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THE STRONG STABILITY OF ALGORITHMS FOR SOLVING SYMMETRIC LINEAR SYSTEMS*

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Abstract. An algorithm for solving linear equations is stable on the class of nonsingular symmetric matrices or on the class of symmetric positive definite matrices if the computed solution solves a system that is near the original problem. Here it is shown that any stable algorithm is also strongly stable on the same matrix class if the computed solution solves a nearby problem that is also symmetric or symmetric positive definite.

Key words. stability, symmetric matrices

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1. Introduction. When applied to a linear system $Ax = b$, a stable algorithm for solving systems of linear equations produces a computed solution $\hat{x}$ that is the solution to a nearby system

$$\hat{A}\hat{x} = \hat{b},$$

where $\|\hat{A} - A\| / \|A\|$ is small and $\|\hat{b} - b\| / \|b\|$ is small, for some norm $\|\cdot\|$. How "small" is small enough depends on the accuracy desired in the solution (and on the condition number of A) [16, pp. 189–191]. A proof of the stability of an algorithm usually involves showing that $\|\hat{A} - A\| / \|A\|$ and $\|\hat{b} - b\| / \|b\|$ are bounded by $p(n)u$, where p is a low degree polynomial, n is the order of A, and u is the unit roundoff (machine precision). We would like $p(n)u << 1$.

In solving structured linear equations, it is often important that the perturbed matrix $\hat{A}$ have the same structure as A. For example, solving electrical network problems gives rise to symmetric systems of linear equations, $Ax = b$. If the computed solution $\hat{x}$ to $Ax = b$ satisfies $\hat{A}\hat{x} = \hat{b}$, but $\hat{A}$ is not symmetric, then the system $\hat{A}\hat{x} = \hat{b}$ could never have arisen from an electrical network problem. But if $\hat{A}$ is symmetric, then we hope that there is an electrical network near our original network that gives rise to the system $\hat{A}\hat{x} = \hat{b}$.

Another situation where it is important that the perturbed matrix remain symmetric is in the analysis of Algorithm 5 in [3]. That algorithm uses a variation of inverse iteration to find the eigenvectors of a certain class of symmetric matrices to high accuracy. The class includes all symmetric positive definite matrices that can be consistently ordered. The error analysis uses a new perturbation theorem about symmetric perturbations of symmetric matrices, and to apply it one needs to know that a nearby symmetric matrix exists which exactly satisfies the equations at each step of inverse iteration.

The term *strongly stable*, developed in [4], is used in this context. An algorithm for solving linear equations is *strongly stable* for a class of matrices **A** if for each A in **A**

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and for arbitrary b the computed solution $\hat{x}$ solves a nearby system $\hat{A}\hat{x} = \hat{b}$ with $\hat{A}$ in $\mathbf{A}$. Note that for stability we do not require $\hat{A}$ to be in $\mathbf{A}$, but for strong stability we do. (Other stability concepts were introduced in [2], [13], [14].)

In [4] it is shown that the following algorithms for solving linear equations are strongly stable for their respective classes of matrices:

(1) Gaussian elimination with partial or complete pivoting on $\mathbf{A}_1 = \{\text{nonsingular matrices}\}$ [16];

(2) Cholesky on $\mathbf{A}_2 = \{\text{symmetric positive definite matrices}\}$ [16];

(3) LDL^T (symmetric Gaussian elimination) on $\mathbf{A}_2$ [16];

(4) Symmetric indefinite algorithm (diagonal pivoting method [5], [6], [9]) on $\mathbf{A}_3 = \{\text{nonsingular symmetric matrices}\}$;

(5) LU decomposition (Gaussian elimination without pivoting) on $\mathbf{A}_4 = \{$strictly column diagonally dominant matrices $(|a_{ii}| > \sum_{j\neq i} |a_{ji}|$ for all $i)\}$ or $\mathbf{A}_5 = \{$strictly column diagonally dominant band matrices$\}$. (See Appendix.)

(6) Gaussian elimination with partial or complete pivoting followed by iterative refinement on $\mathbf{A}_6 = \{$nonsingular matrices with an arbitrary but fixed sparsity pattern and which are not too ill conditioned$\}$. (See [2], [13], [14] for discussion.)

In [4] it was noted that while Gaussian elimination with partial pivoting and Gaussian elimination with complete pivoting are stable on $\mathbf{A}_2$ and $\mathbf{A}_3$ and Aasen's method [1], [10] is stable on $\mathbf{A}_3$, it does not follow from their error analyses that these algorithms are strongly stable. Thus, the strong stability of these algorithms on $\mathbf{A}_2$ and $\mathbf{A}_3$, respectively, was left as an open question.

