Completeness of Star-Continuity

We argued in the previous lecture that the equational theory of each of the following classes of interpretations is contained in the next in the list: Kleene algebras, star-continuous Kleene algebras, closed semirings, S-algebras, N-algebras, relational models, language models, Reg_Σ, and $\mathsf{Reg}_\Sigma, R_\Sigma$.

In this lecture we show that all these classes from star-continuous Kleene algebra onward have the same equational theory. It suffices to show that any equation that holds under the canonical interpretation $R_\Sigma : \mathsf{RExp}_\Sigma \to \mathsf{Reg}_\Sigma$ holds in all star-continuous Kleene algebras. Later on we will add Kleene algebras to this list as well.

Lemma 0.1 *For any regular expressions $s, t, u \in \mathsf{RExp}_\Sigma$, the following property holds in any star-continuous Kleene algebra K:*

$$stu \quad = \quad \sup_{x \in R_\Sigma(t)} sxu.$$

In other words, if K is star-continuous, then under any interpretation $I : \mathsf{RExp}_\Sigma \to K$, the supremum of the set

$$\{I(sxu) \mid x \in R_\Sigma(t)\}$$

exists and is equal to $I(stu)$. In particular, in the special case $s = u = 1$,

$$t \quad = \quad \sup_{x \in R_\Sigma(t)} x.$$

Note that the star-continuity axiom is a special case of this lemma for t of the form a^*, $a \in \Sigma$. This lemma expresses a natural extension of the star-continuity property to all expressions t. Later on we will do the same thing with the axioms of Kleene algebra; there we will extend the axioms, which give the ability to solve one linear inequality, to a theorem that gives the solution to any finite system of inequalities.

Proof. Let K be an arbitrary star-continuous Kleene algebra. We proceed by induction on the structure of t. There are three base cases, corresponding to the regular expressions $a \in \Sigma$, 1, and 0. For $a \in \Sigma$, we have $R_\Sigma(a) = \{a\}$ and

$$\sup_{x \in R_\Sigma(a)} sxu \quad = \quad sau.$$

The case of 1 is similar, since $R_\Sigma(1) = \{\varepsilon\}$. Finally, since $R_\Sigma(0) = \varnothing$ and since 0 is the least element in K and therefore the supremum of the empty set,

$$\sup_{x \in R_\Sigma(0)} sxu \quad = \quad \sup \varnothing \quad = \quad 0 \quad = \quad s0u.$$

There are three cases to the inductive step, one for each of the operators $+, \cdot, ^*$. We give a step-by-step argument for the case $+$, followed by a justification of each step.

$$\begin{aligned}
s(t_1 + t_2)u &= st_1u + st_2u && (1)\\
&= \sup_{x \in R_\Sigma(t_1)} sxu + \sup_{y \in R_\Sigma(t_2)} syu && (2)\\
&= \sup_{z \in R_\Sigma(t_1) \cup R_\Sigma(t_2)} szu && (3)\\
&= \sup_{z \in R_\Sigma(t_1 + t_2)} szu. && (4)
\end{aligned}$$

Equation (1) follows from the distributive laws of Kleene algebra; (2) follows from the induction hypothesis on t_1 and t_2; (3) follows from the general property of Kleene algebras that if A and B are two sets whose suprema $\sup A$ and $\sup B$ exist, then the supremum of $A \cup B$ exists and is equal to $\sup A + \sup B$ (this requires proof—see below); finally, equation (4) follows from the fact that the interpretation map R_Σ is a homomorphism.

The general property used in equation (3) states that if A and B are two subsets of a Kleene algebra whose suprema $\sup A$ and $\sup B$ exist, then the supremum $\sup A \cup B$ of $A \cup B$ exists and is equal to $\sup A + \sup B$.

