There has been some disagreement regarding the proper axiomatization of Kleene algebra. Many inequivalent axiomatizations have been proposed [5, 7, 8, 16, 17], all serving roughly the same purpose. It is important to understand the relationships between these classes in order to extract the axiomatic essence of Kleene algebra. In the next two lectures we present some of these alternative axiomatizations and discuss the relationships among them. But be forewarned: some of these approaches are inelegant, ambiguous, or even incorrect.

Recall that our official definition is that a Kleene algebra is an idempotent semiring satisfying

$$1 + xx^* \le x^* \tag{1}$$
$$1 + x^*x \le x^* \tag{2}$$
$$b + ax \le x \;\Rightarrow\; a^*b \le x \tag{3}$$
$$b + xa \le x \;\Rightarrow\; ba^* \le x. \tag{4}$$

Star-Continuity

A Kleene algebra is called *star-continuous* (or sometimes *star-complete*) if it satisfies the axiom

$$xy^*z = \sup_{n\ge 0} xy^nz, \tag{5}$$

where $y^0 = 1$, $y^{n+1} = yy^n$, and sup refers to the *supremum* or *least upper bound* with respect to the natural order $x \le y \overset{\text{def}}{=} x+y = y$. This property says that the (possibly infinite) set $\{xy^nz \mid n \ge 0\}$ has a least upper bound with respect to $\le$, and that least upper bound is xy^*z. The property (5) is called *star-continuity*. Star-continuous Kleene algebras have been used to model programs in Dynamic Logic [7].

Every star-continuous idempotent semiring is a Kleene algebra, since one can easily show that in any idempotent semiring, the star-continuity condition (5) implies the axioms (1)–(4) of Kleene algebra. However, as we shall see later, there exist Kleene algebras that are not star-continuous, although most naturally occurring ones are.

The property (5) is actually an infinitary condition. It is equivalent to infinitely many inequalities

$$xy^nz \le xy^*z,\ n \ge 0, \tag{6}$$

which together say that xy^*z is an upper bound for all xy^nz, $n \ge 0$, along with the infinitary Horn formula

$$(\bigwedge_{n\ge 0}(xy^nz \le w)) \;\Rightarrow\; xy^*z \le w, \tag{7}$$

which says that it is the least such upper bound.

Another way to view (5) is as a combination of the statement that y^* is the supremum of y^n, $n \ge 0$, along with two infinitary distributivity properties, one on the left and one on the right.

To show that every star-continuous idempotent semiring is a Kleene algebra, we first show that (1) holds.

$$1 + xx^* = 1 + \sup_n xx^n = x^0 + \sup_n x^{n+1} = \sup_n x^n = x^*.$$

The general property we have used in the third step is that if A and B are any subsets of an upper semilattice such that $\sup A$ and $\sup B$ exist, then $\sup A \cup B$ exists and is equal to $\sup A + \sup B$. The proof of (2) is symmetric.

To show (3), assume that $b + ax \le x$. We would like to show that $a^*b \le x$. By star-continuity, it suffices to show that for all $n \ge 0$, $a^n b \le x$. This is easily shown by induction on n. For the basis $n = 0$, we have $a^0 b = b \le x$ from our assumption. Now assuming $a^n b \le x$, we have $a^{n+1}b = aa^n b \le ax$ by monotonicity, and $ax \le x$ by our assumption. Again, the proof of (4) is symmetric.

Closed Semirings

In the design and analysis of algorithms, a related family of structures called *closed semirings* form an important algebraic abstraction. They give a unified framework for deriving efficient algorithms for transitive closure and all-pairs shortest paths in graphs and constructing regular expressions from finite automata [1, 11, 19]. Very fast algorithms for all these problems can be derived as special cases of a single general algorithm over an arbitrary closed semiring.

Closed semirings are defined in terms of a countable summation operator $\sum$ as well as $\cdot$, 0, and 1; the operator * is defined in terms of $\sum$. Under the operations of (finite) $+$, $\cdot$, *, 0, and 1, any closed semiring is a star-continuous Kleene algebra. In fact, in the treatment of [1, 11], the sole purpose of $\sum$ seems to be to define *. A more descriptive name for closed semirings might be *ω-complete idempotent semirings*.

We will define a *closed semiring* to be an idempotent semiring in which every countable set A has a supremum $\sum A$ with respect to the natural order $\le$, and such that for any countable set A,

$$x \cdot (\sum A) \cdot z = \sum_{y \in A} xyz. \tag{8}$$

The presence of x and z in (8) ensure a kind of infinite distributivity property on the left and right.

In any closed semiring, one can define * by

$$x^* \stackrel{\text{def}}{=} \sum_{n \ge 0} x^n,$$

where $x^0 = 1$ and $x^{n+1} = xx^n$. By infinite distributivity,

$$xy^*z = \sum_n xy^n z,$$

thus any closed semiring is a star-continuous Kleene algebra.