Here we will extend the list of strongly stable "situations" developed in [4]. In particular, we show that if a method is stable for the class of nonsingular symmetric matrices or the class of symmetric positive definite matrices, then it is strongly stable for the same class.

2. Constructing a symmetric perturbed system. If $A = A^T$, $(A + E)z = b$, $z \neq 0$, where E might be nonsymmetric, then we shall construct $F = F^T$ such that $(A + F)z = b$ and $\|F\|$ is within a small constant of $\|E\|$ for the 2-norm and the Frobenius norm. We shall do this in two different ways. The first will use the Powell-Symmetric-Broyden (PSB) update [12]; the second will use a construction via the QR decomposition; in either case we shall show that z is the exact solution of a symmetric perturbed system. We include both since the analyses are instructive in their own right.

The problem of nearby symmetric systems has already been addressed in the theory for quasi-Newton methods. For the first approach we shall use the following [7], [8, p. 196].

THEOREM 1. *If H_c is symmetric, $s_c \neq 0$, then the unique solution to*

$$\text{minimize}\,\{\|H - H_c\|_F : H = H^T, Hs_c = y_c\}$$

is given by the PSB-*update*:

$$H_+ = H_c + \frac{(y_c - H_c s_c)s_c^T + s_c(y_c - H_c s_c)^T}{s_c^T s_c} - \frac{\langle y_c - H_c s_c, s_c\rangle s_c s_c^T}{(s_c^T s_c)^2}.$$

Here, $\|\ \ \|_F$ is the Frobenius norm and $\langle u, v\rangle = u^T v$. We will use this to prove the following theorem.

THEOREM 2. *If $A = A^T$, $(A + E)z = b$, $r \equiv b - Az$, $z \neq 0$, then*

$$\hat{F} = \frac{rz^T + zr^T}{z^T z} - \frac{(z^T r)}{(z^T z)^2} zz^T$$

satisfies $(A + \hat{F})z = b$, $\hat{F} = \hat{F}^T$, *and* $\|\hat{F}\| \leqq 3\|E\|$ *for the* 2-*norm and the Frobenius norm. Furthermore,* $\hat{F}$ *is the unique solution to*

$$\text{minimize}\,\{\|F\|_F: F=F^T, (A+F)z=b\}.$$

Proof. In Theorem 1, take $H_c = A$, $s_c = z$, $y_c = b$. Then $y_c - H_c s_c = b - Az = r$. Thus, the unique $\hat{F}$ minimizing $\{\|F\|_F: (A+F)z=b, F=F^T\}$ is the PSB update

$$\hat{F} = \frac{rz^T + zr^T}{z^Tz} - \frac{(z^Tr)}{(z^Tz)^2} zz^T.$$

Thus,

$$\|\hat{F}\|_2 \leqq \|\hat{F}\|_F \leqq \frac{\|rz^T\|_F + \|zr^T\|_F}{z^Tz} + \frac{|z^Tr|}{(z^Tz)^2}\|zz^T\|_F.$$

But

$$\|uv^T\|_F = \|uv^T\|_2 = \|u\|_2\|v\|_2 \,[10, \text{p. } 16].$$

Hence,

$$\|\hat{F}\|_2 \leqq \|\hat{F}\|_F \leqq \frac{2\|r\|_2\|z\|_2}{\|z\|_2^2} + \frac{\|z\|_2^3\|r\|_2}{\|z\|_2^4} = 3\frac{\|r\|_2}{\|z\|_2}.$$

However, $r \equiv b - Az = Ez$, so $\|r\|_2 \leqq \|E\|_2\|z\|_2$. Thus,

$$\|\hat{F}\|_2 \leqq \|\hat{F}\|_F \leqq 3\|E\|_2 \leqq 3\|E\|_F. \qquad \square$$

Now, we shall construct a symmetric perturbed system by an approach via the *QR* decomposition which will give a slightly sharper bound. But first we need the following lemma.

LEMMA 1. *Given any two unit vectors u and v, there exists a symmetric orthogonal matrix P such that* $Pu = v$.

