Using the Ellipse to Fit and Enclose Data Points

A First Look at Scientific Computing and Numerical Optimization

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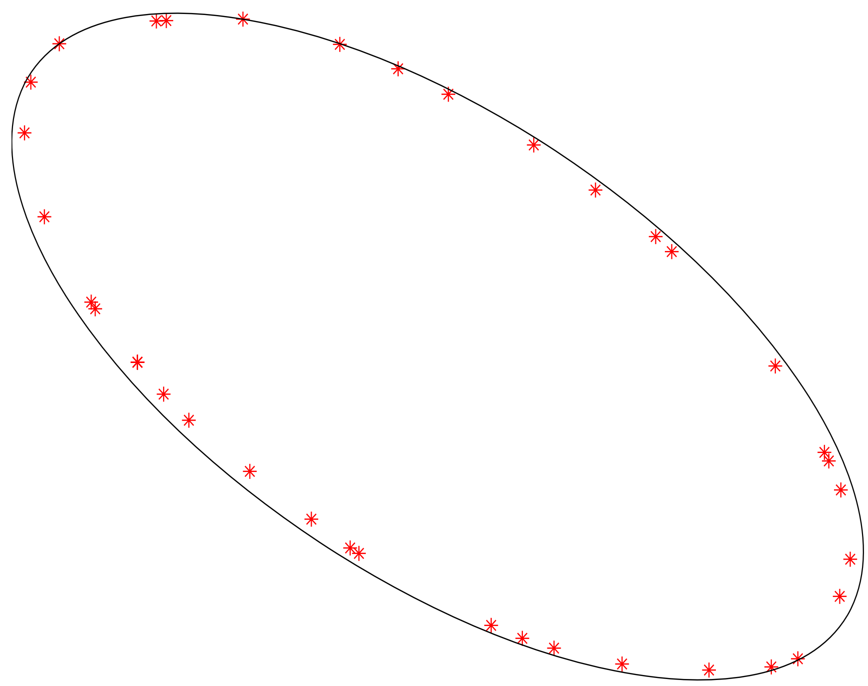
Problem 1: Ellipse Enclosing

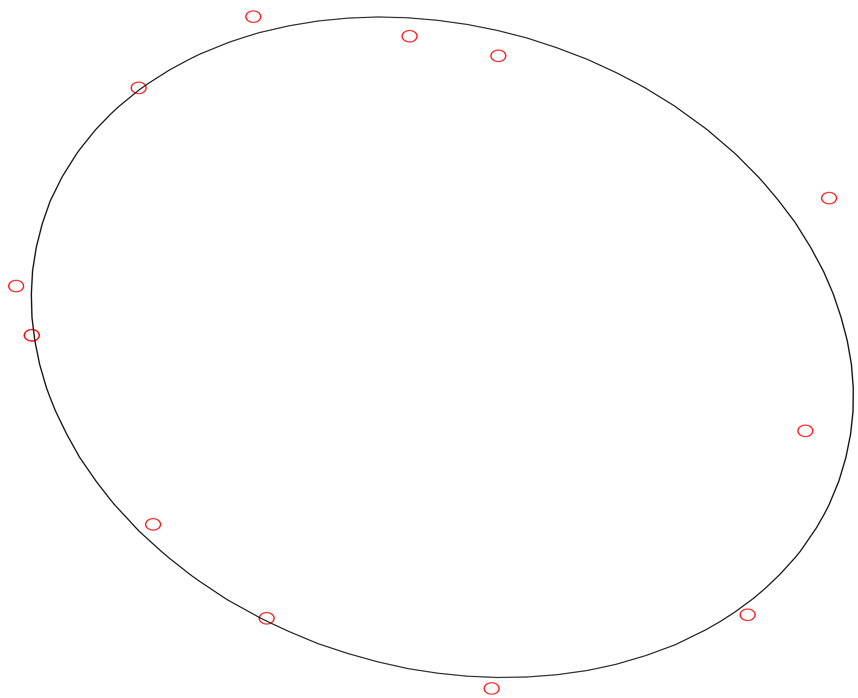
Suppose $\mathcal{P}$ a convex polygon with n vertices $P_1, \ldots, P_n$. $P_1, \ldots, P_n$. Find the **smallest** ellipse $\mathcal{E}$ that encloses $P_1, \ldots, P_n$.



What do we mean by "smallest"?

Problem 2: Ellipse Fitting

Suppose $\mathcal{P}$ a convex polygon with n vertices $P_1, \ldots, P_n$. Find an ellipse $\mathcal{E}$ that is as **close** as possible to $P_1, \ldots, P_n$.



What do we mean by "close"?

Subtext

Talking about the these problems is a nice way to introduce the field of scientific computing.

Problems 1 and 2 pose research issues, but we will keep it simple.

We can go quite far towards solving these problems using 1-D minimization and simple linear least squares.

Outline

- **Representation**

  We consider three possibilities with the ellipse.

- **Approximation**

  We can measure the size of an ellipse by area or perimeter. The latter is more complicated and requires approximation.

- **Dimension**

  We use heuristics to reduce search space dimension in the enclosure problem. Sometimes it is better to be approximate and fast than fool-proof and slow.

- **Distance**

  We consider two ways to measure the distance between a point set and an ellipse, leading to a pair of radically different best-fit algorithms.

Part I. $\rightarrow$ **Representation**

Approximation

Dimension

Distance

Representation

In computing, choosing the right representation can simplify your algorithmic life.

We have several choices when working with the ellipse:

1. *The Conic Way*
2. *The Parametric Way*
3. *The Foci/String Way*

The Conic Way

The set of points (x, y) that satisfy

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

defines an ellipse if

$$B^2 - 4AC < 0.$$

This rules out hyperbolas like $xy + 1 = 0$ and $3x^2 - 2y^2 + 1 = 0$.

This rules out parabolas like $x^2 + y = 0$ and $-3y^2 + x + 2 = 0$.

To avoid *degenerate* ellipses like $3x^2 + 4y^2 + 1 = 0$ we also require

$$\frac{D^2}{4A} + \frac{E^2}{4C} - F > 0$$

The Conic Way (Cont'd)

Without loss of generality we may assume that $A = 1$.

An ellipse is the set of points (x, y) that satisfy

$$x^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

subject to the constraints

$$B^2 - 4C < 0$$

and

$$\frac{D^2}{4} + \frac{E^2}{4C} - F > 0$$

An ellipse has five parameters.

The Parametric Way

This is an ellipse with center (h, k) and semiaxes a and b:

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

i.e., the set of points $(x(t), y(t))$ where

$$\begin{aligned} x(t) &= h + a\cos(t) \\ y(t) &= k + b\sin(t) \end{aligned}$$

and $0 \le t \le 2\pi$.

Where is the fifth parameter?

The Parametric Way (Cont'd)

The tilt of the ellipse is the fifth parameter.

