A Schur Decomposition for Hamiltonian Matrices

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ABSTRACT

A Schur-type decomposition for Hamiltonian matrices is given that relies on unitary symplectic similarity transformations. These transformations preserve the Hamiltonian structure and are numerically stable, making them ideal for analysis and computation. Using this decomposition and a special singular-value decomposition for unitary symplectic matrices, a canonical reduction of the algebraic Riccati equation is obtained which sheds light on the sensitivity of the nonnegative definite solution. After presenting some real decompositions for real Hamiltonian matrices, we look into the possibility of an orthogonal symplectic version of the QR algorithm suitable for Hamiltonian matrices. A finite-step initial reduction to a Hessenberg-type canonical form is presented. However, no extension of the Francis implicit-shift technique was found, and reasons for the difficulty are given.

1. INTRODUCTION

A matrix $M\in\mathbb{C}^{2n\times 2n}$ is said to be *Hamiltonian* [11] if $JM=(JM)^H$, where

$$J=\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{1.1}$$

*Work supported by the Natural Sciences and Engineering Research Council of Canada, Grant A8652.

†Work partially supported by the National Science Foundation, MCS 8004106.

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Here, I_n denotes the $n\times n$ identity and the superscript H conjugate transpose. If we partition M conformably with J, then we find

$$M=\begin{bmatrix} A & N \\ K & -A^H \end{bmatrix}, \qquad N^H=N, \quad K^H=K.$$

Throughout this paper, M will denote this block matrix.

The eigensystem of M has many easily verified properties. In particular, if

$$M\begin{bmatrix} y \\ z \end{bmatrix}=\lambda\begin{bmatrix} y \\ z \end{bmatrix} \qquad y,z\in\mathbb{C}^n, \quad y^Hy+z^Hz\neq 0,$$

then

$$\left[z^H \mid -y^H\right]M=-\bar{\lambda}\left[z^H\mid -y^H\right], \tag{1.2}$$

$$y^HKy+z^HNz=(\lambda+\bar{\lambda})y^Hz, \tag{1.3}$$

$$\operatorname{Re}(\lambda)\neq 0 \quad\Rightarrow\quad y^Hz\in\mathbb{R}, \tag{1.4}$$

$$N>0 \text{ and } K>0 \quad\Rightarrow\quad \operatorname{Re}(\lambda)\neq 0 \tag{1.5}$$

$$N\geqslant 0,\quad K\geqslant 0, \text{ and } \operatorname{Re}(\lambda)=0 \quad\Rightarrow\quad Ay=\lambda y \text{ and } A^Hz=-\lambda z \tag{1.6}$$

Here, $F>0$ ($F\geqslant 0$) means that F is positive (nonnegative) definite. For a good set of references to the Hamiltonian-matrix literature, see the paper by Laub and Meyer [4].

Our interest in Hamiltonian matrices stems from the fact that if

$$\begin{bmatrix} A & N \\ K & -A^H \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix}=\begin{bmatrix} Y \\ Z \end{bmatrix}W, \qquad Y,Z,W\in\mathbb{C}^{n\times n}, \tag{1.7}$$

and Y is nonsingular, then $X=-ZY^{-1}$ solves the following matrix Riccati equation:

$$-XNX+XA+A^HX+K=0. \tag{1.8}$$

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$$\left[z^H\,|-y^H\right]M=-\bar{\lambda}\left[z^H\,|-y^H\right], \tag{1.2}$$

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$$\mathrm{Re}(\lambda)\neq 0 \quad\Rightarrow\quad y^Hz\in\mathbb{R}, \tag{1.4}$$

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This matrix quadratic equation frequently arises in optimal-control applications, and when it does, there typically exist matrices B and C such that the following three conditions hold:

Nonnegative definiteness (NND):

$$N = BB^H, \qquad K = C^H C.$$

Stabilizability (S):

$$\text{If} \quad w^H A = \lambda w^H \, (w \neq 0) \text{ and } \mathrm{RE}(\lambda) \geqslant 0, \quad \text{then} \quad w^H B \neq 0.$$

Detectability (D):

$$\text{If} \quad Ax = \lambda x \, (x \neq 0) \text{ and } \mathrm{Re}(\lambda) \geqslant 0, \quad \text{then} \quad Cx \neq 0.$$

These conditions ensure, via (1.6) and (1.2), that M has precisely n eigenvalues in the open left half plane. Moreover, if the columns of $\begin{bmatrix} Y \\ Z \end{bmatrix}$ span the associated invariant subspace, then it is known, and will be shown in Section 4, that the matrix $X = -ZY^{-1}$ exists and satisfies $X^H = X \geqslant 0$. It is this solution to the Riccati equation that is normally required and to which we direct all our attention in the sequel.

The most reliable method for carrying out the above invariant-subspace computations makes use of the well-known QR algorithm for eigenvalues and is described by Laub [3]. The crux of his technique involves the calculation of M's Schur decomposition, i.e., a unitary matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad Q_{ij} \in \mathbb{C}^{n \times n},$$

is found such that

$$Q^H M Q = T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad T_{ij} \in \mathbb{C}^{n \times n},$$

is upper triangular with the eigenvalues of T_{11} in the open left half plane. Since

$$M \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} T_{11},$$

the desired Riccati equation solution is given by $X = -Q_{21} Q_{11}^{-1}$.

Although Laub's method relies on numerically stable unitary transformations, it has the defect of not preserving the Hamiltonian form of M during the computations; the QR algorithm treats M as just another general matrix. This shortcoming is the motivation for the present paper. Our intention is to examine a class of unitary transformations which preserve Hamiltonian structure under similarity. Using these transformations, we prove Hamiltonian matrix "versions" of both the Schur and Hessenberg decompositions, giving an algorithm in the latter case.