The regular sets $\mathsf{Reg}\,\Sigma$ do not form a closed semiring: if A is nonregular, the countable set $\{\{x\} \mid x \in A\}$ has no supremum. However, the power set of Σ^* does form a closed semiring.

Similarly, the family of all binary relations on a set forms a closed semiring under the relational operations described in Lecture 4 and set union for $\sum$.

Our definition of closed semiring as given above is somewhat stronger than those found in the literature on design and analysis of algorithms [1, 11]. In those works, the operator $\sum$ is not viewed as a supremum, but as an infinitary summation operator. According to [1], a closed semiring is an idempotent semiring equipped with a summation operator $\sum$ defined on countable *sequences* (not sets) that satisfies infinitary associativity and distributivity. Infinitary idempotence and commutativity are not assumed. Also, the relation between the between finitary $+$ and infinitary $\sum$ is not explicitly mentioned in [1], but can be inferred from the use of the notation $x_0 + x_1 + x_2 + \cdots$ for the infinitary sum and infinitary associativity. The element x^* is defined to be $1 + x + x^2 + \cdots$.

Infinitary associativity is defined as follows. If $(x_n \mid n \ge 0)$ is any countable sequence of elements, then for any way of partitioning the index set $\mathbb{N}$ into intervals, the sum $\sum_i x_i$ is the same as the sum of the sums of the intervals. If an interval is finite, then its sum is computed with $+$. If an interval is infinite, then its sum is computed with $\sum$. Note that any such partition must consist either of

- infinitely many finite intervals, or
- finitely many intervals, all of which are finite except the last, which is infinite.

Infinitary distributivity says that

$$x \cdot (\sum_i y_i) \cdot z = \sum xy_iz.$$

This is not the same as (8), since it says nothing about suprema.

The axiomatization in [11] postulates infinitary commutativity as well. Infinitary commutativity says that for any partition of the index set (not necessarily into intervals), the sum of the sums of the partition elements is the same as the sum of the original sequence.

Infinitary idempotence says that if all $x_i = x$, then $\sum_i x_i = x$. This does not follow from the axiomatizations of [1, 11], nor does the equation $x^{**} = x^*$. It can be shown that $0^* = 1$, but not that $1^* = 1$.

To see this, consider an idempotent semiring with elements $\mathbb{N} \cup \{\infty\}$. Define finitary addition $+$ to be max in the natural order on $\mathbb{N}$, with ∞ being the largest element. Multiplication is ordinary multiplication in $\mathbb{N}$, extended to ∞ as follows:

$$\infty \cdot x = x \cdot \infty \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

The constants 0 and 1 in the semiring are the natural numbers 0 and 1, respectively.

To define $\sum$ in this algebra, define the *support* of an infinite sequence $x = (x_n \mid n \geq 0)$ to be the set

$$\mathsf{supp}\, x \stackrel{\text{def}}{=} \{n \mid x_n \neq 0\}.$$

We define

$$\sum_n x_n \stackrel{\text{def}}{=} \begin{cases} \sum_{n \in \mathsf{supp}\, x} x_n, & \text{if } \mathsf{supp}\, x \text{ is finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

One can show that infinitary associativity, commutativity, and distributivity are satisfied, and $0^* = 1$. However, $0^{**} = 1^* = \infty$, so $\sum$ is not idempotent (since $1^* = 1 + 1 + 1 + \cdots$) and $0^{**} \neq 0^*$.

It is conjectured that the axiomatization of [1] does not imply infinitary commutativity. In particular, it is conjectured that

$$x_0 + x_1 + x_2 + \cdots = (x_0 + x_2 + x_4 + \cdots) + (x_1 + x_3 + x_5 + \cdots)$$

is not provable.

One can show that our official definition of closed semirings in terms of suprema of countable sets and infinitary distributivity is equivalent to a countable summation operator $\sum$ satisfying infinitary associativity, commutativity, idempotence, and distributivity. Surely supremum is associative, commutative, and idempotent, and the axiom (8) gives distributivity as well.

Conversely, if $\sum$ is infinitely associative, commutative, and idempotent, then its value on a given sequence is independent of the order and multiplicity of elements occurring in the sequence. Thus we might as well define $\sum$ on finite or countable subsets instead of sequences. In this view, $\sum$ gives the supremum with respect to the natural order $\leq$. To see this, let A be a nonempty finite or countable set of elements. If $x \in A$, then

$$x + \sum A = \sum (A \cup \{x\}) = \sum A,$$

thus $x \leq \sum A$; and if $x \leq y$ for all $x \in A$, then $x + y = y$ for all $x \in A$, thus

$$(\sum A) + y = (\sum_{x \in A} x) + (\sum_{x \in A} y) = \sum_{x \in A} (x + y) = \sum_{x \in A} y = y,$$

therefore

$$\sum A \leq y.$$

Thus $\sum$ gives the supremum of countable sets.