Rotate the ellipse counter-clockwise by τ radians:

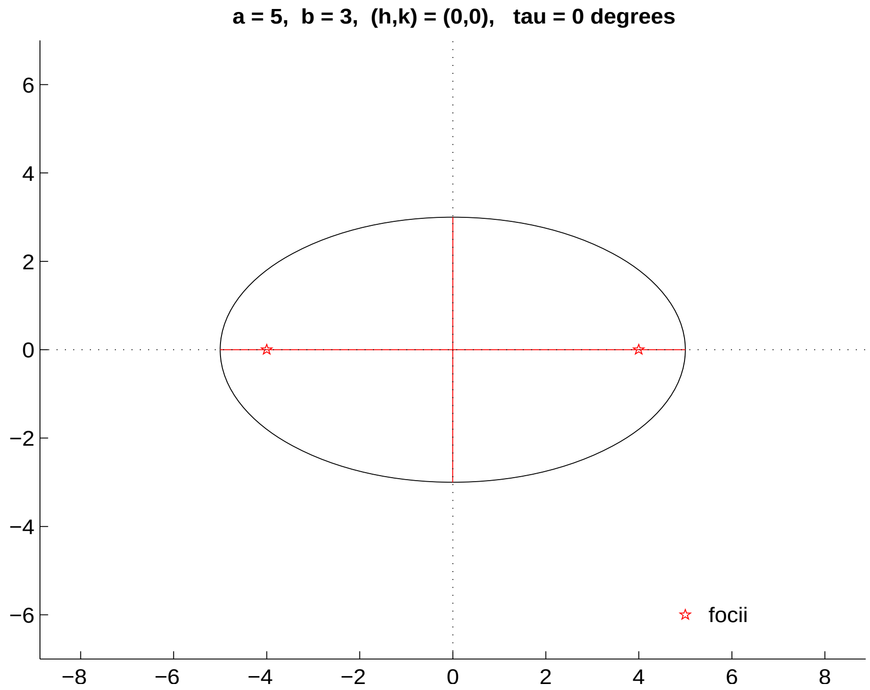
$$\begin{aligned} x(t) &= h + \cos(\tau)\,[\,a\cos(t)\,] - \sin(\tau)\,[\,b\sin(t)\,] \\ y(t) &= k + \sin(\tau)\,[\,a\cos(t)\,] + \cos(\tau)\,[\,b\sin(t)\,] \end{aligned}$$

In matrix-vector notation:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix} + \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}$$

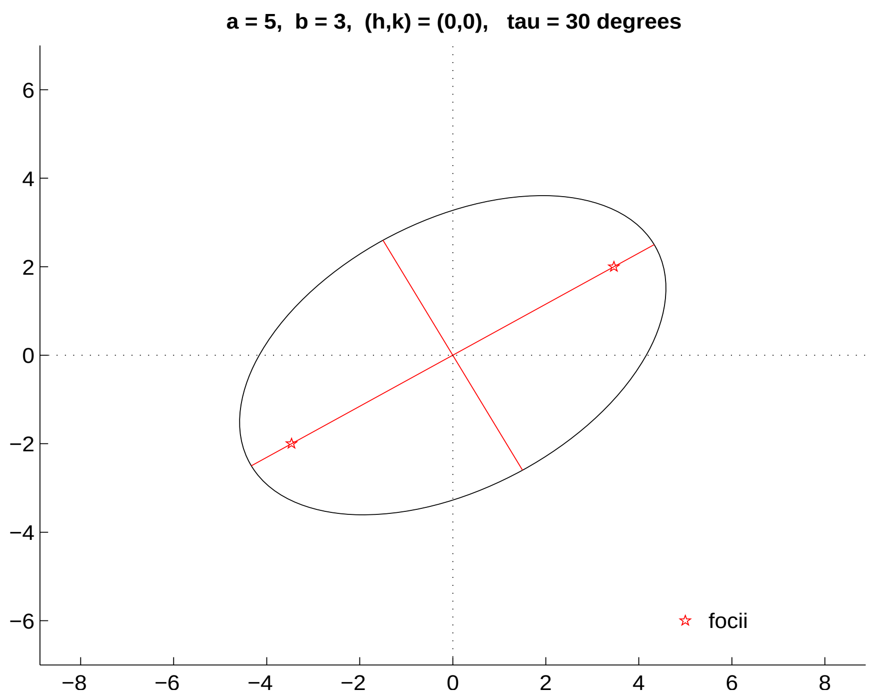
Example 1: No tilt and centered at (0,0)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 5\cos(t) \\ 3\sin(t) \end{bmatrix}$$



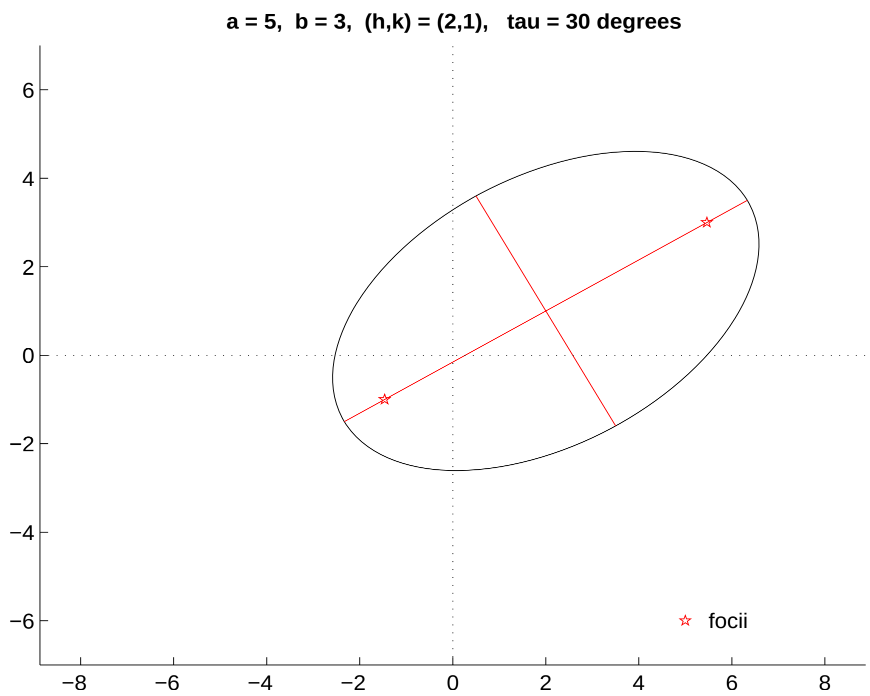
Example 2. $30^{\mathbf{o}}$ tilt and centered at (0,0)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} \begin{bmatrix} 5\cos(t) \\ 3\sin(t) \end{bmatrix}$$



Example 3. 30^{o} tilt and centered at (2,1)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} \begin{bmatrix} 5\cos(t) \\ 3\sin(t) \end{bmatrix}$$



The Foci/String Way

Suppose points $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ are given and that s is a positive number greater than the distance between them.

