

A Unitary Method for the ESPRIT Direction-of-Arrival Estimation Algorithm

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ABSTRACT

ESPRIT is an interesting new method for solving the Direction-of-Arrival estimation problem. It involves some rather tricky matrix manipulations. We show how these calculations can be carried out using only unitary transformations of the data. No inverses or cross-products are required making the new method extremely robust.

1. INTRODUCTION

Matrix methods that are based upon unitary transformations have proven to be both useful and important in many application areas including signal processing. They offer stability and are able to handle ill-conditioned problems gracefully. In this paper we show how the ESPRIT method for direction-of-arrival (DOA) estimation can be implemented using only unitary transformations. We summarize the ESPRIT assumptions below. The reader requiring more background should consult the papers by Paulraj, Roy, and Kailath (1986) and Roy, Paulraj, and Kailath (1986,1987) as well as the beautifully written PhD dissertation by Roy (1987).

The current work is similar in spirit to Speiser and Van Loan (1984) where a completely unitary approach to the MUSIC DOA estimation algorithm is given. MUSIC is due to Schmidt (1979,1981,1986).

Whether or not it is necessary to resort to our rather complicated procedure in a given application is unclear at this time. The problem of assessing DOA estimation algorithms and their implementations requires precise knowledge of DOA sensitivity, something that is not fully understood, at least by the author. The notion of an ill-conditioned DOA problem needs to be quantified before we can legitimately favor one procedure over another. Our contribution here is merely to present an ESPRIT implementation that seems about as robust as possible because of its sole reliance upon unitary transformations and its avoidance of inverses and cross-product matrices. Whether or not one can get by with a cheaper, non-unitary, normal equation-type technique in certain ESPRIT applications should be addressed by future research.

The corresponding situation in the ordinary least square problem is completely understood and worth recalling. In least squares there are important classes of problems for which the method of normal equations is perfectly adequate. See Golub and Van Loan (1983,p142). One does not always need the full force of the singular value decomposition. Likewise, we suspect that our unitary approach is not always needed in ESPRIT applications. This is certainly suggested by the computational results presented in Roy (1987). However, it may be the case that our implementation can handle a wider range of problems than other techniques.

2. BACKGROUND LINEAR ALGEBRA

Our algorithm relies upon a range of unitary matrix decompositions all of which are discussed in Golub and Van Loan (1983). The QR and singular value decompositions of an m-by-n matrix A are well-known:

$$\begin{aligned} A &= QR & Q^H Q &= I_n, \quad R \text{ (n-by-n) upper triangular} \\ A &= UEV^H & U^H U &= I_m, \quad V^H V = I_n, \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_n) \end{aligned}$$

Codes exist in LINPACK.

The CS, the generalized singular value, and the generalized Schur decompositions are less well-known and so we summarize them for easy reference.

Theorem 1 (CS Decomposition (CSD))

If

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \quad \begin{matrix} m_1 \\ m_2 \end{matrix} \quad \begin{matrix} m_1 \geq n, & m_2 \geq n \end{matrix}$$

n

has orthonormal columns, then there exist unitary U_1 (m_1 -by- m_1), U_2 (m_2 -by- m_2), and V (n -by- n) such that

$$\begin{aligned} U_1^H Q_1 V &= C = \text{diag}(c_k) & c_k &= \cos(\theta_k) \\ U_2^H Q_2 V &= S = \text{diag}(s_k) & s_k &= \sin(\theta_k) \end{aligned}$$

where $0 \leq \theta_1 \leq \dots \leq \theta_n \leq \pi/2$.

Proof

See Davis and Kahan(1970) or Stewart(1977).

Algorithms for computing the CSD are given in Stewart(1983) and Van Loan (1985).