As mentioned above, our interest in these things has to do with solving the Riccati equation. By presenting unitary, structure preserving reductions of this problem, we hope to lay the groundwork for future algorithmic and perturbation-theory developments. Although many authors before us have offered analyses of the Riccati problem and the associated matrix M [2, 3, 6], we think that the unitary-matrix approach should prove to be as useful in this setting as it has in other application areas.

2. UNITARY SYMPLECTIC MATRICES

A matrix $Q\in\mathbb{C}^{2n\times n}$ is said to be symplectic [11] if $Q^HJQ=J$, where J is defined by (1.1). If Q is symplectic and M is a Hamiltonian matrix, then $M_0=QMQ^{-1}$ is also a Hamiltonian matrix:

$$Q^HJM_0Q=Q^HJQM=JM=(JM)^H=Q^H(JM_0)^HQ.$$

Let $\mathfrak{Q}$ denote the set of all unitary symplectic matrices. Note that $Q\in\mathfrak{Q}$ implies $QJ=JQ$, from which we conclude

$$\mathfrak{Q}=\left\{Q\in\mathbb{C}^{2n\times 2n}\,|\,Q=\begin{bmatrix} Q_{11} & Q_{12}\\ -Q_{12} & Q_{11}\end{bmatrix}, Q^HQ=I_n, Q_{11}, Q_{12}\in\mathbb{C}^{n\times n}\right\}.$$

It is clear that $\mathfrak{Q}$ is closed under multiplication and conjugate transposition.

We now identify two subsets of $\mathfrak{Q}$ that are important for both practical and theoretical reasons. The first subset is made up of *Householder symplectic* matrices, which have the form

$$H(k,u)=\begin{bmatrix} P & 0\\ 0 & P\end{bmatrix}, \qquad P\in\mathbb{C}^{n\times n},$$

where

$$P=I_n-\frac{2uu^H}{u^Hu}$$

and

$$u^H=[0,\ldots,0,\bar{u}_k,\ldots,\bar{u}_n]\neq 0.$$

The other subset is comprised of the *Jacobi symplectic* matrices which have the structure

$$J(k,c,s)=\begin{bmatrix} C & S \\ -S & C \end{bmatrix}, \qquad C,S\in\mathbb{C}^{n\times n},$$

where

$$C=\mathrm{diag}(\underbrace{1,\ldots,1}_{k-1},c,1,\ldots,1), \qquad S=\mathrm{diag}(\underbrace{0,\ldots,0}_{k-1},s,0,\ldots,0)$$

and

$$|c|^2+|s|^2=1, \qquad \bar{c}s\in\mathbb{R}.$$

Observe that the cosine and sine in a Jacobi symplectic matrix have the form $c=\omega\tilde{c}$ and $s=\omega\tilde{s}$, where $\tilde{c}$ and $\tilde{s}$ are real and satisfy $\tilde{c}^2+\tilde{s}^2=1$ and where $\omega\in\mathbb{C}$.

We now present three algorithms which show how these special members of $\mathcal{Q}$ can be used to zero specified entries in a vector. The verification that these algorithms perform as described is left to the reader, who may wish to review the zeroing capabilities of Householder and Jacobi transformations in Wilkinson [9] before reading further.

ALGORITHM 1. Given y and z in $\mathbb{C}^n$ and k $(1\leqslant k\leqslant n)$, the following algorithm constructs a unitary $Q\in\mathcal{Q}$ such that if

$$Q\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} w \\ x \end{bmatrix},$$

then for all $i>k$, $x_i=0$, while for all $i<k$, $y_i=w_i$ and $z_i=x_i$:

1. Let $\alpha=(|z_k|^2+\cdots+|z_n|^2)^{1/2}$, and let $z_k=|z_k|e^{i\Theta}$ define $\theta\in\mathbb{R}$.
2. If $\alpha=0$, then set $Q=I_{2n}$. Otherwise set $Q=H(k,u)$, where

$$u^H=(0,\ldots,0,\bar{z}_k+\alpha e^{-i\Theta},\bar{z}_{k+1},\ldots,\bar{z}_n).$$

ALGORITHM 2. Given y and z in $\mathbb{C}^n$, k $(1 \leqslant k \leqslant n)$, and the assumption that $\bar{y}_k z_k \in \mathbb{R}$, the following algorithm computes a unitary $Q \in \mathcal{Q}$ such that if

$$Q\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} w \\ x \end{bmatrix},$$

then $x_k = 0$ and for all $i \neq k$, $y_i = w_i$, and $z_i = x_i$:

1. If $z_k = 0$, then set $c = 1$ and $s = 0$. Otherwise, define c and s by the equations $\tau = y_k / z_k$, $s = (1+\tau^2)^{-1/2}$, and $c = \tau s$.
2. Set $Q = J(k, c, s)$.

(*Note:* If $\bar{y}_k z_k$ is not real, then no $J(k,c,s)$ exists such that the $(n+k)$th component of $J(k,c,s)\begin{bmatrix} y \\ z \end{bmatrix}$ is zero.)

ALGORITHM 3. Given y and z in $\mathbb{C}^n$, k $(1 \leqslant k \leqslant n)$, and the assumption that $\sum_{i=k}^{n} \bar{y}_i z_i \in \mathbb{R}$, the following algorithm constructs a unitary $Q \in \mathcal{Q}$ such that if

$$Q\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} w \\ x \end{bmatrix},$$

then $w_i = 0$ for all $i > k$, $x_i = 0$ for all $i \geqslant k$, and for all $i < k-1$, $z_i = x_i$ and $y_i = w_i$.