The set of points (x, y) that satisfy

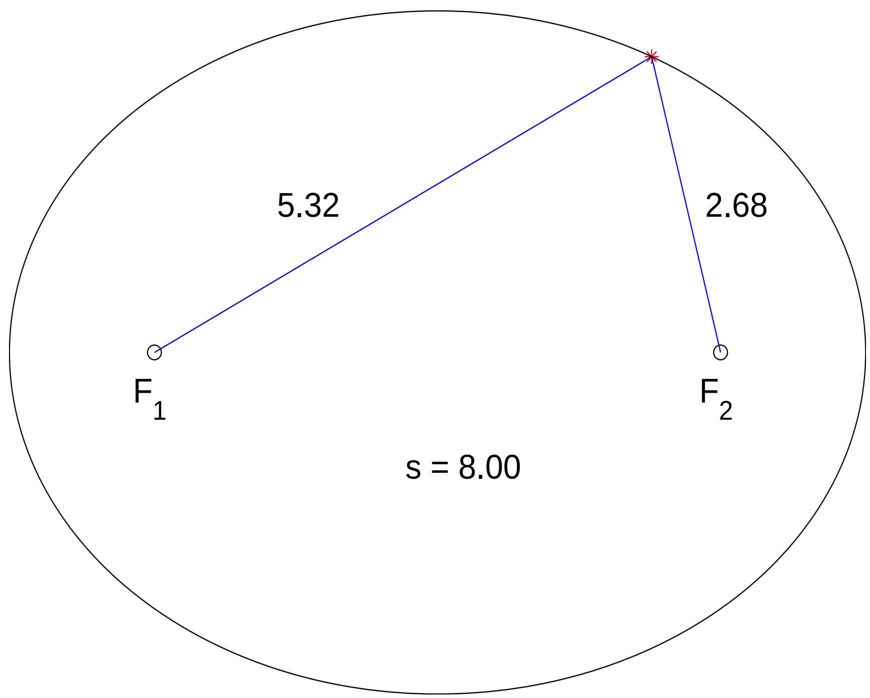
$$\sqrt{(x - x_1)^2 + (y - y_1)^2} \; + \; \sqrt{(x - x_2)^2 + (y - y_2)^2} \; = \; s$$

defines an ellipse.

The points F_1 and F_2 are the foci of the ellipse.

The sum of the distances to the foci is a constant designated by s and from the "construction" point of view can be thought of as the "string length."

The Foci/String Way (Cont'd)



Ellipse Construction: Anchor a piece of string with length s at the two foci. With your pencil circumnavigate the foci always pushing out against the string.

Conversions: Conic $\longrightarrow$ Parametric

If $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ specifies an ellipse and we define the matrices

$$M_0 = \begin{bmatrix} F & D/2 & E/2 \\ D/2 & A & B/2 \\ E/2 & B/2 & C \end{bmatrix} \qquad M = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

then

$$\begin{aligned} a &= \sqrt{-\det(M_0)/(\det(M)\lambda_1)} & \qquad b &= \sqrt{-\det(M_0)/(\det(M)\lambda_2)} \\ h &= (BE - 2CD)/(4AC - B^2) & \qquad k &= (BD - 2AE)/(4AC - B^2) \\ \tau &= \operatorname{arccot}((A - C)/B)/2 \end{aligned}$$

where λ_1 and λ_2 are the eigenvalues of M ordered so that $|\lambda_1 - A| \leq |\lambda_1 - C|$. (This ensures that $|\lambda_2 - C| \leq |\lambda_2 - A|$.)

Conversions: Parametric $\longrightarrow$ Conic

If $c = \cos(\tau)$ and $s = \sin(\tau)$, then the ellipse

$$\begin{aligned} x(t) &= h \;+\; c\,[\,a\cos(t)\,] \;-\; s\,[\,b\sin(t)\,] \\ y(t) &= k \;+\; c\,[\,a\cos(t)\,] \;+\; c\,[\,b\sin(t)\,] \end{aligned}$$

is also specified by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where

$$\begin{aligned} A &= (bc)^2 + (as)^2 \\ B &= -2cs(a^2 - b^2) \\ C &= (bs)^2 + (ac)^2 \\ D &= -2Ah - kB \\ E &= -2Ck - hB \\ F &= -(ab)^2 + Ah^2 + Bhk + Ck^2 \end{aligned}$$

Conversions: Parametric $\longrightarrow$ Foci/String

Let $\mathcal{E}$ be the ellipse

$$x(t) \;=\; h \;+\; \cos(\tau)\,[\,a\cos(t)\,] \;-\; \sin(\tau)\,[\,b\sin(t)\,]$$

$$y(t) \;=\; k \;+\; \sin(\tau)\,[\,a\cos(t)\,] \;+\; \cos(\tau)\,[\,b\sin(t)\,]$$

If $c = \sqrt{a^2 - b^2}$ then $\mathcal{E}$ has foci

$$F_1 = (h - \cos(\tau)c, k - \sin(\tau)c) \qquad\qquad F_2 = (h + \cos(\tau)c, k + \sin(\tau)c)$$

and string length

$$s = 2a$$

Conversions: Foci/String $\longrightarrow$ Parametric

If $F_1 = (\alpha_1, \beta_1)$, $F_2 = (\alpha_2, \beta_2)$, and s defines an ellipse, then

$$
\begin{aligned}
a &= s/2 \\
b &= \sqrt{s^2 - ((\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2))}\,/\,2 \\
h &= (\alpha_1 + \alpha_2)/2 \\
k &= (\beta_1 + \beta_2)/2 \\
\tau &= \arctan((\beta_2 - \beta_1)/(\alpha_2 - \alpha_1))
\end{aligned}
$$

Summary of the Representations

Parametric: h , k, a, b, and τ.

$$\mathcal{E} = \left\{ (x,y) \;\middle|\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix} + \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix},\ 0 \le t \le 2\pi \right\}$$

Conic: B, C, D, E, and F

$$\mathcal{E} = \{\, (x,y) \mid x^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \,\}$$

Foci/String: α_1, β_1, α_2, β_2, and s.

$$\mathcal{E} = \left\{ (x,y) \;\middle|\; \sqrt{(x-\alpha_1)^2 + (y-\beta_1)^2} \;+\; \sqrt{(x-\alpha_2)^2 + (y-\beta_2)^2} = s. \right\}$$

Representation

Part II. $\rightarrow$ **Approximation**

Dimension

Distance

The Size of an Ellipse

How big is an ellipse $\mathcal{E}$ with semiaxes a and b?

Two reasonable *metrics*:

$$\begin{aligned} \text{Area}(\mathcal{E}) &= \pi ab \\ \text{Perimeter}(\mathcal{E}) &= \textstyle\int_0^{2\pi} \sqrt{(a\sin(t))^2 + (b\cos(t))^2}\, dt \end{aligned}$$

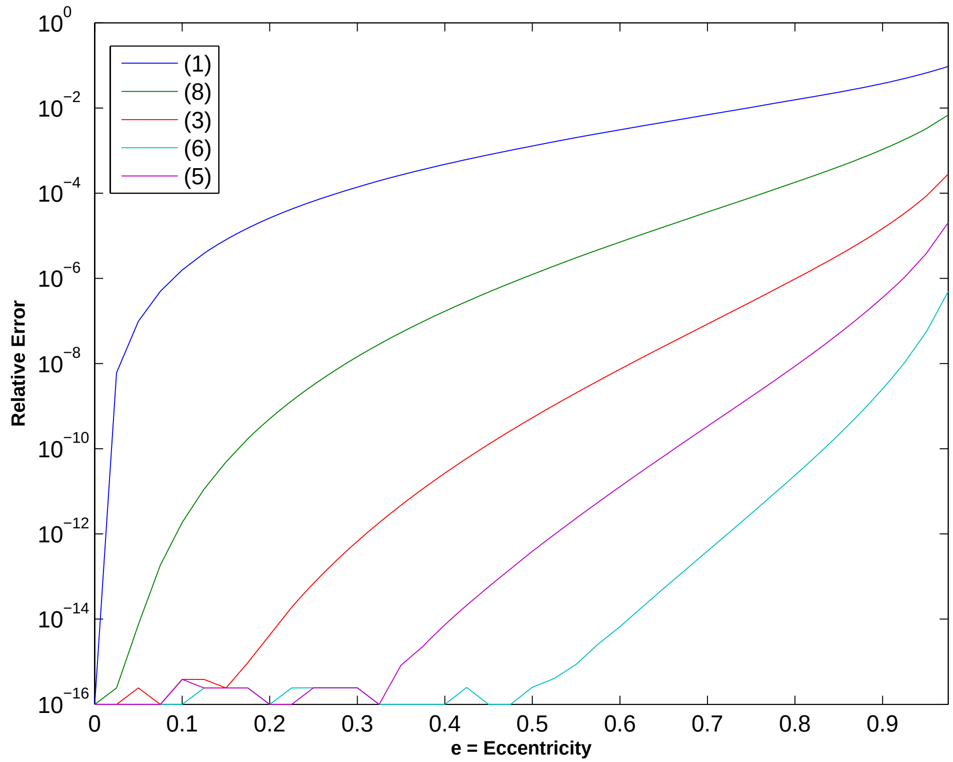
There is no simple closed-form expression for the perimeter of an ellipse.