Paige and Saunders(1981) were the first to point out a connection between the CSD and the generalized singular value problem

$$A_1^H A_1 x = \mu^2 A_2^H A_2 x$$

Generalized singular value problems arise in several important signal and image processing problems, see Van Loan and Speiser (1986). The μ which make $A_1^H A_1 - \mu^2 A_2^H A_2$ singular are called the generalized singular values of (A_1, A_2) and they may be found through a judiciously chosen CSD:

Theorem 2 (Generalized Singular Value Decomposition (GSVD))

Suppose $A_1 \in C^{m_1 \times n}$ and $A_2 \in C^{m_2 \times n}$ have full column rank. If

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R \quad Q_1 \in C^{m_1 \times n}, Q_2 \in C^{m_2 \times n}, R \in C^{n \times n}$$

is a QR factorization and

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V^H$$

is a CSD with U_1, U_2 , and V unitary and $C = \text{diag}(\cos(\theta_k))$ and $S = \text{diag}(\sin(\theta_k))$, then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V^H R$$

and

$$A_1^H A_1 x_k = \mu_k^2 A_2^H A_2 x_k \quad k = 1:n$$

where $X = [x_1, \dots, x_n] = R^{-1}V$ and $\mu_k = \text{ctn}(\theta_k)$.

Proof

See Speiser and Van Loan (1984).

The full column rank assumptions are for expository purposes only. These conditions ensure that there are no infinite generalized singular values.

There is an important corollary of the GSVD that is central to our ESPRIT method. It is a representation of $A_1 A_1 - \mu^2 A_2 A_2$ when μ is the smallest generalized singular value.

Corollary 2.1

If in Theorem 2 we have $0 \leq \theta_1 \leq \dots \leq \theta_r < \theta_{r+1} = \dots = \theta_n < \pi/2$ and we set $\mu_{\min} = \text{ctn}(\theta_n)$, then

$$A_1^H A_1 - \mu_{\min}^2 A_2^H A_2 = \sum_{k=1}^r (c_k^2 - s_k^2) w_k w_k^H$$

where $W = R^H V = [w_1, \dots, w_n]$.

Proof

From Theorem 2, $A_1 = U_1 C W^H$ and $A_2 = U_2 S W^H$. Thus, we have the expansions

$$A_1^H A_1 = W C C^H W^H = \sum_{k=1}^n c_k^2 w_k w_k^H \quad A_2^H A_2 = W S S^H W^H = \sum_{k=1}^n s_k^2 w_k w_k^H$$

Thus,

$$A_1^H A_1 - \mu_{\min}^2 A_2^H A_2 = \sum_{k=1}^n (c_k^2 - \mu_{\min}^2 s_k^2) w_k w_k^H$$

The corollary follows since $c_k^2 - \mu_{\min}^2 s_k^2 = 0$ for $k=1:r$.

Finally we mention the generalized Schur decomposition which applies to the generalized eigenvalue problem $A_1 x = \lambda A_2 x$.

Theorem 3 (Generalized Schur Decomposition)

If A_1 and A_2 are complex n -by- n matrices, then there exist unitary Q and Z such that $Q^H A_1 Z = T$ and $Q^H A_2 Z = R$ are upper triangular. If $\lambda = t_{kk}/r_{kk}$ is defined, then $\det(A_1 - \lambda A_2) = 0$.

Proof

See Golub and Van Loan (1983, p. 253).

The QZ algorithm of Moler and Stewart (1973) can be used to compute this factorization. An implementation of QZ may be found in EISPACK.

3. THE ESPRIT MATRIX PROBLEM

We now specify the matrix problem central to ESPRIT. Details may be found in Roy (1987). First, some definitions:

m = number of sensor doublets (known)
 d = number of sources (unknown)
 N = number of sensor output snapshots (known)

The sensor output $z \in C^{2m \times 1}$ is modelled as follows:

$$z = \begin{bmatrix} A_1 \\ A_2 \Phi \end{bmatrix} s + n_z$$

where

A = m -by- d steering vector matrix (unknown, rank d)
 Φ = d -by- d diagonal unitary matrix (sought)
 s = d -by-1 signal vector (unknown)
 n_z = $2m$ -by-1 noise vector (unknown)

Without going into details, the sought-after DOAs are easy functions of Φ 's diagonal entries. Thus, the problem is to find Φ . This unitary diagonal matrix is buried in the expected value of z which we assume has the following form:

$$E(zz^H) = \begin{bmatrix} ASA^H & AS\phi^H A^H \\ A\phi SA^H & A\phi S\phi^H A^H \end{bmatrix} + \sigma^2 \Sigma$$