1. Use Algorithm 1 to construct $Q_1 \in \mathcal{Q}$ such that

$$Q_1\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad f, g \in \mathbb{C}^n, \quad g_{k+1} = \cdots = g_n = 0.$$

2. Use Algorithm 2 to construct $Q_2 \in \mathcal{Q}$ such that

$$Q_2\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} r \\ p \end{bmatrix}, \qquad r, p \in \mathbb{C}^n, \quad p_k = 0.$$

(*Note:* $f_k g_k = \sum_{i=k}^{n} \bar{y}_i z_i$ is real, so that this step is defined.)

3. Use Algorithm 1 to construct $Q_3 \in \mathcal{Q}$ such that

$$Q_3\begin{bmatrix} r \\ p \end{bmatrix} = \begin{bmatrix} w \\ x \end{bmatrix}, \qquad w, x \in \mathbb{C}^n, \quad w_{k+1} = \cdots = w_n = 0.$$

4. Set $Q = Q_3 Q_2 Q_1$.

We mention in passing that the matrix Q that emerges from Algorithm 3 has the following structure:

$$Q=\begin{bmatrix} I_{k-1} & 0 & 0 & 0 \\ 0 & Q_{11} & 0 & Q_{12} \\ 0 & 0 & I_{k-1} & 0 \\ 0 & -Q_{12} & 0 & Q_{11} \end{bmatrix}, \qquad Q_{11}, Q_{12} \in \mathbb{C}^{(n-k+1)\times(n-k+1)}.$$

We next establish a special variant of the singular-value decomposition for matrices in $\mathcal{Q}$. The result is a specialization of a theorem due to Stewart [8] and will be useful in Section 4, where we analyze the Riccati equation. Before we proceed, we remind the reader of the "ordinary" singular-value decomposition theorem which states that if $F\in\mathbb{C}^{n\times n}$, then there exist unitary U and V in $\mathbb{C}^{n\times n}$ such that

$$U^HFV=\operatorname{diag}(\mu_1,\dots,\mu_n), \qquad 0\leqslant\mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_n.$$

The μ_i are referred to as singular values. See [7] for details.

THEOREM 2.1 (Symplectic SVD). *If*

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix}, \qquad Q_{11}, Q_{12} \in \mathbb{C}^{n\times n},$$

is unitary, then there exist unitary U and V in $\mathbb{C}^{n\times n}$ such that

$$\operatorname{diag}(U^H,U^H)\,Q\operatorname{diag}(V,V)=\begin{bmatrix} \Sigma & \Delta \\ -\Delta & \Sigma \end{bmatrix}, \tag{2.1}$$

where

$$\Sigma=\operatorname{diag}(\sigma_1,\dots,\sigma_n), \qquad 0\leqslant\sigma_1\leqslant\cdots\leqslant\sigma_n\leqslant 1,$$

and

$$\Delta=\operatorname{diag}(\delta_1,\dots,\delta_n), \qquad \delta_i=\pm(1-\sigma_i^2)^{1/2}.$$

Proof. Let $U_1^H Q_{11} V_1 = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ be the singular-value decomposition of Q_{11}. Write

$$\Sigma = \mathrm{diag}(d_1 I_{m_1}, \ldots, d_k I_{m_k}), \qquad m_1 + \cdots + m_k = n,$$

where $0 \leqslant d_1 < d_2 < \cdots < d_k$, and partition $W = U_1^H Q_{12} V_1$ conformably:

$$W = U_1^H Q_{12} V_1 = \begin{bmatrix} W_{11} & \cdots & W_{1k} \\ \vdots & & \vdots \\ W_{k1} & \cdots & W_{kk} \end{bmatrix}.$$

Since

$$\mathrm{diag}(U_1^H, U_1^H)\, Q\, \mathrm{diag}(V_1, V_1) = \begin{bmatrix} \Sigma & W \\ -W & \Sigma \end{bmatrix}$$

is unitary, it follows that

$$\Sigma W = W^H \Sigma, \qquad \Sigma W^H = W \Sigma, \tag{2.2a}$$

$$\Sigma^2 + W^H W = I_n, \qquad \Sigma^2 + W W^H = I_n. \tag{2.2b}$$

By comparing blocks in (2.2a) we find

$$d_i W_{ij} = W_{ji}^H d_j, \qquad d_i W_{ji}^H = W_{ij} d_j. \tag{2.3a}$$

Thus, $d_i^2 W_{ij} = d_i W_{ji}^H d_j = W_{ij} d_j^2$, from which we conclude that $W_{ij} = 0$ whenever $i \neq j$. It then follows from (2.2b) that

$$W_{ii}^H W_{ii} = (1 - d_i^2) I_{m_i}, \qquad W_{ii} W_{ii}^H = (1 - d_i^2) I_{m_i}. \tag{2.3b}$$

We now determine unitary $Y = \mathrm{diag}(Y_{11}, \ldots, Y_{kk})$ and $Z = \mathrm{diag}(Z_{11}, \ldots, Z_{kk})$ such that $Y^H W Z$ is diagonal. If $d_i \neq 0$, then from (2.3a) we see W_{ii} is Hermitian. Let Y_{ii} be a unitary matrix comprised of its eigenvectors, and set $Z_{ii} = Y_{ii}$. If $d_1 = 0$, then from (2.3b) W_{11} is unitary. In this case, set $Y_{11} = W_{11}$ and $Z_{11} = I_{m_1}$. It then follows that

$$Y^H W Z = \mathrm{diag}(Y_{11}^H W_{11} Z_{11}, \ldots, Y_{kk}^H W_{kk} Z_{kk}) \equiv \Delta$$

is diagonal. Moreover, it is easy to show from the block structure of Σ that $Y^H \Sigma Z = \Sigma$. Equation (2.1) now follows by setting $U = U_1 Y$ and $V = V_1 Z$. The relations between the σ_i and the δ_i follow from the equation $\Sigma^2 + \Delta^2 = I_n$. ■

We conclude this section with a corollary that assures us that without loss of generality, we may consider only $J(k, c, s)$ and $H(k, u)$ matrices in the course of doing computations with unitary symplectic transformations.

