Introduction to Kleene Algebra	**Lecture 16**
CS786 Spring 2002	**Thursday, March 28, 2002**

Parikh's Theorem in Commutative KA, continued

In this lecture we use the infrastructure developed in the previous lecture to prove

Theorem 0.1 *Every system of inequalities*

$$f_i(x_1, \ldots, x_n) \quad \leq \quad x_i, \quad 1 \leq i \leq n, \tag{1}$$

where the f_i are polynomials in $K[x_1, \ldots, x_n]$ over a commutative Kleene algebra K, has a unique least solution in K^n; moreover, the components of the solution are given by polynomials in the coefficients of the f_i.

We first prove the result for $n = 1$, then extend it to arbitrary n. Many arguments in this section are inspired by those of Pilling [2] (see also [1]) but generalized to apply to arbitrary commutative Kleene algebras.

Theorem 0.2 *Let K be a commutative Kleene algebra and let $f(x) \in K[x]$. The unique least solution of the inequality $f(x) \leq x$ is*

$$f'(f(0))^* \cdot f(0). \tag{2}$$

Moreover, this holds uniformly over all homomorphic images of K.

For example, the context-free language $A = \{a^n b^n \mid n \geq 0\}$ is generated by the grammar $S \to aSb \mid \varepsilon$, which translates to the one-dimensional system $axb + 1 \leq x$. Letting $f(x) = axb + 1$, we get $f'(x) = ab$ and $f(0) = 1$, thus (2) gives $(ab)^*$. This is a regular expression describing a regular set letter-equivalent to A.

Proof. First we argue that (2) is a solution to $f(x) \leq x$. It follows by a straightforward inductive argument that for any polynomial $h(x)$,

$$ac \leq bc \quad \to \quad h(a)c \leq h(b)c. \tag{3}$$

Applying this with $b = f(0)$, $c = f'(b)^*$, and $a = bc$,

$$\begin{aligned}
f(a) &= f(bc) \\
&= f(0) + f'(bc)bc && \text{by Taylor's theorem (Theorem \textbf{??} of Lecture \textbf{??})} \\
&= b + f'(bc)bc \\
&\leq b + f'(b)bc && \text{by (3)} \\
&= b + f'(b)f'(b)^*b \\
&= f'(b)^*b && \text{by Kleene algebra} \\
&= a.
\end{aligned}$$

Now we show that (2) is the least solution. Suppose y is any solution; thus $f(y) \leq y$. We wish to show that

$$f'(f(0))^*f(0) \;\leq\; y.$$

By Kleene algebra, it suffices to show

$$f(0) + f'(f(0)) \cdot y \;\leq\; y.$$

But by monotonicity, $f(0) \leq f(y) \leq y$, and

$$\begin{aligned}
& f(0) + f'(f(0)) \cdot y \\
&\leq f(0) + f'(y) \cdot y && \text{by monotonicity} \\
&= f(y) && \text{by Taylor's theorem (Theorem \textbf{??} of Lecture \textbf{??})} \\
&\leq y.
\end{aligned}$$

The expression (2) gives the least solution of $f(x) \leq x$ uniformly over all homomorphic images of K because the axioms of Kleene algebra used in the proof hold universally under any interpretation. □

The uniformity condition of Theorem 0.2 may seem obvious, but it is actually a rather subtle point. The issue is that equations are preserved under homomorphisms, but in general Horn formulas (equational implications) are not. The homomorphic image $h(e)$ of a solution e of an inequality $f(x) \leq x$ is a solution of the homomorphic image of the inequality, because the inequality is equivalent to an equation $f(x) + x = x$; but that $h(e)$ is the *least* solution does not follow from the fact that e is least, since this property requires a Horn formula.

Proof of Theorem 0.1. We iterate the one-dimensional solution as follows. Consider the two-dimensional system

$$\begin{aligned}
f(x, y) &\leq x \\
g(x, y) &\leq y.
\end{aligned} \tag{4}$$

Viewing $K[x,y]$ as $K[x][y]$, first compute the least solution to the one-dimensional system $g(x,y) \leq y$ in $K[x]$; call it $h(x)$. Then compute the least solution a of $f(x,h(x)) \leq x$ in K.

We claim that $(a,h(a))$ is the desired least solution to (4) in K^2. Surely $f(a,h(a)) \leq a$ by the one-dimensional argument. Moreover, by the uniformity observation, we also have $g(a,h(a)) \leq h(a)$, since it is the image of $g(x,h(x)) \leq h(x)$ under the evaluation homomorphism $x \mapsto a$.