To compute perimeter we must resort to *approximation.*

Some Formulas for Perimeter Approximation

$$
\text{Perimeter}(\mathcal{E}) \approx \begin{cases}
\pi\,(a+b) & (1) \\
\pi(a+b)\cdot\dfrac{3-\sqrt{1-h}}{2} & (2) \\
\pi(a+b)\cdot(1+h/8)^2 & (3) \\
\pi(a+b)\cdot(3-\sqrt{4-h}) & (4) \\
\pi(a+b)\cdot\dfrac{64-3h^2}{64-16h} & (5) \\
\pi(a+b)\cdot(1+\dfrac{3h}{10+\sqrt{4-3h}}) & (6) \\
\pi(a+b)\cdot\dfrac{256-48h-21h^2}{256-112h+3h^2} & (7) \\
\pi\sqrt{2(a^2+b^2)} & (8) \\
\pi\sqrt{2(a^2+b^2)-\dfrac{(a-b)^2}{2}} & (9)
\end{cases}
\qquad h = \left(\frac{a-b}{a+b}\right)^2
$$

Relative Error as a Function of Eccentricity



$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \qquad e = 0 \Rightarrow \text{circle},\ e = .99 \Rightarrow \text{cigarlike}$$

Perimeter via Quadrature

Apply a quadrature rule to

$$\text{Perimeter}(\mathcal{E}) \;=\; \int_0^{2\pi} \sqrt{(a\sin(t))^2 + (b\cos(t))^2}\; dt$$

For example:

```
  function P = Perimeter(a,b,N)
% Rectangle rule with N rectangles
t = linspace(0,2*pi,N+1);
h = 2*pi/N;
P = h*sum(sqrt((a*cos(t)).^2 + (b*sin(t)).^2));
```

How do you chose N?

What is the error?

Efficiency and Accuracy

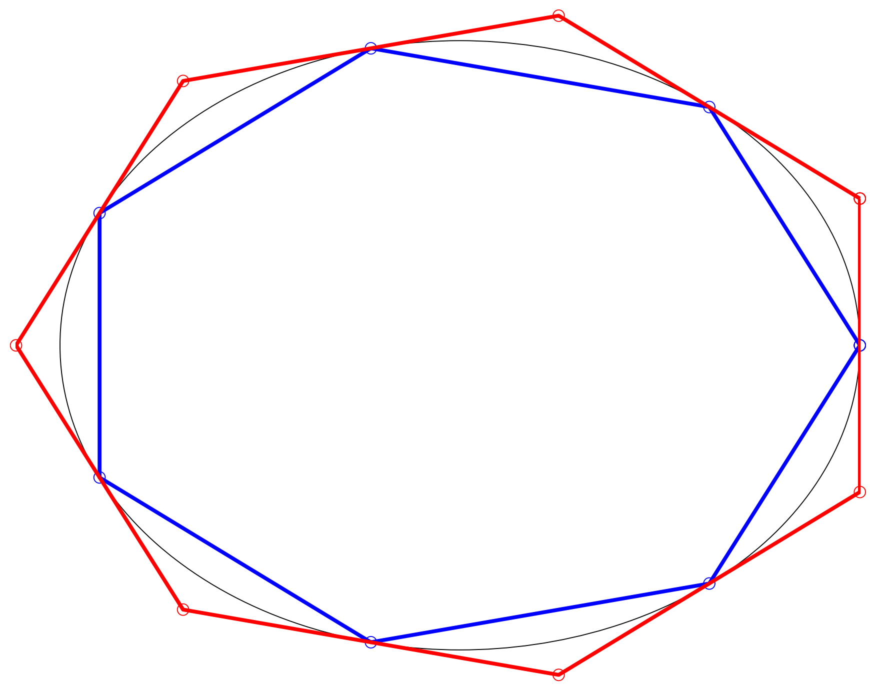
Compared to a formula like $\pi(a+b)$, the function `Perimeter(a,b,N)` is much more expensive to evaluate.

The relative error of `Perimeter(a,b,N)` is about $O(1/N^2)$.

Can we devise an approximation with an easily computed rigorous error bound?

Computable Error Bounds

For given n, define inner and outer polygons by the points $(a\cos(k\delta), b\sin(k\delta)$ for $k = 0{:}n-1$ and $\delta = 2\pi/n$.



$$\begin{pmatrix}\text{Inner Polygon}\\ \text{Perimeter}\end{pmatrix} \;\le\; \text{Perimeter}(\mathcal{E}) \;\le\; \begin{pmatrix}\text{Outer Polygon}\\ \text{Perimeter}\end{pmatrix}$$

Relative Error as a Function of n

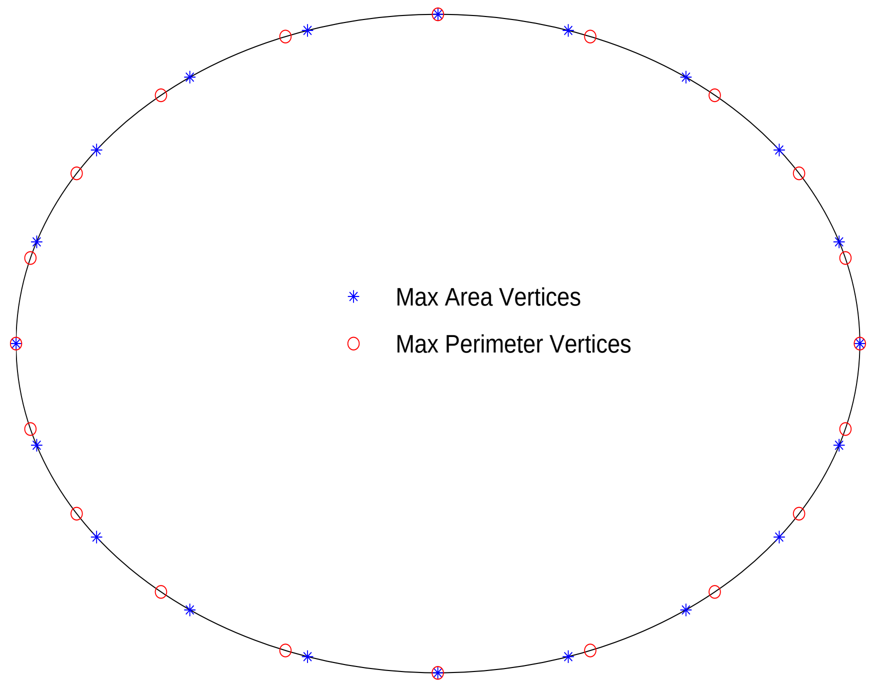
e	$n = 10$	$n = 10^2$	$n = 10^3$	$n = 10^4$	$n = 10^5$	$n = 10^6$
0.00	$1.590 \cdot 10^{-1}$	$1.551 \cdot 10^{-3}$	$1.550 \cdot 10^{-5}$	$1.550 \cdot 10^{-7}$	$1.555 \cdot 10^{-9}$	$1.114 \cdot 10^{-11}$
0.50	$1.486 \cdot 10^{-1}$	$1.449 \cdot 10^{-3}$	$1.448 \cdot 10^{-5}$	$1.448 \cdot 10^{-7}$	$1.448 \cdot 10^{-9}$	$1.461 \cdot 10^{-11}$
0.90	$1.164 \cdot 10^{-1}$	$1.157 \cdot 10^{-3}$	$1.156 \cdot 10^{-5}$	$1.156 \cdot 10^{-7}$	$1.156 \cdot 10^{-9}$	$1.163 \cdot 10^{-11}$
0.99	$6.755 \cdot 10^{-2}$	$1.015 \cdot 10^{-3}$	$1.015 \cdot 10^{-5}$	$1.015 \cdot 10^{-7}$	$1.015 \cdot 10^{-9}$	$1.003 \cdot 10^{-11}$