COROLLARY 2.2. *If $Q \in \mathcal{Q}$, then Q is the product of Householder symplectic and Jacobi symplectic matrices.*

Proof. Note that in (2.1),

$$\begin{bmatrix} \Sigma & -\Delta \\ \Delta & \Sigma \end{bmatrix} = \prod_{k=1}^{n} J(k, \sigma_k, -\delta_k).$$

Thus, it suffices to show that the corollary holds for matrices of the form $\mathrm{diag}(V,V)$ where $V^H V = I_n$. Let $P_{n-1} \cdots P_1 V = R$ be the Householder upper triangularization of V. (See [7] for details.) It follows that R has the form $R = \mathrm{diag}(e^{i\Theta_k})$ and therefore

$$\mathrm{diag}(V,V) = \mathrm{diag}(P_1, P_1) \cdots \mathrm{diag}(P_{n-1}, P_{n-1}) \cdot \prod_{k=1}^{n} J(k, e^{i\Theta_k}, 0). \quad ■$$

3. UNITARY DECOMPOSITIONS FOR HAMILTONIAN MATRICES

We now turn to the problem of reducing a given Hamiltonian matrix to some "illuminating" canonical form using unitary symplectic similarity transformations. The following theorem constitutes our main result along these lines.

THEOREM 3.1 (The Schur-Hamiltonian decomposition). *If*

$$M = \begin{bmatrix} A & N \\ K & -A^H \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$$

is a Hamiltonian matrix whose eigenvalues have nonzero real part, then there exists a unitary

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix}, \qquad Q_{11}, Q_{12} \in \mathbb{C}^{n \times n},$$

such that

$$Q^H M Q = \begin{bmatrix} T & R \\ 0 & -T^H \end{bmatrix}, \qquad T, R \in \mathbb{C}^{n\times n}, \tag{3.1}$$

where T is upper triangular and $R^H = R$. Q can be chosen so that the eigenvalues of T are in the left half plane.

The matrix in (3.1) is said to be in *Schur-Hamiltonian* form.

Proof. We use induction. For the $n=1$ case, M has the form

$$M = \begin{bmatrix} \alpha_1 + i\alpha_2 & \eta \\ \kappa & -\alpha_1 + i\alpha_2 \end{bmatrix}, \qquad \alpha_1, \alpha_2, \kappa, \eta \in \mathbb{R},$$

and the assumption that M has no purely imaginary eigenvalues ensures that $\kappa\eta$ is positive. It then follows that real c and s exist such that if $Q = J(1, c, s)$, then

$$Q^H \begin{bmatrix} \alpha_1 & \eta \\ \kappa & -\alpha_1 \end{bmatrix} Q = \begin{bmatrix} \lambda_1 & \rho \\ 0 & -\lambda_1 \end{bmatrix}, \qquad \lambda_1 \leqslant 0.$$

It follows that $Q^H M Q$ has the Schur-Hamiltonian form (3.1).

We now assume that the decomposition exists for Hamiltonian matrices of order $2(n-1)$ that have no purely imaginary eigenvalues. Suppose

$$M\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}, \qquad y, z \in \mathbb{C}^n$$

with $y^H y + z^H z \neq 0$ and $\mathrm{Re}(\lambda) < 0$. By (1.4) we have $y^H z \in \mathbb{R}$, and so, using Algorithm 3, there exists a $P \in \mathcal{Q}$ such that

$$P\begin{bmatrix} y \\ z \end{bmatrix} = \alpha \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \qquad \alpha \neq 0,$$

where e_1 is the first column of I_n. From the equation $(PMP^H)e_1 = \lambda e_1$ we conclude that

$$PMP^H = \left[\begin{array}{cc|cc} \lambda & w^H & \beta & r^H \\ 0 & A_1 & s & N_1 \\ \hline 0 & v^H & \gamma & u^H \\ 0 & K_1 & q & D_1 \end{array}\right] \begin{array}{l} \}1 \\ \}n-1 \\ \}1 \\ \}n-1 \end{array}$$

$$\qquad\qquad \underbrace{}_{1} \quad \underbrace{}_{n-1} \quad \underbrace{}_{1} \quad \underbrace{}_{n-1}$$

However, since this matrix is also Hamiltonian, we must have $v=0$, $K_1^H=K_1$, $\beta=\bar{\beta}$, $s=r$, $N_1^H=N_1$, $\gamma=-\bar{\lambda}$, $u=0$, $q=-w$, and $D_1=-A_1^H$. Thus, the eigenvalues of M are λ and $-\bar{\lambda}$ together with the eigenvalues of the Hamiltonian matrix M_1 defined by

$$M_1=\begin{bmatrix} A_1 & N_1 \\ K_1 & -A_1^H \end{bmatrix}.$$

By induction, there exists a unitary symplectic Z of the form

$$Z=\begin{bmatrix} Z_{11} & Z_{12} \\ -Z_{12} & Z_{11} \end{bmatrix}, \qquad Z_{11}, Z_{12} \in \mathbb{C}^{(n-1)\times(n-1)},$$

such that

$$Z^H\begin{bmatrix} A_1 & N_1 \\ K_1 & -A_1^H \end{bmatrix}Z=\begin{bmatrix} T_1 & R_1 \\ 0 & -T_1^H \end{bmatrix},$$

where T_1 is upper triangular with eigenvalues in the left half plane. It is easy to verify that if

$$Q=P\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & Z_{11} & 0 & Z_{12} \\ \hline 0 & 0 & 1 & 0 \\ 0 & -Z_{12} & 0 & Z_{11} \end{array}\right],$$

then Q^HMQ has the form specified by the theorem. ■

This result amounts to a Schur-like decomposition for Hamiltonian matrices. However, unlike the ordinary Schur decomposition, it may fail to exist if M has purely imaginary eigenvalues. For example, it is easy to verify that no unitary symplectic similarity transformation can reduce

$$M=\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \qquad \lambda_1=i, \quad \lambda_2=-i,$$

to upper triangular form. Thus, the most we can assert about the reduction of general Hamiltonian matrices is the following:

COROLLARY 3.2. *If $M \in \mathbb{C}^{2n\times 2n}$ is a Hamiltonian matrix, then there exists a unitary symplectic $Q\in\mathbb{C}^{2n\times 2n}$ such that*

$$Q^H M Q = \begin{bmatrix} T_{11} & T_{12} & R_{11} & R_{12} \\ 0 & T_{22} & R_{21} & R_{22} \\ 0 & 0 & -T_{11}^H & 0 \\ 0 & K_{22} & -T_{12}^H & -T_{22}^H \end{bmatrix} \begin{matrix} \}p \\ \}q \\ \}p \\ \}q \end{matrix}, \qquad p+q=n,$$
$$\qquad\qquad \underbrace{}_{p} \quad \underbrace{}_{q} \quad \underbrace{}_{p} \quad \underbrace{}_{q}$$

where T_{11} is upper triangular and $\begin{bmatrix} T_{22} & R_{22} \\ K_{22} & -T_{22}^H \end{bmatrix}$ *is a Hamiltonian matrix with purely imaginary eigenvalues.*

Let us refer to those Hamiltonian matrices that can be reduced to the form (3.1) as *Schur-reducible* Hamiltonian matrices. An important class of Schur-reducible Hamiltonian matrices is identified in the following result:

COROLLARY 3.3. *If*

$$M = \begin{bmatrix} A & N \\ K & -A^H \end{bmatrix}$$

with $N^H = N \geqslant 0$ and $K^H = K \geqslant 0$, then M is Schur-reducible.

Proof. Let $N_i \to N$ and $K_i \to K$ be sequences of positive definite matrices, and define

$$M_i = \begin{bmatrix} A & N_i \\ K_i & -A^H \end{bmatrix}.$$

It follows from (1.5) and Theorem 3.1 that for each j, M_j is Schur-reducible. Suppose $Q_j^H M_j Q_j$ ($Q_j \in \mathcal{Q}$) is in Schur-Hamiltonian form and that Q is a limit point of the (bounded) sequence $\{Q_j\}$. It follows that Q is in $\mathcal{Q}$ and that $Q^H M Q$ is in Schur-Hamiltonian form. ∎

4. THE RICCATI EQUATION

The results of the previous two sections can be used to render an interesting reduction of the Riccati equation (1.8). Assume first that $N \geqslant 0$ and

$K \geqslant 0$. In light of Corollary 3.3 there exists a unitary

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix}$$

such that

$$Q^H M Q=\begin{bmatrix} T & R \\ 0 & -T^H \end{bmatrix}. \tag{4.1}$$

Using the symplectic SVD (Theorem 2.2), we can find $n \times n$ unitary matrices U and V such that

$$\begin{aligned} U^H Q_{11} V &= \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n), \qquad 0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_n \leqslant 1, \\ U^H Q_{12} V &= \Delta = \operatorname{diag}(\delta_1, \dots, \delta_n), \end{aligned} \tag{4.2}$$

and thus the equation

$$\begin{bmatrix} A & N \\ K & -A^H \end{bmatrix} \begin{bmatrix} Q_{11} \\ -Q_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} \\ -Q_{12} \end{bmatrix} T \tag{4.3}$$

transforms to

$$\begin{bmatrix} \hat{A} & \hat{N} \\ \hat{K} & -\hat{A}^H \end{bmatrix} \begin{bmatrix} \Sigma \\ -\Delta \end{bmatrix} = \begin{bmatrix} \Sigma \\ -\Delta \end{bmatrix} \hat{T}, \tag{4.4}$$

where $\hat{A} = U^H A U$, $\hat{N} = U^H N U$, $\hat{K} = U^H K U$, and $\hat{T} = V^H T V$. Moreover, it is easy to verify that if

$$-\hat{X}\hat{N}\hat{X} + \hat{X}\hat{A} + \hat{A}^H \hat{X} + \hat{K} = 0,$$

then $X = U\hat{X}U^H$ solves the original Riccati equation (1.8). In particular, if the diagonal matrix Σ is nonsingular, then

$$X = U \operatorname{diag}(\delta_1/\sigma_1, \dots, \delta_n/\sigma_n) U^H \tag{4.5}$$

is a Hermitian solution.

Recall the conditions of stabilizability (S) and detectability (D) given in Section 1. We now show that if (S) and (D) both hold, as they frequently do in optimal control, then the matrix X in (4.5) not only exists but satisfi $X \geqslant 0$.