Here,

$$S = E(ss^H)$$

and

$$\sigma^2 \Sigma = E(n_z n_z^H)$$

are assumed to be positive definite. We also assume the existence of full column rank data matrices

$$A_z \in C^{2N \times 2m} \quad B \in C^{m \times 2m}$$

such that

$$E(zz^H) = A_z^H A_z \quad \sigma^2 \Sigma = B^H B \quad (1)$$

In practice, these equations hold only approximately which complicates the implementation of our ESPRIT method. We discuss this further in the final section. In any case, we are now able to specify the central matrix problem of ESPRIT:

Given the matrices $A_z \in C^{2N \times 2m}$ and $B \in C^{m \times 2m}$ that satisfy $A_z^H A_z = C_{zz} + \sigma^2 B^H B$ with

$$C_{zz} = \begin{bmatrix} ASA^H & AS\phi^H A^H \\ A\phi SA^H & A\phi S\phi^H A^H \end{bmatrix},$$

find ϕ .

4. COMPUTING σ^2 AND C_{zz}

Once we know σ^2 then we can compute C_{zz} from the equation

$$C_{zz} = A_z^H A_z - \sigma^2 B^H B$$

The following theorem shows how to do this and moreover gives a useful representation for C_{zz} as well.

Theorem 4

Suppose $A_z \in C^{2N \times 2m}$ and $B \in C^{m \times 2m}$ have full column rank, satisfy (1), and have the GSVD

$$\begin{bmatrix} A_z \\ B \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V^H R$$

with $C = \text{diag}(\cos(\theta_k))$ and $S = \text{diag}(\sin(\theta_k))$. If $\mu_1 \geq \dots \geq \mu_{2m}$ are the generalized singular values ($\mu_k = \cos(\theta_k)/\sin(\theta_k)$) and $W = [w_1, \dots, w_{2m}] = R^H V$, then

$$\mu_1 \geq \dots \geq \mu_d > \mu_{d+1} = \dots = \mu_{2m} = \sigma$$

and

$$C_{zz} = \sum_{k=1}^d (c_k^2 - \sigma^2 s_k^2) w_k w_k^H$$

Proof

If $A_z^H A_z x = \mu^2 B^H B x$, then since $A_z^H A_z = C_{zz} + \sigma^2 B^H B$ we have

$$\mu^2 = x^H (A_z^H A_z) x / x^H (B^H B) x = \sigma^2 + x^H C_{zz} x / x^H (B^H B) x$$

Thus, $\mu^2 \geq \sigma^2$ always with equality iff $C_{zz} x = 0$. Now

$$C_{zz} = \begin{bmatrix} A \\ A\phi \end{bmatrix} S \begin{bmatrix} A \\ A\phi \end{bmatrix}^H$$

has rank d and so $\dim[\text{null}(C_{zz})] = 2m-d$. Thus, as a generalized singular value of (A_z, B) , σ must have multiplicity $2m-d$. The characterization of C_{zz} in terms of W 's columns follows directly from Corollary 2.1.

5. COMPUTING ϕ

From Theorem 4 we know that

$$C_{zz} = \begin{bmatrix} ASA^H & AS\phi^H A^H \\ A\phi SA^H & A\phi S \phi^H A^H \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} D \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^H \quad (2)$$

where $D = \text{diag}(c_k^2 - \sigma_k^2 s_k^2) \in C^{d \times d}$ is nonsingular and

$$[w_1, \dots, w_d] = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{matrix} m \\ m \end{matrix}$$

By equating blocks in (2) we find that

$$AS(I - \lambda\phi^H)A^H = ASA^H - \lambda AS\phi^H A^H = W_1 D W_1^H - \lambda W_1 D W_2^H$$

Note that ordinarily,

$$\text{rank}(W_1 D W_1^H - \lambda W_1 D W_2^H) = d$$

However, if λ is a diagonal entry of ϕ then the diagonal matrix $I - \lambda\phi$ is singular and so

$$\text{rank}(W_1 D W_1^H - \lambda W_1 D W_2^H) < d. \quad (3)$$

Thus, the problem is to find the d complex numbers $\lambda_1, \dots, \lambda_d$ that "reduce the rank" of $W_1 D W_1^H - \lambda W_1 D W_2^H$. This can be accomplished by computing the QR factorization of W_1 and then solving a d -by- d generalized eigenvalue problem. Indeed, suppose that $Z \in C^{m \times m}$ is unitary so

$$Z^H W_1 = \begin{bmatrix} W_{11} & & \\ & & \\ 0 & & \end{bmatrix} \begin{matrix} d \\ \\ m-d \end{matrix}$$

is upper triangular. Since $ASA^H = W_1 D W_1^H$, it follows that the last $m-d$ columns of Z span $\text{null}(A^H)$. Likewise, $W_2 D W_2^H = A\phi S \phi^H A^H$ implies that

$$Z^H W_2 = \begin{bmatrix} W_{12} & & \\ & & \\ 0 & & \end{bmatrix} \begin{matrix} d \\ \\ m-d \end{matrix}$$