Proof. If u and v are parallel, then P is a multiple of the identity. If u and v are not parallel, P can be taken to be a Householder matrix that reflects in a plane containing $u + v$ and is orthogonal to the plane containing u and v. $\square$

THEOREM 3. *If* $A = A^T$, $(A + E)z = b$, $z \neq 0$, *then there exists* $\tilde{F} = \tilde{F}^T$ *such that* $(A + \tilde{F})z = b$, $\|\tilde{F}\|_2 \leqq \|E\|_2$ *and* $\|\tilde{F}\|_F \leqq \sqrt{2}\|E\|_F$. (*The bounds are sharp.*)

Proof. We need to determine $\tilde{F}$ so that

$$\tilde{F}^T = \tilde{F} \text{ and } \tilde{F}z = r,$$

where $r \equiv b - Az = Ez$. If $r = 0$, let $\tilde{F} \equiv 0$. Suppose $r \neq 0$.

$$\text{Let } X \equiv [z|r] = QR, \quad \text{where } R \equiv [\hat{z}|\hat{r}], \hat{z} \equiv \begin{bmatrix} \hat{z}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and } \hat{r} \equiv \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

be the *QR* decomposition of X. Note that, expressing $\tilde{F} = QFQ^T$, it is sufficient to determine F so that

$$(QFQ^T)^T = QFQ^T \quad \text{and} \quad QFQ^Tz = r,$$

or, more simply, so that

$$F^T = F \quad \text{and} \quad F\hat{z} = \hat{r}.$$

These can both be satisfied by choosing $F = \operatorname{diag}(F_{11}, 0)$ if F_{11} can be determined so that

$$F_{11}^T = F_{11} \quad \text{and} \quad F_{11}\begin{bmatrix} \hat{z}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \end{bmatrix}.$$

Since $z \neq 0$ and $r \neq 0$, $\hat{z}_1 \neq 0$, and $\hat{r}_1 \neq 0$ or $\hat{r}_2 \neq 0$. Let

$$u = \begin{bmatrix} \hat{z}_1 \\ 0 \end{bmatrix} \Big/ |\hat{z}_1|$$

and

$$v = \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \end{bmatrix} \Big/ \left\| \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \end{bmatrix} \right\|_2.$$

By Lemma 1, there exists $P = P^T = P^{-1}$ such that $Pu = v$. Let $F_{11} \equiv \alpha P$, where $\alpha \equiv \|\hat{r}\|_2 / \|\hat{z}\|_2 = \|r\|_2 / \|z\|_2$. Then $Fz = r$ and $F^T = F$.

$$\|F\|_2 = \alpha \|P\|_2 = \alpha = \frac{\|r\|_2}{\|z\|_2} = \frac{\|Ez\|_2}{\|z\|_2} \leqq \|E\|_2.$$

If $\|Ez\|_2 = \|E\|_2 \|z\|_2$, then $\|E\|_2 = \|F\|_2$ and the bound is sharp.

Since F_{11} is a multiple of a 2×2 orthogonal matrix, $\|F_{11}\|_F = \sqrt{2}\|F_{11}\|_2$. Thus,

$$\|F\|_F = \sqrt{2}\|F\|_2 \leqq \sqrt{2}\|E\|_2 \leqq \sqrt{2}\|E\|_F.$$

Setting $\tilde{F} = QFQ^T$ gives us the result. □

However, the $\hat{F}$ constructed in Theorem 2 minimizes

$$\{\|F\|_F : F = F^T, (A+F)z = b\},$$

and, hence

$$\|\hat{F}\|_F \leqq \|\tilde{F}\|_F \leqq \sqrt{2}\|E\|_F.$$

Thus, Theorem 3 gives us the following Corollary.

COROLLARY. *The matrix $\hat{F}$ in Theorem 2 satisfies* $\|F\|_F \leqq \sqrt{2}\|E\|_F$.

(*Note*: In [11] Higham gives a result similar to this Corollary.)

3. Applications. Gaussian elimination with pivoting and Aasen's method are stable for symmetric systems [10]. But, while the computed solution $\hat{x}$ solves a nearby system

$$(A+E)\hat{x} = b,$$

it is *not* the case that the matrix E is symmetric, at least not from the traditional backwards error analyses. Our results show that there is a symmetric F with $\|F\|_2 \leqq \|E\|_2$ and $\|F\|_F \leqq \sqrt{2}\|E\|_F$ so that

$$(A+F)\hat{x} = b.$$

Thus, Gaussian elimination with pivoting and Aasen's method are strongly stable when applied to symmetric systems. In [4], only the diagonal pivoting method [5], [6], [9] was shown to be strongly stable on symmetric systems. More generally, we have Theorem 4.

THEOREM 4. *If a method is stable for (nonsingular) symmetric matrices, then it is strongly stable for (nonsingular) symmetric matrices.*

Finally we make some observations about strong stability of algorithms for symmetric *positive definite* systems. The BFGS update [8, p. 201] and the DFP update [8, p. 205] do not give an F near E in the symmetric positive definite case. However, we can make an existence argument as follows.