THEOREM 4.1. *Suppose that $N = BB^H$ and $K = C^H C$ and that the matr Q in (4.1) is chosen so that T has eigenvalues in the left half plane. Let th unitary matrices U and V be given by (4.2). If (S) and (D) hold, then th matrix X in (4.5) exists, solves the Riccati equation, and satisfies $X \geqslant 0$.*

Proof. We first show that (S) implies that σ_1 is positive. We do this b contradiction. Suppose for some $j \geqslant 1$ that $0 = \sigma_1 = \cdots = \sigma_j < \sigma_{j+1}$. Thu diag$(\delta_1, \ldots, \delta_j) = I_j$, and $\Sigma_2 \equiv \text{diag}(\sigma_{j+1}, \ldots, \sigma_n)$ is nonsingular. With obviou notation, (4.4) can be partitioned as follows:

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{N}_{11} & \hat{N}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{N}_{12}^H & \hat{N}_{22} \\ \hat{K}_{11} & \hat{K}_{12} & -\hat{A}_{11}^H & -\hat{A}_{21}^H \\ \hat{K}_{12}^H & \hat{K}_{22} & -\hat{A}_{12}^H & -\hat{A}_{22}^H \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_2 \\ -I_j & 0 \\ 0 & -\Delta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_2 \\ -I_j & 0 \\ 0 & -\Delta_2 \end{bmatrix} \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{bmatrix}$$

By comparing (1,1) blocks in this equation we find that $\hat{N}_{11} = 0$. Sinc $\hat{N} \geqslant 0$, it follows that $\hat{N}_{12} = 0$ as well. Comparison of the (2,1) (3,1), and (4,1 blocks respectively reveals that $\hat{T}_{21} = 0$, $\hat{A}_{11}^H = -\hat{T}_{11}^H$, and $\hat{A}_{12} = 0$. If $0 \neq w \in \mathbb{C}$ satisfies $\hat{T}_{11} w = \lambda w$, then it follows that $\text{Re}(\lambda) \leqslant 0$, since the eigenvalues of $\hat{T}$ are the eigenvalues of $\hat{T}_{11}$ and $\hat{T}_{22}$ collectively. This implies that

$$\hat{A}^H \begin{bmatrix} w \\ 0 \end{bmatrix} = -\lambda \begin{bmatrix} w \\ 0 \end{bmatrix}, \qquad \hat{N} \begin{bmatrix} w \\ 0 \end{bmatrix} = 0.$$

However, if we define x by

$$x = U \begin{bmatrix} w \\ 0 \end{bmatrix},$$

then we have $A^H x = -\lambda x$ and $B^H x = 0$. This contradicts (S), and hence $\sigma_1 > 0$.

Now conditions (D) ensures via (1.3) and (1.6) that M has no purely imaginary eigenvalues. Thus, the eigenvalues of T are in the open left hal plane. However, if the matrix X is defined by (4.5), then $A - NX = Q_{11} T Q_{11}^{-1}$ and

$$X(A - NX) + (A - NX)^H X = -(K + XNX).$$

It follows from Lyapunov theory that $X \geqslant 0$.

It should be apparent at this stage that if σ_1 is small, then numerical difficulties can be expected to arise in the course of solving the Riccati equation. To begin with, if follows from (4.5) that

$$\|X\|_2 = \left|\frac{\delta_1}{\sigma_1}\right| = \frac{\left(1-\sigma_1^2\right)^{1/2}}{\sigma_1} = \cot \sigma_1 .$$

Suppose X is computed with a computer having machine precision ε. Since rounding errors of order ε/σ_1 can be expected to contaminate the result, we see that serious inaccuracies can arise whenever σ_1 is small.

Another way to understand the difficulties posed by a small σ_1 is based on the observation that an $O(\sigma_1)$ perturbation of M can lead to a breakdown of Laub's method [3] (or any other technique that requires the calculation of M's "stable" invariant subspace). To illustrate this important point, define the Hamiltonian M_0 by

$$M_0 = ZQ^H M Q Z^H,$$

where Q is chosen in (4.1) so that T's eigenvalues are in the left half plane and where Z is defined by the quantities in (4.2) as follows:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ -Z_{12} & Z_{11} \end{bmatrix},$$

$$Z_{11} = U\operatorname{diag}(0, \sigma_2, \ldots, \sigma_n) V^H, \qquad Z_{12} = U\operatorname{diag}(1, \delta_2, \ldots, \delta_n) V^H.$$

Note that

$$M_0 \begin{bmatrix} Z_{11} \\ -Z_{12} \end{bmatrix} = \begin{bmatrix} Z_{11} \\ -Z_{12} \end{bmatrix} T,$$

but since Z_{11} is singular, the matrix $X = Z_{12} Z_{11}^{-1}$ is undefined. We conclude that Laub's method would fail when applied to M_0, a matrix that is within $O(\sigma_1)$ of M:

$$\|M - M_0\|_2 = \|(Q-Z)Q^H M - ZQ^H M(Q-Z)Z^H\|_2 \leqslant 2\sqrt{2}\,\sigma_1 \|M\|_2 .$$

However, the precise connection between a small σ_1 and nearness to unstabilizability is unclear at this time.

5. ORTHOGONAL DECOMPOSITIONS FOR REAL HAMILTONIAN MATRICES

Much of what we have presented carries over to the case of real Hamiltonian matrices. For example, if M is real and Q is orthogonal and symplectic, then Q^TMQ is Hamiltonian. Obvious real analogs exist for Algorithms 1, 2, and 3, Theorems 2.1, 3.1, and 4.1, and Corollaries 2.2, 3.2, and 3.3.

In this section we consider the canonical forms for real Hamiltonian matrices that can be attained via orthogonal symplectic similarity transformations. Recall that in the ordinary eigenvalue problem, the real Schur decomposition theorem states that if $F\in\mathbb{R}^{n\times n}$, then there exists an orthogonal $Q\in\mathbb{R}^{n\times n}$ such that Q^TFQ is upper quasitriangular, i.e., upper traingular with possible 2×2 blocks along the diagonal. By confining the complex conjugate eigenvalues of F to these blocks, complex arithmetic is avoided.

