Introduction to Kleene Algebra	**Lecture 6**
CS786 Spring 2002	**Tuesday, February 12, 2002**

Relations between Algebras

The notion of free algebra described in the previous lecture is an example of a more general phenomenon called *adjunction*. An adjunction is a way of describing a particular relationship between categories of algebraic structures.

An adjunction typically arises when some category C of algebras has more structure than another category D, but there is a canonical way to extend any D-algebra to a C-algebra. The construction normally constitutes a functor $\mathsf{F} : \mathsf{D} \to \mathsf{C}$, and there is normally a corresponding *forgetful functor* $\mathsf{G} : \mathsf{C} \to \mathsf{D}$ going in the opposite direction that forgets the extra structure.

We will show that adjuctions characterize the relationships among the following categories:

- KA^*, the category of star-continuous Kleene algebras and Kleene algebra morphisms;
- CS, the category of closed semirings and ω-continuous semiring morphisms (semiring morphisms that preserve suprema of countable sets); and
- SA, the category of S-algebras and continuous semiring morphisms (semiring morphisms that preserve arbitrary suprema).

Adjunction

Formally, let $\mathsf{F} : \mathsf{D} \to \mathsf{C}$ and $\mathsf{G} : \mathsf{C} \to \mathsf{D}$ be functors between two categories C and D (think of D as the category with less structure). We say that F is a *left adjoint* of G and that G is a *right adjoint* of F if for any D-algebra A and C-algebra B, there are maps $\iota_A : A \to \mathsf{F}\,A$ and $\eta_B : B \to \mathsf{G}\,B$ such that

(i) for any D-homomorphism $h : A \to \mathsf{G}B$, there is a unique C-homomorphism $\widehat{h} : \mathsf{F}A \to B$ such that $h = \eta_B \circ \widehat{h} \circ \iota_A$;

(ii) for any C-homomorphism $g : \mathsf{F}\,A \to B$, the map $\eta_B \circ g \circ \iota_A : A \to \mathsf{G}\,B$ is a D-homomorphism; and

(iii) for any D-homomorphism $h : A \to B$, $(\mathsf{F}\,h) \circ \iota_A = \iota_B \circ h$, and similarly for η and G.

$$
\begin{tikzcd}
\mathsf{C} \arrow[d, shift left, "\mathsf{G}"] & \mathsf{F}\,A \arrow[r, "\widehat{h}"] & B \arrow[d, "\eta_B"] \\
\mathsf{D} \arrow[u, shift left, "\mathsf{F}"] & A \arrow[u, "\iota_A"] \arrow[r, "h"'] & \mathsf{G}\,B
\end{tikzcd}
$$

In all instances we will consider, G is a *forgetful functor*, which means that $\mathsf{G}\,g$ is set-theoretically the same function as g, and η is the identity function.

In the free construction of the last lecture, the two categories are the category KA^* of star-continuous Kleene algebras and Kleene algebra homomorphisms and the category Set of sets and set functions. The free construction is the left adjoint of the forgetful functor that associates to every star-continuous Kleene algebra its underlying set.

$$
\begin{tikzcd}
\mathsf{KA}^* \arrow[d, shift left, "\mathsf{G}"] & \mathsf{Reg}_\Sigma \arrow[r, "\widehat{h}"] & K \arrow[d, "id"] \\
\mathsf{Set} \arrow[u, shift left, "\mathsf{F}"] & \Sigma \arrow[u, "R_\Sigma"] \arrow[r, "h"'] & K
\end{tikzcd}
$$

In general, a category of algebras has free algebras iff the forgetful functor to Set, which forgets all algebraic structure, has a left adjoint.

As observed, every closed semiring C gives a star-continuous Kleene algebra $\mathsf{K}\,C$ by defining $x^* = \sum_n x^n$. Also, if $h : C \to C'$ is an ω-continuous semiring morphism between closed semirings, then h must preserve *, therefore is a Kleene algebra morphism $\mathsf{K}\,h : \mathsf{K}\,S \to \mathsf{K}\,S'$. Similarly, every S-algebra S is a closed semiring $\mathsf{G}\,S$, and every complete semiring morphism is ω-complete. Thus we have a forgetful functors $\mathsf{G} : \mathsf{SA} \to \mathsf{CS}$ and $\mathsf{K} : \mathsf{CS} \to \mathsf{KA}^*$.

In the other direction, not every star-continuous Kleene algebra is a closed semiring. However, it is possible to construct, in a canonical way, a closed semiring $\mathsf{C}\,K$ extending any star-continuous Kleene algebra K. Similarly, although not every closed semiring is an S-algebra, every closed semiring can be extended to one.