$$e = \text{eccentrity} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

Error mildly decreases with eccentricity.

Summary of the Area vs. Perimeter Issue

The "inverse" of the enclosing ellipse problem is the problem of inscribing the largest possible polygon in an ellipse.



The choice of objective function, Area($\mathcal{E}$) or Perimeter($\mathcal{E}$), matters. For the enclosing ellipse problem we will have to make a choice.

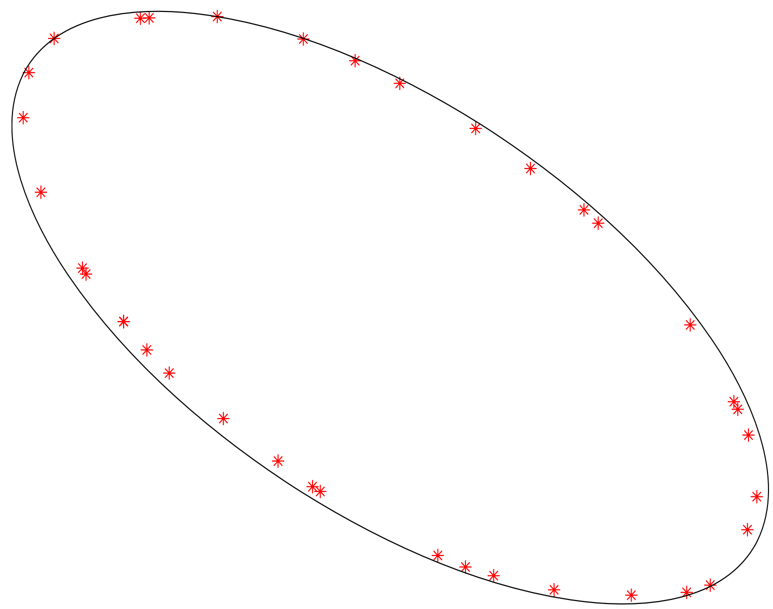
Representation

Approximation

Part III. $\rightarrow$ **Dimension**

Distance

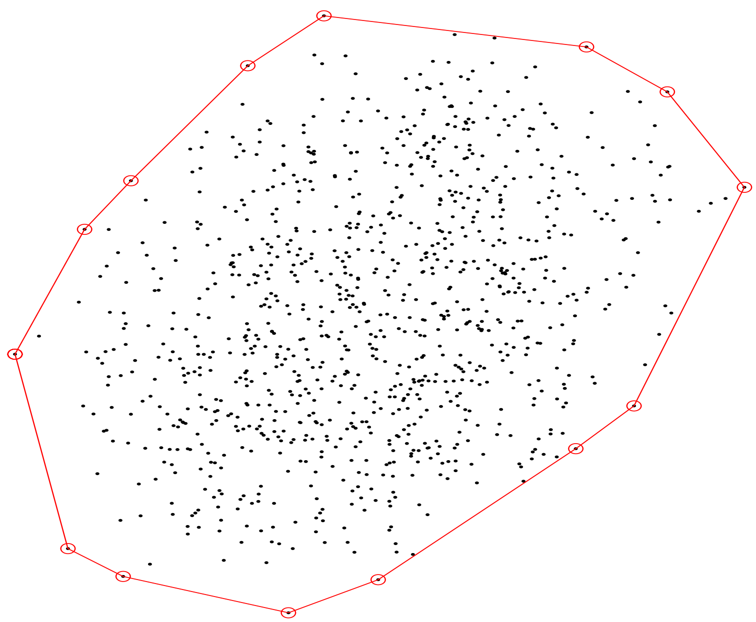
Enclosing Data with an Ellipse



Given a point set $\mathcal{P} = \{\ (x_1, y_1),\ \ldots,\ (x_n, y_n)\}$, *minimize Area(*$\mathcal{E}$*) subject to the constraint that* $\mathcal{E}$ *encloses* $\mathcal{P}$.

Everything that follows could be adapted if we used Perimeter($\mathcal{E}$).

Simplification: Convex Hull



The ConvHull($\mathcal{P}$) is a subset of $\mathcal{P}$ which when connected in the right order define a convex polyon that encloses $\mathcal{P}$.

The minimum enclosing ellipse for ConvHull($\mathcal{P}$) is the same as the minimum enclosing ellipse for $\mathcal{P}$. This greatly reduces the “size” of the problem.

Checking Enclosure

Is the point (x, y) inside the ellipse $\mathcal{E}$?

Compute the distances to the foci $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$ and compare the sum with the string length s.

In other words, if

$$\sqrt{(x-\beta_1)^2 + (y-\beta_1)^2} \; + \; \sqrt{(x-\alpha_2)^2 + (y-\beta_2)^2} \; \leq \; s$$

then (x, y) is inside $\mathcal{E}$.

The "Best" $\mathcal{E}$ given Foci F_1 and F_2

If $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$ are fixed, then area is a function of the string length s. In particular, if

$$d = \sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2}$$

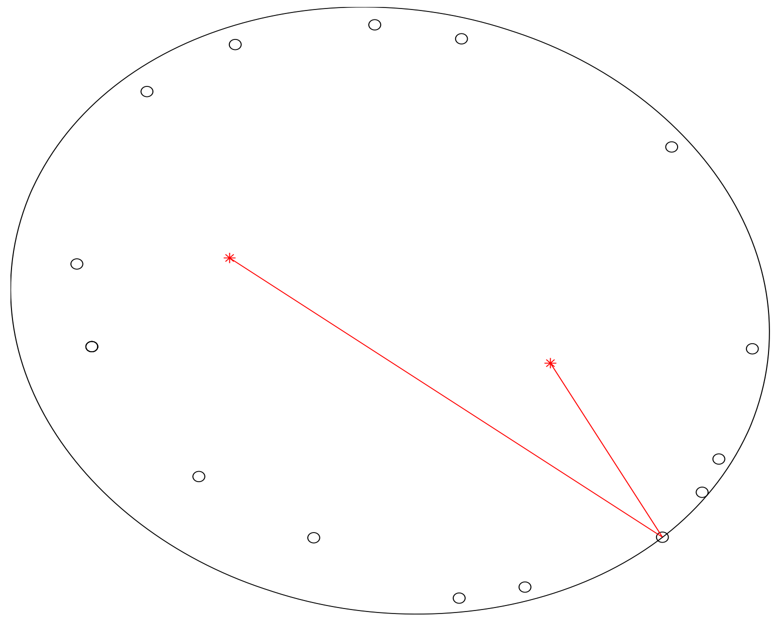
then it can be shown that

$$\text{Area}(\mathcal{E}) \;=\; \pi \, \frac{s}{4} \, \sqrt{s^2 - d^2}$$

If $\mathcal{E}$ is to enclose $\mathcal{P} = \{\ (x_1, y_1),\ \ldots,\ (x_n, y_n)\}$, and have minimal area, we want the smallest possible s, i.e.,

$$s(F_1, F_2) \;=\; \max_{1 \le i \le n} \; \sqrt{(x_i - \alpha_1)^2 + (y_i - \beta_1)^2} \;+\; \sqrt{(x_i - \alpha_2)^2 + (y_i - \beta_2)^2}$$