THEOREM 5.1. (*The real Schur-Hamiltonian decomposition*) *Suppose*

$$M=\begin{bmatrix} A & N \\ K & -A^T \end{bmatrix}, \qquad A, N, K\in\mathbb{R}^{n\times n},$$

where $N^T=N$ and $K^T=K$. If M has no nonzero purely imaginary eigenvalues, then there exists an orthogonal

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix}, \qquad Q_{11}, Q_{12}\in\mathbb{R}^{n\times n},$$

such that

$$Q^TMQ=\begin{bmatrix} T & R \\ 0 & -T^T \end{bmatrix}, \qquad T, R\in\mathbb{R}^{n\times n}, \tag{5.1}$$

where T is upper quasitriangular and $R^T=R$. Q can be chosen such that the eigenvalues of T are in the left half plane and such that each 2×2 block on the diagonal of T is associated with a complex conjugate pair of eigenvalues.

Proof. The proof is identical with the proof of Theorem 3.1 except in the handling of the complex conjugate eigenvalues, which we now describe. Suppose

$$M\begin{bmatrix} y \\ z \end{bmatrix}=\lambda\begin{bmatrix} y \\ z \end{bmatrix},$$

where

$$\lambda=\gamma+i\mu, \quad \gamma,\mu\in\mathbb{R}, \quad \mu\neq 0, \quad \text{so } \gamma\neq 0,$$
$$y=u+iv, \quad u,v\in\mathbb{R}^n,$$
$$z=r+is, \quad r,s\in\mathbb{R}^n.$$

From these equations we have

$$M\begin{bmatrix} u & v \\ r & s \end{bmatrix}=\begin{bmatrix} u & v \\ r & s \end{bmatrix}\begin{bmatrix} \gamma & \mu \\ -\mu & \gamma \end{bmatrix}. \tag{5.2}$$

Let Z_1 be an orthogonal symplectic matrix determined by Algorithm 3 such that

$$Z_1\begin{bmatrix} u & v \\ r & s \end{bmatrix}=\begin{bmatrix} \alpha e_1 & f \\ 0 & g \end{bmatrix}, \qquad f,g\in\mathbb{R}^n, \quad 0\neq\alpha\in\mathbb{R}.$$

Since $\begin{bmatrix} \alpha e_1+if \\ ig \end{bmatrix}$ is an eigenvector of $Z_1MZ_1^T$, it follows from (1.4) that $(\alpha e_1+if)^H(ig)$ is real and hence $g_1=0$.

Again using Algorithm 3, we can find an orthogonal symplectic Z_2 such that

$$Z_2\begin{bmatrix} \alpha e_1 & f \\ 0 & g \end{bmatrix}=\begin{bmatrix} W \\ 0 \end{bmatrix}, \qquad W=\begin{bmatrix} \alpha & \eta \\ 0 & \kappa \end{bmatrix}\in\mathbb{R}^{2\times 2}.$$

If W were singular, then we would have $\eta u=\alpha v$ and $\eta r=\alpha s$. But this would imply that $\mu=0$ because $y=(i+\eta/\alpha)u$ and $z=(1+\eta/\alpha)r$. Thus, W must be nonsingular. Defining $P=Z_2Z_1$ and using (5.2), we obtain

$$(PMP^T)\begin{bmatrix} I_2 \\ 0 \end{bmatrix}=\begin{bmatrix} I_2 \\ 0 \end{bmatrix}W\begin{bmatrix} \gamma & \mu \\ -\mu & \gamma \end{bmatrix}W^{-1}\equiv\begin{bmatrix} L \\ 0 \end{bmatrix}.$$

The theorem now follows by induction, since

$$PMP^T=\left[\begin{array}{cc|cc} L & U^T & K & S^T \\ 0 & A_1 & S & N_1 \\ \hline 0 & 0 & -L^T & 0 \\ 0 & K_1 & -U & -A_1^T \end{array}\right]\begin{array}{l} \}2 \\ \}n-2 \\ \}2 \\ \}n-2 \end{array} \qquad \blacksquare.$$

6. ALGORITHMS

We conclude this paper with some remarks about the computation of the real Schur-Hamiltonian form. Ideally, we would like an analog of the QR algorithm consisting of a finite-step initial reduction followed by a Francis-type iteration. (See Stewart [7] for a discussion of the QR algorithm.) Of course, both phases of the proposed algorithm should rely exclusively on orthogonal symplectic similarity transformations.

We have been able to produce an initial reduction for this problem with an analog of the Householder reduction to Hessenberg form. (A matrix is *upper Hessenberg* if it is zero below its first subdiagonal.) In particular, if $N=N^T$, $K=K^T$, and A are in $\mathbb{R}^{n\times n}$, then there exists an orthogonal symplectic Q such that

$$Q^T\begin{bmatrix} A & N \\ K & -A^T \end{bmatrix} Q = \begin{bmatrix} H & R \\ D & -H^T \end{bmatrix}, \qquad H, R, D \in \mathbb{R}^{n\times n}, \tag{6.1}$$

where H is upper Hessenberg, $R^T=R$, and $D=\operatorname{diag}(d_1,\ldots,d_n)$.

To see how this can be accomplished, suppose that we have computed orthogonal symplectic matrices $P_1,\ldots,P_{k-1}$ such that

$$M_{k-1} = (P_1\cdots P_{k-1})^T M (P_1\cdots P_{k-1})$$

$$= \left[\begin{array}{cc|cc} H_{11} & H_{12} & N_{11} & N_{12} \\ ue_k^T & H_{22} & N_{21} & N_{22} \\ \hline D_{11} & e_k v^T & -H_{11}^T & -e_k u^T \\ ve_k^T & D_{22} & -H_{12}^T & -H_{22}^T \end{array}\right] \begin{matrix} \}k \\ \}n-k \\ \}k \\ \}n-k \end{matrix},$$

where H_{11} is upper Hessenberg, D_{11} is diagonal, and e_k is the kth column of I_k. By using Algorithm 3 we can construct an orthogonal symplectic matrix P_k such that

$$P_k^T\begin{bmatrix} 0 \\ u \\ 0 \\ v \end{bmatrix} \equiv \left[\begin{array}{cc|cc} I_k & 0 & 0 & 0 \\ 0 & Q_{11}^{(k)} & 0 & Q_{12}^{(k)} \\ \hline 0 & 0 & I_k & 0 \\ 0 & -Q_{12}^{(k)} & 0 & Q_{11}^{(k)} \end{array}\right]\begin{bmatrix} 0 \\ u \\ 0 \\ v \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ e_k \\ 0 \\ 0 \end{bmatrix}.$$