Furthermore, any Kleene algebra homomorphism $h : K \to K'$ extends naturally to an ω-complete semiring morphism $\mathsf{C}\,g : \mathsf{C}\,K \to \mathsf{C}\,K'$, and every ω-complete semiring morphism $g : C \to C'$ extends to a complete semiring morphism $\mathsf{S}\,g : \mathsf{S}\,C \to \mathsf{S}\,C'$. The functors $\mathsf{C} : \mathsf{KA}^* \to \mathsf{CS}$ and $\mathsf{S} : \mathsf{CS} \to \mathsf{SA}$ are left adjoints to K and G, respectively.

$$
\begin{tikzcd}
\mathsf{KA}^* \arrow[r, shift left, "\mathsf{C}"] & \mathsf{CS} \arrow[l, shift left, "\mathsf{K}"] \arrow[r, shift left, "\mathsf{S}"] & \mathsf{SA} \arrow[l, shift left, "\mathsf{G}"]
\end{tikzcd}
$$

Completion by Star-Ideals

The basic construction used here is known as *completion by star-ideals* and was used by Conway to extend a star-continuous Kleene algebra to an S-algebra [1, Theorem 1, p. 102]. Thus Conway's construction is equivalent to the composition $\mathsf{S} \circ \mathsf{C}$. The construction C, which shows that every star-continuous Kleene algebra is embedded in a closed semiring, can be described as a completion by *countably generated* star-ideals.

Definition 0.1 (Conway [1]) Let K be a star-continuous Kleene algebra. A *star-ideal* is a subset I of K such that

- I is nonempty
- I is closed under $+$
- I is closed downward under $\leq$
- if $ab^n c \in I$ for all $n \geq 0$, then $ab^* c \in I$.

$\square$

A nonempty set A *generates* a star-ideal I if I is the smallest star-ideal containing A. We write $<A>$ to denote the star-ideal generated by A. A star-ideal is *countably generated* if it has a countable generating set. If A is a singleton $\{x\}$, then we abbreviate $<\{x\}>$ by $<x>$. Such an ideal is called *principal with generator x*.

Let K be a star-continuous Kleene algebra. We define a closed semiring $\mathsf{C}K$ as follows. The elements of $\mathsf{C}K$ will be the countably generated star-ideals of K. For any countable set of countably generated star-ideals I_n, define

$$\sum_n I_n \quad = \quad <\bigcup_n I_n>.$$

This ideal is countably generated, since if A_n is countable and generates I_n for $n \geq 0$, then $\bigcup_n A_n$ is countable and generates $\sum_n I_n$. The operator $\sum$ is associative, commutative, and idempotent, since $\bigcup$ is.

For any pair of elements I, J, define

$$I \cdot J \quad = \quad <\{ab \mid a \in A,\ b \in B\}>.$$

This ideal is countably generated if I and J are, since

$$\begin{aligned} <A> \cdot <B> &= <\{ab \mid a \in <A>,\ b \in <B>\}> \\ &= <\{ab \mid a \in A,\ b \in B\}>, \end{aligned}$$

and $\{ab \mid a \in A,\ b \in B\}$ is countable if A and B are (Exercise **??**).

The ideal $<0> = \{0\}$ is included in every ideal and is thus an additive identity. It is also a multiplicative annihilator:

$$\begin{aligned} <0> \cdot I &= <\{ab \mid a \in <0>,\ b \in I\}> \\ &= <\{ab \mid a \in \{0\},\ b \in I\}> \\ &= <0>. \end{aligned}$$

The ideal $<1>$ is a multiplicative identity:

$$\begin{aligned} <1> \cdot I &= <\{ab \mid a \in <1>,\ b \in I\}> \\ &= <\{ab \mid a \in \{1\},\ b \in I\}> \quad \text{by Exercise \textbf{??}} \\ &= <I> \\ &= I. \end{aligned}$$

Finally, the distributive laws hold:

$$\begin{aligned} I \cdot \sum_n J_n &= <\{ab \mid a \in I,\ b \in \sum_n J_n\}> \\ &= <\{ab \mid a \in I,\ b \in <\bigcup_n J_n>\}> \\ &= <\{ab \mid a \in I,\ b \in \bigcup_n J_n\}> \quad \text{by Exercise \textbf{??}} \\ &= <\bigcup_n \{ab \mid a \in I,\ b \in J_n\}> \\ &= <\bigcup_n <\{ab \mid a \in I,\ b \in J_n\}>> \\ &= <\bigcup_n \{ab \mid a \in I,\ b \in J_n\}> \\ &= \sum_n I \cdot J_n, \end{aligned}$$

and symmetrically.

Closed Semirings and S-algebras

The other half of the factorization of Conway's construction embeds an arbitrary closed semiring into an S-algebra. In comparison to the previous construction, this construction is much less interesting. We give the main construction and omit formal details.

Recall that closed semirings and S-algebras are both idempotent semirings with an infinite summation operator $\sum$ satisfying infinitary associativity, commutativity, idempotence, and distributivity laws. The only difference is that closed semirings allow only countable sums, whereas S-algebras allow arbitrary sums. Morphisms of closed semirings are the ω-continuous semiring morphisms and those of S-algebras are the continuous semiring morphisms.

To embed a given closed semiring C in an S-algebra $\mathsf{S}\,C$, we complete C by ideals. An *ideal* is a subset $A \subseteq C$ such that

- A is nonempty
- A is closed under countable sum
- A is closed downward under $\leq$.

Take $\mathsf{S}\,C$ to be the set of ideals of C with the following operations:

$$\begin{aligned} \sum_\alpha I_\alpha &= \langle \bigcup_\alpha I_\alpha \rangle \\ I \cdot J &= \langle \{ab \mid a \in I,\ b \in J\} \rangle \\ 0 &= \langle 0 \rangle \\ 1 &= \langle 1 \rangle. \end{aligned}$$

The arguments from here on are quite analogous to those of the previous section.

References

[1] John Horton Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, London, 1971.