To show $(a,h(a))$ is the least solution, suppose (b,c) is any other solution. Then $f(b,c) \leq b$ and $g(b,c) \leq c$. Using the uniformity observation with the evaluation morphism $x \mapsto b$, we have that $h(b)$ is the least solution of $g(b,y) \leq y$. Then $h(b) \leq c$. But by monotonicity, $f(b,h(b)) \leq f(b,c) \leq b$. Since a is the least solution to $f(x,h(x)) \leq x$, we have that $a \leq b$. Again by monotonicity, $h(a) \leq h(b) \leq c$. Thus $(a,h(a)) \leq (b,c)$.

By iterating this process inductively, we can obtain the existence of a solution to any $n \times n$ system. $\square$

A Closed Form Solution

The iterated construction of the previous section does not give a symmetric closed-form expression for any dimension greater than one. In this section we provide a symmetric closed-form solution.

Let K be a commutative Kleene algebra and consider an $n \times n$ system

$$\mathbf{f}(\mathbf{x}) \quad \leq \quad \mathbf{x} \tag{5}$$

where $\mathbf{x} = x_1, \ldots, x_n$ and $\mathbf{f} = \mathbf{f}(\mathbf{x}) = f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}) \in K[\mathbf{x}]$. Let $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ be the Jacobian of the system (5) as defined in Lecture **??**. Define

$$\begin{aligned} \mathbf{a}_0 &\stackrel{\text{def}}{=} \mathbf{f}(\mathbf{0}) \\ \mathbf{a}_{k+1} &\stackrel{\text{def}}{=} \tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)^* \mathbf{a}_k. \end{aligned}$$

Theorem 0.3 *For sufficiently large finite N, the n-vector $\mathbf{a}_N$ is the least solution to (5). Moreover, this solution is uniform over all homomorphic images of K.*

Below we will derive an explicit single-exponential bound on N as a function of n.

Proof. We will prove the first statement of the theorem; the uniformity property will follow by the same considerations as in the proof of Theorem 0.2. The proof proceeds by induction on n. The basis $n = 1$ was given in Theorem 0.2.

Now suppose $n \geq 2$. Partition n as $m + (n - m)$ where $1 \leq m < n$. For an n-vector $\mathbf{b} = b_1, \ldots, b_n$, write

$$
\begin{aligned}
\mathbf{b}^{\text{▀}} &\stackrel{\text{def}}{=} b_1, \ldots, b_m \\
\mathbf{b}^{\text{▄}} &\stackrel{\text{def}}{=} b_{m+1}, \ldots, b_n,
\end{aligned}
$$

and for an $n \times n$ matrix M, write

$$
M^{\text{▘}}, \quad M^{\text{▝}}, \quad M^{\text{▖}}, \quad M^{\text{▗}}
$$

for the upper left $m \times m$, upper right $m \times (n - m)$, lower left $(n - m) \times m$, and lower right $(n - m) \times (n - m)$ submatrices of M, respectively. To simplify notation, define

$$
\begin{aligned}
\mathbf{y} &\stackrel{\text{def}}{=} \mathbf{x}^{\text{▀}}, & \mathbf{g} &\stackrel{\text{def}}{=} \mathbf{f}^{\text{▀}}, \\
\mathbf{z} &\stackrel{\text{def}}{=} \mathbf{x}^{\text{▄}}, & \mathbf{h} &\stackrel{\text{def}}{=} \mathbf{f}^{\text{▄}}.
\end{aligned}
$$

In this notation, we can rewrite (5) as

$$
\begin{aligned}
\mathbf{g}(\mathbf{y}, \mathbf{z}) &\leq \mathbf{y} \\
\mathbf{h}(\mathbf{y}, \mathbf{z}) &\leq \mathbf{z}.
\end{aligned} \tag{6}
$$

Also,

$$
\begin{aligned}
\frac{\partial \mathbf{g}}{\partial \mathbf{y}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\text{▘}}, & \frac{\partial \mathbf{g}}{\partial \mathbf{z}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\text{▝}}, \\
\frac{\partial \mathbf{h}}{\partial \mathbf{y}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\text{▖}}, & \frac{\partial \mathbf{h}}{\partial \mathbf{z}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\text{▗}}.
\end{aligned}
$$

Now define

$$
\begin{aligned}
\mathbf{c}_0(\mathbf{y}) &\stackrel{\text{def}}{=} \mathbf{h}(\mathbf{y}, \mathbf{0}) \\
\mathbf{c}_{k+1}(\mathbf{y}) &\stackrel{\text{def}}{=} \frac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \mathbf{c}_k(\mathbf{y})
\end{aligned}
$$