To prove this, we must show two things:

(i) $\sup A + \sup B$ is an upper bound for $A \cup B$; that is, for any $x \in A \cup B$, $x \leq \sup A + \sup B$; and

(ii) $\sup A + \sup B$ is the least such upper bound; that is, for any other upper bound y of the set $A \cup B$, $\sup A + \sup B \leq y$.

To show (i),

$$\begin{aligned}
x \in A \cup B \quad &\to \quad x \in A \text{ or } x \in B\\
&\to \quad x \leq \sup A \text{ or } x \leq \sup B\\
&\to \quad x \leq \sup A + \sup B.
\end{aligned}$$

To show (ii), let y be any other upper bound for $A \cup B$. Then

$$\begin{aligned}
\forall x \in A \cup B\ x \leq y \quad &\to \quad \forall x \in A\ x \leq y \text{ and } \forall x \in B\ x \leq y\\
&\to \quad \sup A \leq y \text{ and } \sup B \leq y\\
&\to \quad \sup A + \sup B \leq y + y \quad = \quad y.
\end{aligned}$$

Now we give a similar chain of equalities for the case of the operator $\cdot$, but omit the justifications.

$$\begin{aligned} s(t_1t_2)u &= st_1(t_2u) \\ &= \sup_{x\,\in\, R_\Sigma(t_1)} sx(t_2u) \\ &= \sup_{x\,\in\, R_\Sigma(t_1)} (sx)t_2u \\ &= \sup_{x\,\in\, R_\Sigma(t_1)} \sup_{y\in R_\Sigma(t_2)} sxyu \\ &= \sup_{x\,\in\, R_\Sigma(t_1),\ y\,\in\, R_\Sigma(t_2)} sxyu \\ &= \sup_{z\,\in\, R_\Sigma(t_1t_2)} szu. \end{aligned}$$

Finally, for the case *, we have

$$\begin{aligned} st^*u &= \sup_{n\geq 0} st^n u \\ &= \sup_{n\geq 0} \sup_{x\in R_\Sigma(t^n)} sxu \\ &= \sup_{x\,\in\,\bigcup_{n\geq 0} R_\Sigma(t^n)} sxu \\ &= \sup_{x\,\in\, R_\Sigma(t^*)} sxu. \end{aligned}$$

$\square$

Theorem 0.2 *Let s, t be regular expressions over Σ. The equation $s = t$ holds in all star-continuous Kleene algebras iff $R_\Sigma(s) = R_\Sigma(t)$.*

Proof. The forward implication is immediate, since Reg_Σ is a star-continuous Kleene algebra. Conversely, by two applications of Lemma 0.1, if $R_\Sigma(s) = R_\Sigma(t)$, then under any interpretation in any star-continuous Kleene algebra,

$$s \;=\; \sup_{x\,\in\, R_\Sigma(s)} x \;=\; \sup_{x\,\in\, R_\Sigma(t)} x \;=\; t.$$

$\square$

Free Algebras

We have shown that the equational theory of the star-continuous Kleene algebras coincides with the equations true in Reg_Σ under the interpretation R_Σ. Another way of saying this is that Reg_Σ is the *free star-continuous Kleene algebra on generators* Σ. The term *free* intuitively means that Reg_Σ is free from any equations except those that it is forced to satisfy in order to be a star-continuous Kleene algebra.

Formally, a member A of a class of algebraic structures $\mathcal{C}$ of the same signature is said to be *free on generators* X *for the class* $\mathcal{C}$ if

- A is generated by X;
- any function h from X into another algebra $B \in \mathcal{C}$ extends to a homomorphism $\widehat{h} : A \to B$.

The extension is necessarily unique, since a homomorphism is completely determined by its action on a generating set.