It then follows that

$$P_k^T M_{k-1} P_k = \left[\begin{array}{cc|cc} H_{11} & \bar{H}_{12} & N_{11} & \bar{N}_{12} \\ \alpha e_i e_k^T & \bar{H}_{22} & \bar{N}_{21} & \bar{N}_{22} \\ \hline D_{11} & 0 & -H_{11}^T & -\alpha e_k e_1^T \\ 0 & \bar{D}_{22} & -\bar{H}_{12}^T & -\bar{H}_{22}^T \end{array}\right],$$

where the bar is used to indicate those submatrices that are affected by the update. It is clear that M_{n-2} has the "Hessenberg-Hamiltonian" form described in (6.1).

The overall computation can be arranged so that A, K, and N are overwritten by H, D, and R respectively. To illustrate this and other computational nuances associated with the reduction, we give a detailed statement of the algorithm along with an assessment of the amount of work as measured in flops. A "flop" is the amount of floating-point arithmetic and subscripting approximately associated with the arithmetic expression $f_{ij} \leftarrow f_{ij} - tg_{ij}$.

For $j=1,\ldots,n-2$,

(a) Determine a Householder matrix U_j of order $n-j$ such that

$$U_j \begin{bmatrix} k_{j+1,j} \\ \vdots \\ k_{nj} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(* denotes an arbitrary nonzero element).
Set $U = \mathrm{diag}(I_j, U_j)$.

$$A \leftarrow UAU \qquad [2(n-j)^2 + 2n(n-j) \text{ flops}],$$
$$N \leftarrow UNU \qquad [2(n-j)^2 + 2j(n-j) \text{ flops}],$$
$$K \leftarrow UKU \qquad [2(n-j)^2 \text{ flops}].$$

(b) Determine c and s such that $c^2 + s^2 = 1$ and

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_{j+1,j} \\ k_{j+1,j} \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} A & N \\ K & -A^T \end{bmatrix} \leftarrow J(j+1,c,s) \begin{bmatrix} A & N \\ K & -A^T \end{bmatrix} J(j+1,c,s)^T.$$

(c) Determine a Householder matrix V_j of order $n-j$ such that

$$V_j\begin{bmatrix} a_{j+1,j} \\ \vdots \\ a_{nj} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Set $V=\operatorname{diag}(I_j, V_j)$.

$$\begin{aligned} A &\leftarrow VAV && [2(n-j)^2+2n(n-j) \text{ flops}], \\ N &\leftarrow VNV && [2(n-j)^2+2j(n-j) \text{ flops}], \\ K &\leftarrow VKV && [2(n-j)^2 \text{ flops}]. \end{aligned}$$

In deriving the flop counts it is assumed that symmetry and zero structure are exploited. When these work assessments are totaled, we find that the entire reduction requires $\frac{16}{3}n^3$ flops.

The orthogonal matrix Q in (6.1) is clearly given by

$$Q=P_1\cdots P_{n-2} \tag{6.2}$$

where each P_j has the form

$$P_j=H(j+1,u)J(j+1,c,s)^T H(j+1,v).$$

Q can be stored in "factored" form, as is so often done in orthogonal matrix computations. That is, the "Householder vectors" u and v can be stored in the positions of the entries that they are designed to reduce. [The sines and cosines of the $n-2$ Jacobi symplectics require $O(n)$ storage.] Approximately $\frac{8}{3}n^3$ flops are needed to compute Q when it is synthesized from right to left in (6.2), assuming, of course, that the symplectic structure of the P_j matrices is exploited. Thus, the entire computation of (6.1) requires $8n^3$ flops and no more than $4n^2$ storage. (A, Q_{11}, and Q_{12} each need n^2 locations, while N and K require $n^2/2$ locations apiece because of symmetry.)

This brings us to the problem of reducing the "condensed" Hessenberg form (6.1) to the Schur-Hamiltonian form (5.1). We have so far been unable to find an algorithm to do this, and here we briefly summarize what some of the difficulties appear to be.

The main problem seems to be that all potentially useful symplectic updates of the condensed form lead to "fill-in" of the nice zero structure. For example, we can compute a shift from the "lower" second-order Hamiltonian matrix

$$\begin{bmatrix} h_{pp} & h_{pn} & r_{pp} & r_{pn} \\ h_{np} & h_{nn} & r_{np} & r_{nn} \\ d_p & 0 & -h_{pp} & -h_{np} \\ 0 & d_n & -h_{pn} & -h_{nn} \end{bmatrix}, \qquad p=n-1,$$

easily enough, but then what do we do with it? Somehow we would like to update the condensed form with an implicit-shift technique in such a way that the new $(n, n-1)$ entry of H is reduced more or less as it is in the QR algorithm. We have yet to figure out how this can be accomplished. However, we remark that once $h_{n,n-1}$ is negligible, it is possible to zero d_n with a symplectic $J(n, c, s)$ and the problem then deflates.

Despite our lack of success generalizing the iterative portion of the QR algorithm, we are reasonably optimistic. Extensions of this algorithm exist for many other unitary eigenvalue decompositions. Consider, for example, the SVD algorithm [1] for singular values and the QZ algorithm [5] for the generalized Schur decomposition. It would therefore be somewhat surprising if no such extension could be found for the Schur-Hamiltonian decomposition.

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Received 3 August 1979; revised 2 January 1981