The "Best" $\mathcal{E}$ given Foci F_1 and F_2 (Cont'd)



Locate the point whose distance sum to the two foci is maximal. This determines s_{opt} and $\mathcal{E}_{opt}$

$$\text{Area}(\mathcal{E}_{opt}) = \frac{\pi}{4} s_{opt} \sqrt{\left(\frac{s_{opt}}{2}\right)^2 - \left(\frac{d}{2}\right)^2}$$

The "Best" $\mathcal{E}$ given Center (h, k) and Tilt τ

Foci location depends on the space between them d:

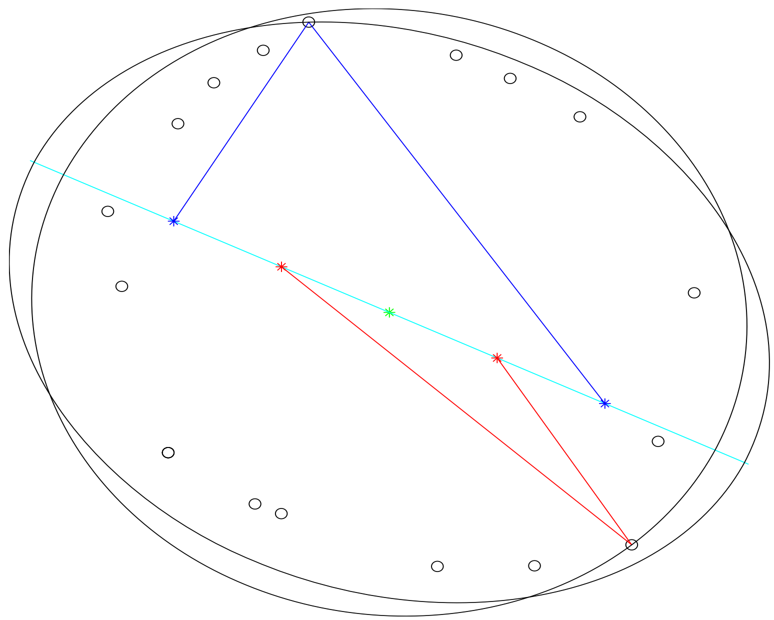
$$
\begin{aligned}
F_1(d) \;&=\; \left(h - \frac{d}{2}\cos(\tau)\;,\; k - \frac{d}{2}\sin(\tau)\right) \\
F_2(d) \;&=\; \left(h + \frac{d}{2}\cos(\tau)\;,\; k + \frac{d}{2}\sin(\tau)\right)
\end{aligned}
$$

Optimum string length $s(F_1(d), F_2(d))$ is also a function of d

The minimum area enclosing ellipse with center (h, k) and tilt τ, is defined by setting $d = d_*$ where d_* minimizes

$$
f_{h,k,\tau}(d) \;=\; \pi\,\frac{s}{4}\sqrt{s^2 - d^2} \qquad s = s(F_1(d), F_2(d))
$$

The "Best" $\mathcal{E}$ given Center (h, k) and Tilt τ (Cont'd)



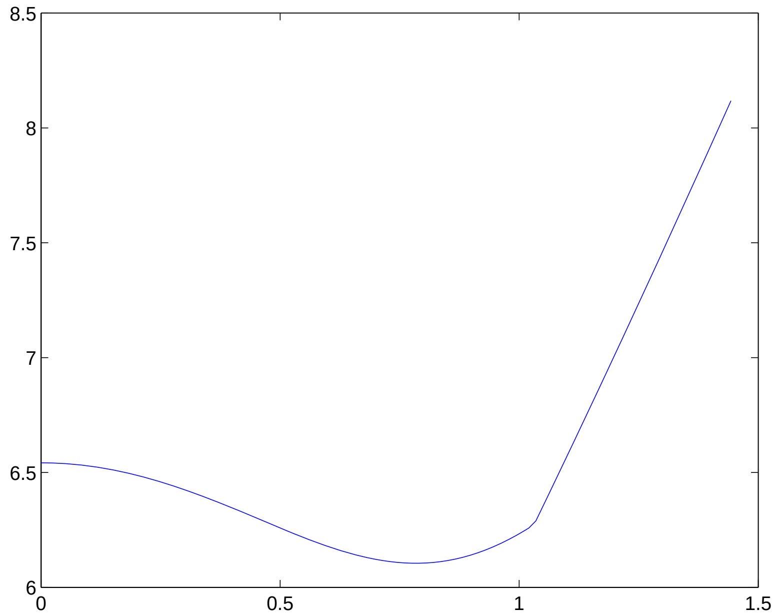
For each choice of separation d along the tilt line, we get a different minimum enclosing ellipse. Finding the best d is a Golden Section Search problem.

The "Best" $\mathcal{E}$ given Center (h, k) and Tilt τ (Cont'd)

Typical plot of the function

$$f_{h,k,\tau}(d) \;=\; \pi \, \frac{s}{4} \sqrt{s^2 - d^2} \qquad s = s(F_1(d), F_2(d))$$

for d ranging from 0 to $\max \left\{ \sqrt{(x(i) - h)^2 + (y(i) - k)^2} \right\}$:



Heuristic Choice for Center(h, k) and Tilt τ

Assume that (x_p, y_p) and (x_q, y_q) are the two points in $\mathcal{P}$ that are furthest apart. (They specify the *diameter* of $\mathcal{P}$.)

Instead of looking for the optimum h, k, and τ we can set

$$\begin{aligned} h &= \frac{x_p + x_q}{2} \\ k &= \frac{y_p + y_q}{2} \\ \tau &= \arctan\left(\frac{y_q - y_p}{x_q - x_p}\right) \end{aligned}$$

and then complete the specification of an approximately optimum $\mathcal{E}$ by determining d_* and $s(F_1(d_*), F_2(d_*))$ as above.

We'll call this the $hk\tau$-heuristic approach. Idea: the major axis tends to be along the line where the points are dispersed the most.

The "Best" $\mathcal{E}$ given Center (h, k)

The optimizing d_* for the function $f_{h,k,d}(d)$ depends on the tilt parameter τ.

Denote this dependence by $d_*(\tau)$.