Thus to say that Reg_Σ is the free star-continuous Kleene algebra on generators Σ says that Reg_Σ is generated by Σ (actually, by $\{\{a\} \mid a \in \Sigma\} = \{R_\Sigma(a) \mid a \in \Sigma\}$), and for any star-continuous Kleene algebra K and map $h : \Sigma \to K$, there is a homomorphism $\widehat{h} : \mathsf{Reg}_\Sigma \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Reg}_\Sigma \arrow[dr, "\widehat{h}"] & \\
& K \\
\Sigma \arrow[uu, "R_\Sigma"] \arrow[ur, "h"'] &
\end{tikzcd} \tag{5}$$

In other words, $\widehat{h} \circ R_\Sigma = h$.

Free algebras, if they exist, are unique up to isomorphism. If A and B are free on generators X for a class $\mathcal{C}$, let i and j be the embeddings of X into A and B, respectively. Since both are free, they extend to homomorphisms $\widehat{\imath} : B \to A$ and $\widehat{\jmath} : A \to B$, respectively.

$$\begin{tikzcd}
A \arrow[dr, shift left, "\widehat{\jmath}"] & \\
X \arrow[u, "i"] \arrow[r, "j"'] & B \arrow[ul, shift left, "\widehat{\imath}"]
\end{tikzcd}$$

Thus $\widehat{\imath} \circ j = i$ and $\widehat{\jmath} \circ i = j$. Then $\widehat{\jmath} \circ \widehat{\imath} \circ j = \widehat{\jmath} \circ i = j$, so for any $x \in X$, $(\widehat{\jmath} \circ \widehat{\imath})(j(x)) = j(x)$. Since B is generated by $\{j(x) \mid x \in X\}$, this says that $\widehat{\jmath} \circ \widehat{\imath} : B \to B$ is the identity—it agrees with the identity on X, and homomorphisms are uniquely determined by their action on a generating set. Symmetrically, $\widehat{\imath} \circ \widehat{\jmath} : A \to A$ is also the identity, so $\widehat{\imath}$ and $\widehat{\jmath}$ are inverses, therefore A and B are isomorphic.

Congruence Relations and the Quotient Construction

Any class of algebras defined by universal equations or equational implications, even infinitary ones, has free algebras. Recall that the axioms for star-continuous Kleene algebra are of this form:

$$\begin{aligned} xy^n z &\leq xy^* z, \quad n \geq 0 \\ \bigwedge_{n\geq 0}(xy^n z \leq w) &\rightarrow xy^* z \leq w. \end{aligned}$$

There is a general technique for constructing free algebras called a *quotient construction.* Here we give a brief general account of the quotient construction and how to apply it to obtain free algebras.

Fix a signature σ. Let A be any σ-algebra. A binary relation $\equiv$ on A is call a *congruence* if

(i) $\equiv$ is an equivalence relation (reflexive, symmetric, transitive);

(ii) $\equiv$ is respected by all the distinguished operations of the signature; that is, if f is n-ary, and if $x_i \equiv y_i$, $1 \leq i \leq n$, then $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$. For example, for the signature of Kleene algebra, this means

$$\begin{aligned} x_1 \equiv y_1 \wedge x_2 \equiv y_2 &\rightarrow x_1 + x_2 \equiv y_1 + y_2 \\ x_1 \equiv y_1 \wedge x_2 \equiv y_2 &\rightarrow x_1 x_2 \equiv y_1 y_2 \\ x \equiv y &\rightarrow x^* \equiv y^*. \end{aligned}$$

The *kernel* of a homomorphism $h : A \rightarrow B$ is

$$\ker h \stackrel{\text{def}}{=} \{(x, y) \mid h(x) = h(y)\}.$$

One can show that the kernel of any homomorphism with domain A is a congruence on A.

Conversely, given any congruence $\equiv$ on A, one can construct a σ-algebra B and an epimorphism $h : A \rightarrow B$ such that $\equiv$ is the kernel of h. This is called the *quotient construction.* For any $x \in A$, define

$$[x] \stackrel{\text{def}}{=} \{y \in A \mid x \equiv y\},$$

the *congruence class* of x. Let

$$A/\equiv \stackrel{\text{def}}{=} \{[x] \mid x \in A\}.$$

One can make this into a σ-algebra by defining

$$f^{A/\equiv}([x_1], \ldots, [x_n]) \stackrel{\text{def}}{=} [f^A(x_1, \ldots, x_n)].$$

The properties of congruence ensure that $f^{A/\equiv}$ is well defined and that the map $x \mapsto [x]$ is an epimorphism $A \rightarrow A/\equiv$.