THEOREM 5. *If A is symmetric positive definite and*

$$(A+E)\hat{x}=b$$

with $\|E\|_2 < \lambda_{\min}(A)$, then there exists a symmetric F so that

$$(A+F)\hat{x}=b, \tag{1}$$

$$\|F\|_2 \leqq \|E\|_2, \tag{2}$$

and

$$\lambda_{\min}(A+F)>0. \tag{3}$$

Proof. Theorem 4 ensures that (1) and (2) hold. From [10, p. 269] or [16, pp. 101–102] we have that

$$\lambda_{\min}(A+F) \geqq \lambda_{\min}(A)+\lambda_{\min}(F) \geqq \lambda_{\min}(A)-\|F\|_2.$$

Since $\|E\|_2 < \lambda_{\min}(A)$ and $\|F\|_2 \leqq \|E\|_2$, we have $\lambda_{\min}(A+F)>0$. □

If A is symmetric positive definite, then $\lambda_{\min}(A) = \|A\|_2$. Hence, Theorem 5 says that if $(A+E)\hat{x} = b$ with $\|E\|_2/\|A\|_2 < 1$, then there exists a symmetric F such that $A+F$ is positive definite, $(A+F)\hat{x} = b$, and $\|F\|_2/\|A\|_2 \leqq \|E\|_2/\|A\|_2$. □

We shall state this more formally in Theorem 6.

THEOREM 6. *If a method is stable for symmetric positive definite matrices, then it is strongly stable for symmetric positive definite matrices.*

4. Conclusions. We have shown that any algorithm for linear equations that is stable on $\mathbf{A}_2$ = {symmetric positive definite matrices} or $\mathbf{A}_3$ = {nonsingular symmetric matrices} will also be strongly stable on the same matrix class.

Appendix. A matrix A is *strictly column (row) diagonally dominant* if $|a_{ii}| > \sum_{j\neq i} |a_{ji}|$ for each i ($|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for each i). Let us perturb A to $\tilde{A} = A + E$. The following lemma shows that if the perturbation E is small enough then $A+E$ is still strictly column (row) diagonally dominant.

LEMMA 2. *If A is strictly column (row) diagonally dominant and if $\|E\|_1 < \delta$, where $\delta \equiv \min_i \{|a_{ii}| - \sum_{j\neq i} |a_{ji}|\}$ (if $\|E\|_\infty < \varepsilon \equiv \min_i \{|a_{ii}| - \sum_{j\neq i} |a_{ij}|\}$), then $A+E$ is strictly column (row) diagonally dominant.*

Proof. We shall prove it for column dominance; the proof for row dominance is similar. Since

$$\sum_j |e_{ij}| \leqq \|E\|_1 < \delta \quad \text{and} \quad \sum_{j\neq i} |a_{ji}| \leqq |a_{ii}| - \delta,$$

we have

$$\begin{aligned}\sum_{j\neq i} |a_{ji}+e_{ji}| &\leqq \sum_{j\neq i} |a_{ji}| + \sum_{j\neq i} |e_{ji}| < |a_{ii}| - \delta + \delta - |e_{ii}| \\ &\leqq |a_{ii}+e_{ii}| \quad \text{for each } i.\end{aligned}$$

□

The following theorem shows that if the machine precision u is small enough then Gaussian elimination without pivoting (LU decomposition) is strongly stable for column strictly diagonally dominant matrices.

THEOREM 7. *Let A be a column strictly diagonally dominant; let z be the computed solution by Gaussian elimination without pivoting. Then there exists an E such that $(A+E)z=b$, where $\|E\|_1 \leqq p(n)ua$, $p(n)$ is a low degree polynomial in n, u is the machine precision, and $a \equiv \max_{i,j}|a_{ij}|$. If, also, $u<\delta/(p(n)a)$, where $\delta \equiv \max_i\{|a_{ii}|-\sum_{j\neq i}|a_{ji}|\}$, then $A+E$ is strictly column diagonally dominant.*

Proof. From [10], [15], [16], there is an E such that $(A+E)z=b$ with $\|E\|_1 < \frac{1}{2}p(n)u\max_{i,j,k}|a_{ij}^{(k)}|$, where p is a polynomial of degree 3 and $a_{ij}^{(k)}$ are the elements in the reduced matrices. From [15, Chap. 3], $\max_{i,j,k}|a_{ij}^{(k)}| \leqq 2a$. If $u<\delta/(p(n)a)$, then $\|E\|_1<\delta$, and by Lemma 2, $A+E$ is strictly column diagonally dominant. □

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