The minimum-area enclosing ellipse with center (h, k) is defined by setting $\tau = \tau_*$ where τ_* minimizes

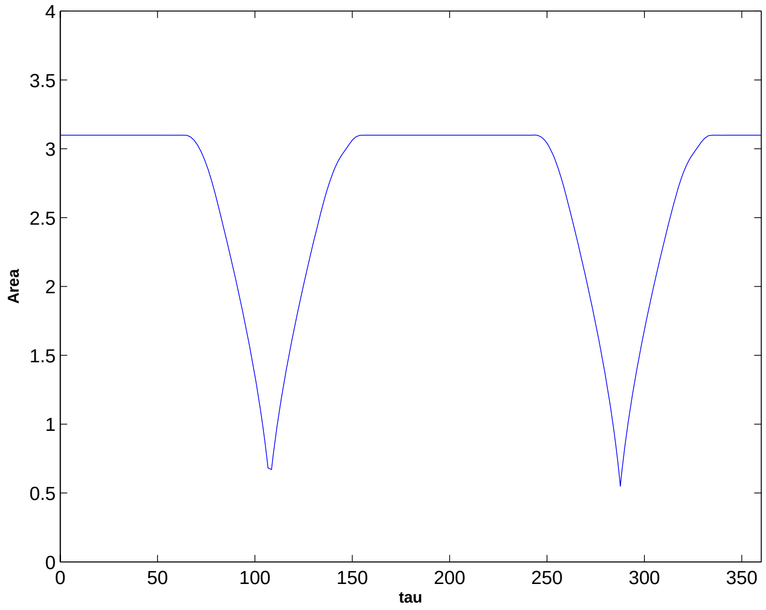
$$\tilde{f}_{h,k}(\tau) \; = \; \pi \, \frac{s}{4} \sqrt{s^2 - d_*(\tau)^2} \qquad s = s(F_1(d_*(\tau)), F_2(d_*(\tau)))$$

The “Best” $\mathcal{E}$ given Center (h, k) (Cont’d)

Typical plot of the function

$$\tilde{f}_{h,k}(\tau) \;=\; \pi\,\frac{s}{4}\,\sqrt{s^2 - d_*(\tau)^2} \qquad s = s(F_1(d_*(\tau)), F_2(d_*(\tau)))$$

across the interval $0 \leq \tau \leq 360^{\text{O}}$:



The “Best” $\mathcal{E}$

The optimizing τ_* for the function $\tilde{f}_{h,k}(\tau)$ depends on the center coordinates h and k.

Denote this dependence by $\tau_*(h,k))$.

The minimum-area enclosing ellipse is defined by setting $(h,k) = (h_*, k_*)$ where h_* and k_* minimize

$$F(h,k) \;=\; \tilde{f}_{h,k}(\tau_*(h,k)) \;=\; \pi\,\frac{s}{4}\sqrt{s^2 - d_*(\tau_*(h,k))^2}$$

where

$$s = s(F_1(d_*(\tau_*(h,k))), F_2(d_*(\tau_*(h,k))))$$

The "Best" $\mathcal{E}$

Through these devices we have reduced to two the dimension of the search for the minimum area enclosing ellipse.

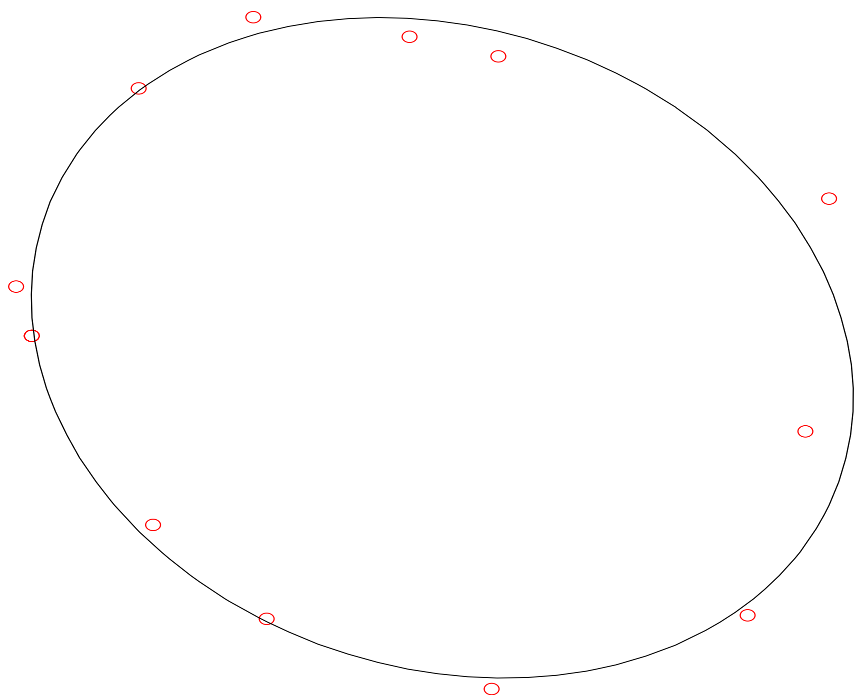
Representation

Approximation

Dimension

Part IV. $\longrightarrow$ Distance

Approximating Data with an Ellipse



We need to define the distance from a point set $\mathcal{P} = \{P_1, \ldots, P_n\}$ to an ellipse $\mathcal{E}$.

Goodness-of-Fit: Conic Residual

The Point set $\mathcal{P}$:

$$\{\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\ \}$$

The Ellipse $\mathcal{E}$:

$$x^2 + Bxy + Cy^2 + Dx + Ey + F \ = \ 0$$

The distance from $\mathcal{P}$ to $\mathcal{E}$:

$$\mathrm{dist}(\mathcal{P}, \mathcal{E}) \ = \ \sum_{i=1}^{n} \left(\alpha_i^2 + \alpha_i \beta_i B + \beta_i^2 C + \alpha_i D + \beta_i E + F\right)^2$$

Sum the squares of what's "left over" when you plug each (α_i, β_i) *into the ellipse equation.*

A Linear Least Squares Problem with Five Unknowns

$$\operatorname{dist}(\mathcal{P}, \mathcal{E}) = \left\| \begin{bmatrix} \alpha_1\beta_1 & \beta_1^2 & \alpha_1 & \beta_1 & 1 \\ \alpha_2\beta_2 & \beta_2^2 & \alpha_2 & \beta_2 & 1 \\ \alpha_3\beta_3 & \beta_3^2 & \alpha_3 & \beta_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1}\beta_{n-1} & \beta_{n-1}^2 & \alpha_{n-1} & \beta_{n-1} & 1 \\ \alpha_n\beta_n & \beta_n^2 & \alpha_n & \beta_n & 1 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \\ E \\ F \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} \right\|_2^2$$

Goodness-of-Fit: Point Proximity

The Point set $\mathcal{P}$:

$$\{\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\ \}$$

The Ellipse $\mathcal{E}$:

$$x(t) \;=\; h \;+\; \cos(\tau)\,[\,a\cos(t)\,] \;-\; \sin(\tau)\,[\,b\sin(t)\,]$$

$$y(t) \;=\; k \;+\; \sin(\tau)\,[\,a\cos(t)\,] \;+\; \cos(\tau)\,[\,b\sin(t)\,]$$

The distance from $\mathcal{P}$ to $\mathcal{E}$:

$$\mathrm{dist}(\mathcal{P}, \mathcal{E}) \;=\; \sum_{i=1}^{n} ((x(t_i) - \alpha_i)^2 + (y(t_i) - \beta_i)^2$$

where $(x(t_i), y(t_i))$ is the closest point on $\mathcal{E}$ to (α_i, β_i), $i = 1{:}n$.