Free Algebras

Now we apply the quotient construction to obtain free algebras. As previously observed, if σ is any signature, the set of well-formed terms $T_\sigma(X)$ over variables X can be regarded as a σ-algebra in which the operations of σ have their syntactic interpretation. For the signature of Kleene algebra, we have been calling the variables Σ, and the terms over Σ are the regular expressions RExp_Σ.

Let Δ be a set of equations or equational implications over $T_\sigma(X)$, and let $\mathbf{Mod}(\Delta)$ denote the class of σ-algebras that satisfy Δ. For example, if Δ consists of the axioms of star-continuous Kleene algebra, then $\mathbf{Mod}(\Delta)$ will be the class of all star-continuous Kleene algebras.

Let $\equiv$ be the smallest congruence on terms in $T_\sigma(X)$ containing all substitution instances of equations in Δ and closed under all substitution instances of equational implications in Δ. The relation $\equiv$ can be built inductively, starting with the substitution instances of equations in Δ and the reflexivity axiom $s \equiv s$ and adding pairs $s \equiv t$ as required by the substitution instances of equational implications in Δ and the symmetry, transitivity, and congruence rules.

One can now show that the quotient $T_\sigma(X)/\!\equiv$ is the free $\mathbf{Mod}(\Delta)$ algebra on generators X. For any algebra A satisfying Δ and any map $h : X \to A$, h extends uniquely to a homomorphism $T_\sigma(X) \to A$, which we also denote by h. The kernel of h is the set of equations satisfied by A under interpretation h. Since A satisfies Δ, the kernel of h contains all the equations of Δ and is closed under the equational implications of Δ. Since $\equiv$ is the smallest such congruence, $\equiv$ refines (is contained in) $\ker h$. Thus we can define $\widehat{h}([x]) \stackrel{\text{def}}{=} h(x)$ and the resulting map $\widehat{h}$ will be well defined. This is the desired homomorphism $T_\sigma(X)/\!\equiv\to A$.

For star-continuous Kleene algebra, the free algebra given by the quotient construction is $\mathsf{RExp}_\Sigma/!\equiv$, where $\equiv$ is the smallest congruence containing all substitution instances of the axioms of idempotent semirings, e.g. $s + (t + u) \equiv (s + t) + u$ for all $s, t, u \in \mathsf{RExp}_\Sigma$, etc., and $st^n u \leqq st^* u$ for all $s, t, u \in \mathsf{RExp}_\Sigma$ and $n \geq 0$ (where $s \leqq t$ is an abbreviation for $s + t \equiv t$), and contains $st^* u \leqq w$ whenever $st^n u \leqq w$ for all $n \geq 0$.

To show that Reg_Σ is free (therefore isomorphic to $\mathsf{RExp}_\Sigma/!\equiv$), let $h : \Sigma \to K$ be an arbitrary function into a star-continuous Kleene algebra K, and extend h to a homomorphism $h : \mathsf{RExp}_\Sigma \to K$. By Theorem 0.2, the set of equations that hold under the interpretation R_Σ, which is $\ker R_\Sigma$, is contained in the set of equations that hold under any interpretation in any star-continuous Kleene algebra; in particular under the interpretation h in K. Thus $\ker R_\Sigma$ refines $\ker h$. This says that we can define $\widehat{h}(R_\Sigma(s)) \stackrel{\text{def}}{=} h(s)$, and the resulting map $\widehat{h} : \mathsf{Reg}_\Sigma \to K$ will be well defined. This is the desired homomorphism making the diagram (5) commute.