By the induction hypthesis, there exists a P such that $\mathbf{c}_P(\mathbf{y})$ is the least solution to the system

$$
\mathbf{h}(\mathbf{y}, \mathbf{z}) \leq \mathbf{z}
$$

uniformly in $\mathbf{y}$. Define

$$
\begin{aligned}
\widehat{\mathbf{g}}(\mathbf{y}) &\stackrel{\text{def}}{=} \mathbf{g}(\mathbf{y}, \mathbf{c}_P(\mathbf{y})) \\
\mathbf{b}_0 &\stackrel{\text{def}}{=} \widehat{\mathbf{g}}(\mathbf{0}) \\
\mathbf{b}_{k+1} &\stackrel{\text{def}}{=} \frac{\partial \widehat{\mathbf{g}}}{\partial \mathbf{y}}(\mathbf{b}_k)^* \mathbf{b}_k.
\end{aligned}
$$

Again by the induction hypthesis, there exists an M such that $\mathbf{b}_M$ is the least solution to the system

$$\widehat{\mathbf{g}}(\mathbf{y}) \quad \leq \quad \mathbf{y}.$$

By the uniformity of the solutions, we have that

$$\begin{array}{rcl} \mathbf{g}(\mathbf{b}_M, \mathbf{c}_P(\mathbf{b}_M)) & \leq & \mathbf{b}_M \\ \mathbf{h}(\mathbf{b}_M, \mathbf{c}_P(\mathbf{b}_M)) & \leq & \mathbf{c}_P(\mathbf{b}_M), \end{array}$$

and $\mathbf{b}_M, \mathbf{c}_P(\mathbf{b}_M)$ is the least solution to (5). Moreover, this is the least solution uniformly over all homomorphic images.

Our task now is to show that for sufficiently large N,

$$\mathbf{b}_M = \mathbf{a}^{\blacksquare}_N, \qquad \mathbf{c}_P(\mathbf{b}_M) = \mathbf{a}^{\square}_N. \tag{7}$$

The inequalities $\geq$ follow from the fact that if $\mathbf{u}$ is any solution to (5), then $\mathbf{a}_k \leq \mathbf{u}$ for all k. This can be shown by induction on k. Certainly

$$\mathbf{a}_0 \quad = \quad \mathbf{f}(\mathbf{0}) \quad \leq \quad \mathbf{f}(\mathbf{u}) \quad \leq \quad \mathbf{u},$$

and by Theorem **??** of Lecture **??**,

$$\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{u})\mathbf{u} \quad \leq \quad \mathbf{u},$$

from which it follows that

$$\begin{array}{rcl} \mathbf{a}_{k+1} & = & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)^* \mathbf{a}_k \\ & \leq & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{u})^* \mathbf{u} \\ & \leq & \mathbf{u}. \end{array}$$

Now we establish a series of inequalities from which the forward inequalities $\leq$ of (7) will follow. First,

$$\mathbf{a}_k \quad \leq \quad \tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)^* \mathbf{a}_k \quad = \quad \mathbf{a}_{k+1},$$

thus $\mathbf{a}_i \leq \mathbf{a}_j$, $i \leq j$, and similarly for $\mathbf{b}_i$ and $\mathbf{c}_i$.

Now we show that

$$\mathbf{g}(\mathbf{a}_k) \quad \leq \quad \mathbf{a}^{\blacksquare}_{k+1} \qquad \mathbf{h}(\mathbf{a}_k) \quad \leq \quad \mathbf{a}^{\square}_{k+1}. \tag{8}$$

By Theorem refthm-Taylor1 of Lecture **??**,

$$
\begin{aligned}
\mathbf{g}(\mathbf{a}_k) &= \mathbf{g}(\mathbf{0}) + \tfrac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{a}_k)\mathbf{a}_k \\
&= \mathbf{f}(\mathbf{0})^{\square} + \tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)^{\square\square}\mathbf{a}_k \\
&= \mathbf{a}_0^{\square} + (\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)\mathbf{a}_k)^{\square} \\
&\leq (\mathbf{a}_k + \tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_k)^*\mathbf{a}_k)^{\square} \\
&= \mathbf{a}_{k+1}^{\square}.
\end{aligned}
$$

The second inequality of (8) follows from a similar argument.