Thus,

$$Z^H [W_1^H D W_1^H - \lambda W_1^H D W_2^H] Z = \begin{bmatrix} W_{11}^H D W_{11}^H - \lambda W_{11}^H D W_{12}^H & 0 \\ 0 & 0 \end{bmatrix}$$

Consequently, if $\det(W_{11}^H D W_{11}^H - \lambda W_{11}^H D W_{12}^H) = 0$, then (3) holds. Since D and W_{11} are nonsingular, this is equivalent to requiring

$$\det(W_{11}^H - \lambda W_{12}^H) = 0 .$$

Thus, the sought-after λ are the generalized eigenvalues of the matrix pencil $W_{11}^H - \lambda W_{12}^H$.

6. SUMMARY AND CONCLUSIONS

Collecting all of the above results we thus have the following implementation of ESPRIT:

- Using the LINPACK QR factorization routine, compute

$$\begin{bmatrix} A \\ Z \\ B \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R$$

Here, Q_1 ($2N$ -by- $2m$), Q_2 (m_b -by- $2m$), and R ($2m$ -by- $2m$) must be saved.

- Using the algorithm in Van Loan (1985), compute the CSD

$$\begin{aligned} U_1^H Q_1 V &= C = \text{diag}(c_k) & c_k &= \cos(\theta_k) \\ U_2^H Q_2 V &= S = \text{diag}(s_k) & s_k &= \sin(\theta_k) \end{aligned}$$

where $0 \leq \theta_1 \leq \dots \leq \theta_{2m} \leq \pi/2$. U_1 and U_2 are not needed but C , S , and V are. Determine d so that

$$c_1 \geq \dots \geq c_d > c_{d+1} = \dots = c_{2m}$$

and let W_1 and W_2 be the m -by- d matrices defined by

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = [w_1, \dots, w_d] \quad R^H V = W = [w_1, \dots, w_n]$$

- Using the LINPACK QR factorization routine again, compute unitary Z (m -by- m) such that

$$Z^H W_1 = \begin{bmatrix} W_{11} \\ 0 \end{bmatrix} \quad \begin{matrix} d \\ m-d \end{matrix}$$

Apply Z to W_2 and get

$$Z^H W_2 = \begin{bmatrix} W_{12} \\ 0 \end{bmatrix} \quad \begin{matrix} d \\ m-d \end{matrix}$$

- Use the EISPACK QZ algorithm to compute d -by- d unitary matrices V_1 and V_2 such that

$$V_1^H W_{11}^H V_2 = F \quad V_1^H W_{12}^H V_2 = G$$

are upper triangular and set $\phi = \text{diag}(f_{kk}/g_{kk})$.

Note the avoidance of cross products and inverses. In practice, the matrices A_z and B do not exactly satisfy

fy (1). One ramification of this is that the determination of d (= the number of sources) in step 2 must be based upon a tolerance. For example, we might set d to be the smallest integer that satisfies

$$\hat{c}_1 \geq \dots \geq \hat{c}_d \geq c_{2m} + \epsilon > c_{d+1} \geq \dots \geq c_{2m}$$

where the "hats" designate computed quantities and $\epsilon > 0$ is a small tolerance that may depend upon the machine precision and the fuzziness of the assumptions (1). Numerous other computational matters will be reported on in a future paper.

ACKNOWLEDGEMENTS

I am very much obliged to Jeff Speiser and Richard Roy for helping me understand the DOA problem and the methods of MUSIC and ESPRIT.

This work was supported by ONR Contract N00014-83-K-640, NSF Contract DCR 86-2310, and the Mathematical Sciences Institute of Cornell which is sponsored by the ARO.

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