternative or iterative commands that contain communication guards. Unfortunately, in general the number of possible configurations is exponential in the number of processes in a program. Because of this, a formal proof is usually too large to aid in understanding a program. Hence, the proof presented here is informal.

The proof will consist of considering each possible blocking point in turn and showing that the assumption that a process can be permanently blocked there results in a contradiction. (This method is suggested in [LeGr81].)

1. Suppose that a process $P_i$ is permanently blocked waiting to send a complete path to some process $P_j$, $j \in T_i$, so $\neg s^j_i \land p_i \neq <>$. Since $P_j$ is not prepared to receive, it must be that $r^j_i$. But from $DLI_i$, we can conclude $\neg r^j_i$, a contradiction.

2. Suppose that $p_i = <> \land pp_i = <>$ and $P_i$ is permanently blocked. Clearly, $P_i$ is ready to receive a partial path from any member of $R_i$. From $DL2_i$ we can infer $(\forall j \in R_i; \neg r^j_i)$, so $P_i$ is also ready to receive a complete path from any member of $R_i$. We know $(\forall j \in R_i; \neg s^j_i)$ by $DLI_j$ and since the postcondition of any $P_j$ includes $s^j_i$, none of the $P_j$ can have terminated. Thus, all the $P_j$ are blocked in their loops and not ready to send, so $(\forall j \in R_i; p_j = <> \land pp_j = <>)$. That is, all the $P_j$ are blocked in the same fashion as $P_i$, and by induction, all processes must be so blocked. But this is impossible, because in $P_S$, initially $pp_S = <S>$, and we observe that thereafter $P_S$ maintains $(pp_S \neq <> \lor p_S \neq <>)$.

3. Suppose $pp_i \neq <>$ and $P_i$ is permanently blocked. By $INV1$, $p_i = <>$, so by $DL2_i$, $(\forall j \in R_i; \neg r^j_i)$ and by $DLI_j$, $j \in R_i$, $(\forall j \in R_i; \neg s^j_i)$. Since $P_i$ is permanently blocked, it must be that no member of $R_i$ will ever be prepared to send to $P_i$. Thus $p_j = <>$ and all the $P_j$ are permanently blocked at the last guarded command in the loop. (They cannot have terminated by $DL2_j$, since the postcondition of $P_j$ includes $s^j_i$.) It cannot be that $pp_j = <>$, as the previous proof shows that $P_j$ would not then be permanently blocked, so $pp_j \neq <>$ and all the $P_j$ are blocked in the same way as $P_i$, and by induction, all processes must be so blocked. But this is impossible, as $P_D$ has
only immediately after receiving a partial path, and since it immediately thereafter sets
\( pp_{D} \neq <> \) and \( p_{D} \) to the complete path, \( P_{D} \) will never block with \( pp_{D} \neq <> \).

(4) Suppose that \( \neg r_{j}^{i} \) for some \( j \in R_{i} \) and \( P_{i} \) is permanently blocked waiting to receive a complete
path from \( P_{j} \). As in the previous proof, \( P_{j} \) has not terminated, as we can deduce \( \neg e_{j}^{i} \) by \( DL1_{j} \).
Since \( P_{i} \) is blocked, \( p_{j} = <> \) and \( P_{j} \) must also be permanently blocked. But if \( pp_{j} = <> \), \( P_{j} \) is
not blocked by (2), whereas if \( pp_{j} \neq <> \), \( P_{j} \) is not blocked by (3), so \( P_{j} \) cannot be permanently
blocked and our assumption that \( P_{i} \) was blocked must be false.

6. Discussion

In [Mart80], A.J. Martin proposed a solution to this problem in terms of a \textit{Broadcast} operation (with which a node can send asynchronously and simultaneously along all output edges) and a
\textit{Select} operation, which chooses one input edge containing a message and reads that message. Messages on a given edge are received in the order sent. \textit{Broadcast} is delayed if sending along some
edge would exceed the "slack bound" (buffering capacity) of that edge. \textit{Select} is delayed if there is
no pending message on any input edge. Martin proved that his algorithm requires a slack bound of at
least two on each edge to avoid deadlock. Since our solution uses synchronous message passing, it
requires a slack bound of zero. Martin's requirement for buffering seems to stem from the fundamentally
asymmetric nature of \textit{Broadcast} and \textit{Select}, since \textit{Broadcast} can send multiple messages,
while \textit{Select} can only receive one at a time.

Martin's algorithm is reproduced below (with minor notation changes):
\[ \text{if } i = S \rightarrow p, r, d, l := \langle S \rangle, \langle S \rangle, D, S; \text{ BROADCAST}(p, d) \]
\[ \text{if } i \neq S \rightarrow r, l := \langle \rangle, \text{NIL} \]
\[ k := 0; \{ K = | R, i | \} \]
\[ \text{do } k < K \rightarrow \text{SELECT}(p, d); \]
\[ \text{if } \text{last}(p) \neq d \land l = \text{NIL} \rightarrow p := p \cdot i; l, r := i, p; \]
\[ \text{BROADCAST}(p, d) \]
\[ \text{if } \text{last}(p) = d \land l \neq d \rightarrow l := d; k := k + 1; r := p; \]
\[ \text{BROADCAST}(p, d) \]
\[ \text{if } \text{last}(p) = d \land l = d \rightarrow k := k + 1 \]
\[ \text{if } \text{last}(p) \neq d \land l \neq \text{NIL} \rightarrow \text{skip} \]
\[ \text{od} \]

At termination, \( r \) holds the path from \( S \) to \( D \). As \( p \) will hold the same path, it seems that \( r \) is mainly needed for the proof.

Our algorithm can be modified to use \text{BROADCAST} and \text{SELECT} operations, with only minor changes to its structure. However, there does not seem to be any easy way to modify Martin's algorithm to use a slack bound of zero.

Our algorithm, like Martin's, has a worst-case complexity of \( O(E) \) messages sent (one partial path and one complete path sent along each link). The best-case complexity is about the same (anywhere between \( E \) and \( 2E \) messages sent, depending on the network), since a complete path must be sent along all links in addition to the partial path sent from \( S \) to \( D \).

Most published distributed path algorithms deal with networks that allow bidirectional communication along any edge. While this is a reasonable model of wired communication systems, it is not always valid in other types, such as certain radio networks. An algorithm that does not depend on bidirectional communication is thus potentially valuable.

Although the algorithm described in this paper is not long, it is far from trivial to prove it correct. The difficulty is justified, however, by the inherent complexity of distributed programs. Attempting to
demonstrate correctness through test cases or informal arguments leaves too many possibilities for error. A formal proof, on the other hand, gives far greater confidence in the correctness of the program.

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References