Now we show by induction on j that

$$
\mathbf{c}_j(\mathbf{a}_k^{\square}) \;\leq\; \mathbf{a}_{j+k+1}^{\square}. \tag{9}
$$

For the basis,

$$
\begin{aligned}
\mathbf{c}_0(\mathbf{a}_k^{\square}) &= \mathbf{h}(\mathbf{a}_k^{\square}, \mathbf{0}) \\
&\leq \mathbf{h}(\mathbf{a}_k) \\
&\leq \mathbf{a}_{k+1}^{\square}.
\end{aligned}
$$

For the induction step,

$$
\begin{aligned}
\mathbf{c}_{j+1}(\mathbf{a}_k^{\square}) &= \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{a}_k^{\square}, \mathbf{c}_j(\mathbf{a}_k^{\square}))^*\mathbf{c}_j(\mathbf{a}_k^{\square}) \\
&\leq \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{a}_k^{\square}, \mathbf{a}_{j+k+1}^{\square})^*\mathbf{a}_{j+k+1}^{\square} \\
&\leq \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{a}_{j+k+1})^*\mathbf{a}_{j+k+1}^{\square} \\
&= (\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_{j+k+1})^{\square\square})^*\mathbf{a}_{j+k+1}^{\square} \\
&\leq (\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_{j+k+1})^*)^{\square\square}\mathbf{a}_{j+k+1}^{\square} \\
&\leq (\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}_{j+k+1})^*\mathbf{a}_{j+k+1})^{\square} \\
&= \mathbf{a}_{j+k+2}^{\square}.
\end{aligned}
$$

Now we show by induction on k that when $m = n - 1$, that is, for $|z| = 1$,

$$
\tfrac{\partial \mathbf{c}_{k+1}}{\partial \mathbf{y}}(\mathbf{y}) \;\leq\; \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})). \tag{10}
$$

The corresponding result for $m < n - 1$ would require some specialized notation even to state. The proof for $m = n - 1$ is considerably simpler, so we henceforth restrict ourselves to that case.

First we note that

$$
\tfrac{\partial \mathbf{c}_0}{\partial \mathbf{y}}(\mathbf{y}) \;=\; \tfrac{\partial}{\partial \mathbf{y}}(\mathbf{h}(\mathbf{y}, \mathbf{0})) \;=\; \tfrac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{y}, \mathbf{0}). \tag{11}
$$

Then

$$
\begin{aligned}
&\tfrac{\partial \mathbf{c}_{k+1}}{\partial \mathbf{y}}(\mathbf{y}) \\
&= \tfrac{\partial}{\partial \mathbf{y}}(\tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \mathbf{c}_k(\mathbf{y})) \\
&= \mathbf{c}_k(\mathbf{y}) \tfrac{\partial}{\partial \mathbf{y}}(\tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^*) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&= \mathbf{c}_k(\mathbf{y}) \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial}{\partial \mathbf{y}}(\tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&= \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \mathbf{c}_k(\mathbf{y}) \tfrac{\partial}{\partial \mathbf{y}}(\tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&\leq \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial}{\partial \mathbf{y}}(\tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})) \mathbf{c}_k(\mathbf{y})) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&\leq \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial}{\partial \mathbf{y}}(\mathbf{h}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&= \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* (\tfrac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})) \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y})) \\
&\qquad + \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \\
&= \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* (\tfrac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})) + \tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y})), \qquad (12)
\end{aligned}
$$

so it suffices to show

$$\tfrac{\partial \mathbf{c}_k}{\partial \mathbf{y}}(\mathbf{y}) \;\leq\; \tfrac{\partial \mathbf{h}}{\partial \mathbf{z}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y}))^* \tfrac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{y}, \mathbf{c}_k(\mathbf{y})).$$

For $k = 0$, this is immediate from (11). For $k > 0$, this follows from (12) and the induction hypothesis.

Now we show by induction on k that

$$\mathbf{b}_k \;\leq\; \mathbf{a}^{\blacksquare}_{(k+1)(P+2)}. \qquad (13)$$

For the basis,

$$
\begin{aligned}
\mathbf{b}_0 &= \widehat{\mathbf{g}}(\mathbf{0}) \\
&= \mathbf{g}(\mathbf{0}, \mathbf{c}_P(\mathbf{0})) \\
&\leq \mathbf{g}(\mathbf{0}, \mathbf{c}_P(\mathbf{a}^{\blacksquare}_0)) \\
&\leq \mathbf{g}(\mathbf{0}, \mathbf{a}^{\blacksquare}_{P+1}) \\
&\leq \mathbf{g}(\mathbf{a}_{P+1}) \\
&\leq \mathbf{a}^{\blacksquare}_{P+2}.
\end{aligned}
$$