Distance from a Point to an Ellipse

Let $\mathcal{E}$ be the ellipse

$$\begin{aligned} x(t) &= h + \cos(\tau)\,[\,a\cos(t)\,] - \sin(\tau)\,[\,b\sin(t)\,] \\ y(t) &= k + \sin(\tau)\,[\,a\cos(t)\,] + \cos(\tau)\,[\,b\sin(t)\,] \end{aligned}$$

To find the distance from point $P = (\alpha, \beta)$ to $\mathcal{E}$ define

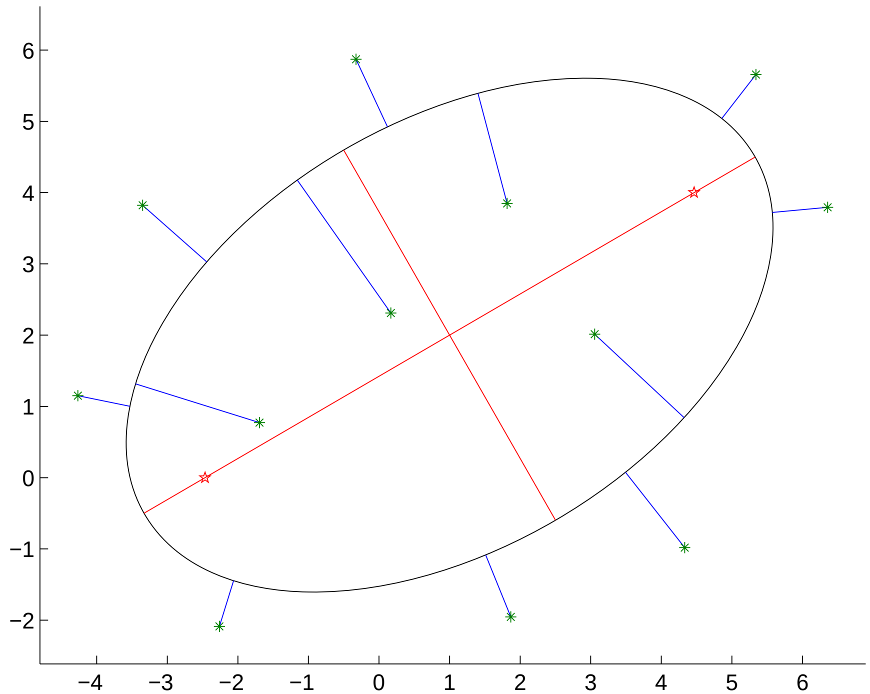
$$d(t) = \sqrt{(\alpha - x(t))^2 + (\beta - y(t))^2}$$

and set

$$\operatorname{dist}(P, \mathcal{E}) = \min_{0 \le t \le 2\pi} d(t)$$

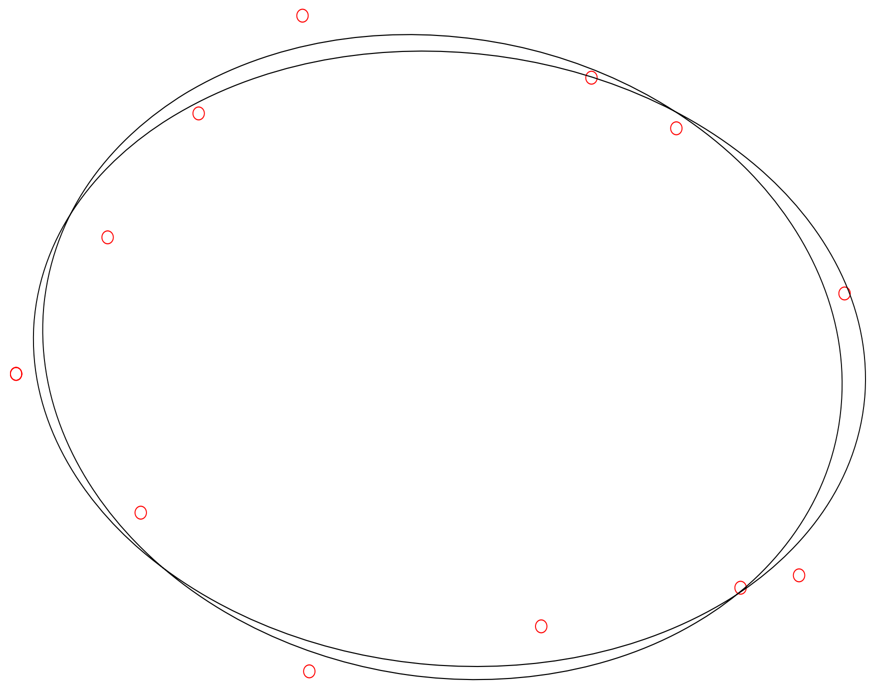
Note that $d(t)$ is a function of a single variable.

“Drop” Perpendiculars



The nearest point on the ellipse $\mathcal{E}$ is in the same “ellipse quadrant”.

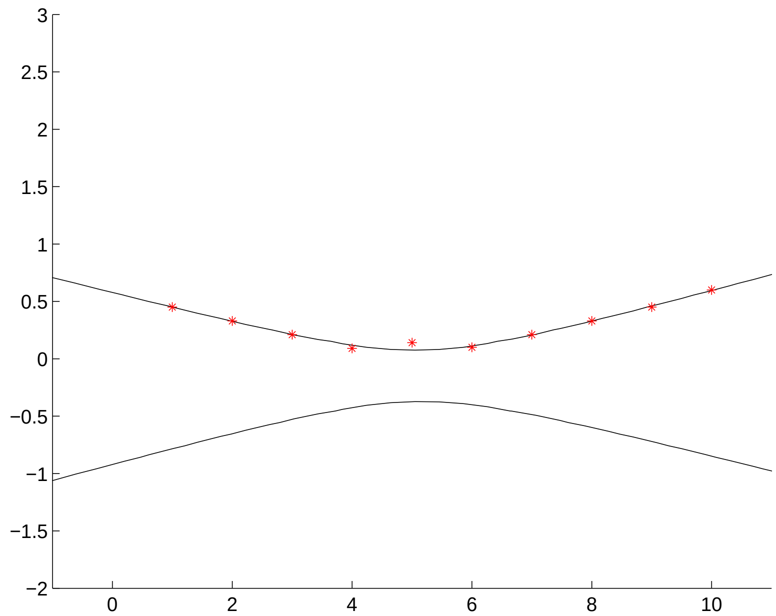
Comparison: Conic Residual vs Point Proximity



The two methods render different best-fitting ellipses.

Conic Residual method is much faster but it may render a hyperbola if the data is "bad".

Conic Residual Method w/o Constraints



There are fast ways to solve the conic residual least squares problem with the constraint $C > B^2/4$ which forces the solution

$$x^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

to define an ellipse.

Overall Conclusions

- **Representation**

  The Parametric Representation if not "friendly" when you want to check if a point is inside an ellipse. The Conic representation led to a very simple algorithm for the best-fit problem.

- **Approximation**

  There are many ways to approximate the perimeter of an ellipse. Although we defined the size of an ellipse in terms of its easily-computed area, it would also be possible to work with perimeter.

- **Dimension**

  We use heuristics to reduce search space dimension in the enclosure problem.

- **Distance**

  We consider two ways to measure the distance between a point set and an ellipse, leading to a pair of radically different best-fit algorithms.