For the induction step,

$$
\begin{aligned}
&\mathbf{b}_{k+1} \\
&\quad= \frac{\partial\hat{\mathbf{g}}}{\partial\mathbf{y}}(\mathbf{b}_k)^*\mathbf{b}_k \\
&\quad= \frac{\partial}{\partial\mathbf{y}}(\mathbf{g}(\mathbf{y},\mathbf{c}_P(\mathbf{y})))^*\mathbf{y}\ |_{\mathbf{y}=\mathbf{b}_k} \\
&\quad= (\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{y},\mathbf{c}_P(\mathbf{y})) \\
&\qquad\quad +\frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{y},\mathbf{c}_P(\mathbf{y}))\frac{\partial\mathbf{c}_P}{\partial\mathbf{y}}(\mathbf{y}))^*\mathbf{y}\ |_{\mathbf{y}=\mathbf{b}_k} \\
&\quad= (\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k)) \\
&\qquad\quad +\frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))\frac{\partial\mathbf{c}_P}{\partial\mathbf{y}}(\mathbf{b}_k))^*\mathbf{b}_k \\
&\quad\le (\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k)) \\
&\qquad\quad +\frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))\frac{\partial\mathbf{h}}{\partial\mathbf{z}}(\mathbf{b}_k,\mathbf{c}_{P-1}(\mathbf{b}_k))^* \\
&\qquad\qquad \cdot\frac{\partial\mathbf{h}}{\partial\mathbf{y}}(\mathbf{b}_k,\mathbf{c}_{P-1}(\mathbf{b}_k)))^*\mathbf{b}_k \\
&\quad\le (\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k)) \\
&\qquad\quad +\frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))\frac{\partial\mathbf{h}}{\partial\mathbf{z}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^* \\
&\qquad\qquad \cdot\frac{\partial\mathbf{h}}{\partial\mathbf{y}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k)))^*\mathbf{b}_k \\
&\quad= (\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^{\begin{smallmatrix}\blacksquare&\square\\\square&\square\end{smallmatrix}} \\
&\qquad\quad +\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^{\begin{smallmatrix}\square&\blacksquare\\\square&\square\end{smallmatrix}}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^{\begin{smallmatrix}\square&\square\\\square&\blacksquare\end{smallmatrix}*} \\
&\qquad\qquad \cdot\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^{\begin{smallmatrix}\square&\square\\\blacksquare&\square\end{smallmatrix}})^*\mathbf{b}_k \\
&\quad\le \frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{b}_k,\mathbf{c}_P(\mathbf{b}_k))^{*\begin{smallmatrix}\blacksquare&\square\\\square&\square\end{smallmatrix}}\mathbf{b}_k \\
&\quad\le \frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+1)(P+2)},\mathbf{c}_P(\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+1)(P+2)}))^{*\begin{smallmatrix}\blacksquare&\square\\\square&\square\end{smallmatrix}}\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+1)(P+2)} \\
&\quad\le \frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+1)(P+2)},\mathbf{a}^{\begin{smallmatrix}\square\\\blacksquare\end{smallmatrix}}_{(k+2)(P+2)-1})^{*\begin{smallmatrix}\blacksquare&\square\\\square&\square\end{smallmatrix}}\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+1)(P+2)} \\
&\quad\le \frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{a}_{(k+2)(P+2)-1})^{*\begin{smallmatrix}\blacksquare&\square\\\square&\square\end{smallmatrix}}\mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+2)(P+2)-1} \\
&\quad\le (\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{a}_{(k+2)(P+2)-1})^*\mathbf{a}_{(k+2)(P+2)-1})^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}} \\
&\quad= \mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(k+2)(P+2)}.
\end{aligned}
$$

It follows from (9) and (13) that

$$
\begin{aligned}
\mathbf{b}_M &\le \mathbf{a}^{\begin{smallmatrix}\blacksquare\\\square\end{smallmatrix}}_{(M+1)(P+2)}, \\
\mathbf{c}_P(\mathbf{b}_M) &\le \mathbf{a}^{\begin{smallmatrix}\square\\\blacksquare\end{smallmatrix}}_{(M+2)(P+2)-1}.
\end{aligned}
$$

Taking $m=n-1$, Theorem 0.2 says that $P=1$ suffices. Thus the N in the statement of the theorem is bounded by $(M+2)(P+2)-1=3M+5$. This gives the following recurrence for N as a function of n:

$$\begin{aligned} N(1) &= 1 \\ N(n+1) &= 3N(n)+5 \end{aligned}$$

with solution $N(n) = (7 \cdot 3^n - 5)/2$. $\square$

References

[1] John Horton Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, London, 1971.

[2] D. L. Pilling. Commutative regular equations and Parikh's theorem. *J. London Math. Soc.*, 6(4):663–666, June 1973.