Conway's Hierarchy

Closed semirings and star-continuous Kleene algebras are strongly related to several classes of algebras defined by Conway in his 1971 monograph [5]. Conway's S-algebras are similar to closed semirings, except that arbitrary sums, not just countable ones, are permitted. A better name for S-algebras might be *complete idempotent semirings*. These structures also make an appearance in the theory of point-free topology under the name *unital quantales*. In Conway's treatment, the operation * is defined as in closed semirings in terms of $\sum$, and again this seems to be the main purpose of $\sum$.

Conway's N-algebras are algebras of signature $(+,\ \cdot,\ ^*,\ 0,\ 1)$ that are subsets of S-algebras containing 0 and 1 and closed under (finite) $+$, $\cdot$, and *. We will show later that the classes of N-algebras and star-continuous Kleene algebras coincide.

An R-algebra is any algebra of signature $(+,\ \cdot,\ ^*,\ 0,\ 1)$ satisfying the equational theory of the N-algebras.

According to the definition in [5], an S-algebra

$$(S,\ \sum,\ \cdot,\ 0,\ 1)$$

is similar to a closed semiring, except that $\sum$ is defined not on sequences but on multisets of elements of S. A *multiset* is a set whose elements have multiplicity; equivalently, it is an equivalence class of sequences, where two sequences are considered equivalent if one is a permutation of the other. In other words, a multiset is like a sequence, except that we ignore the order of the elements. However, there is no cardinality restriction on the multiset. One consequence of this approach is that $\sum$ is too big to be represented in Zermelo-Fraenkel set theory! Since $\sum$ is a function that must be defined on multisets of arbitrary cardinality, it cannot be a set itself. However, as with closed semirings, the value that $\sum$ takes on a given multiset is independent of the multiplicity of the elements, so $\sum$ might as well be defined on subsets of S instead of multisets. So defined, $\sum A$ gives the supremum of A with respect to the order $\leq$. We will also assume the axiom $\sum\{a\} = a$, which is omitted in [5].

Thus, the only essential difference between S-algebras and closed semirings is that closed semirings are only required to contain suprema of countable sets, whereas S-algebras must contain suprema of all sets. Thus every S-algebra is automatically a closed semiring and every continuous semiring morphism (semiring morphism preserving all suprema) is automatically ω-continuous (preserves all countable suprema), and these notions coincide on countable algebras.

In a subsequent lecture we will show some very strong relationships among these classes of algebras. We will eventually show that the R-algebras, Kleene algebras, star-continuous Kleene algebras (a.k.a. N-algebras), closed semirings, and S-algebras each contain the next in the list, and all inclusions are strict. Moreover, each star-continuous Kleene algebra extends in a canonical way to a closed semiring, and each closed semiring to an S-algebra, by a construction known as *ideal completion*.

Other Approaches

There are many other approaches besides these, which we will not consider in this course.

Many authors consider Kleene algebra as synonymous with relation algebra and are not opposed to adding other relational operators such as residuation and complementation. Relation algebras were first studied by Tarski and his students and colleagues [6, 14, 15, 20]; see also [9, 12, 13, 18]. Bloom and Ésik [2–4] study a related structure called *iteration theories*.

In [16, 17], Pratt gives two definitions of Kleene algebras in the context of dynamic algebra. In [16], Kleene algebras are defined to be the Kleenean component of separable dynamic algebras; in [17], this class is enlarged to contain all subalgebras of such algebras.

Generalizations of Kleene's and Parikh's Theorems have been given by Kuich [10] in *ℓ-complete semirings*, which are similar to S-algebras in all respects except that idempotence of $\sum$ is replaced by a weaker condition called ℓ-completeness.

Characterizing the Equational Theory

Most of the early work on Kleene algebra was directed toward characterizing the equational theory of the regular sets. These are equations such as $(x + y)^* = x^*(yx^*)^*$ and $x(yx)^* = (xy)^*x$ that hold under the standard interpretation of regular expressions as regular sets of strings.

It turns out that this theory is quite robust and can be characterized in many different ways. We have defined several different but related classes of algebras: Kleene algebras, star-continuous Kleene algebras, closed semirings, and Conway's S-algebras, N-algebras, and R-algebras, all defined axiomatically, as well as relational and language-theoretic algebras defined model-theoretically. All these classes of models have the same equational theory over the signature $+$, $\cdot$, *, 0, 1 of Kleene algebra, and it is the same as the equational theory of the regular sets.

Let us say more carefully what we are talking about. Let F denote the signature $+$, $\cdot$, *, 0, 1 of Kleene algebra. In general, a *signature* consists of a set of function symbols and their arities (number of inputs); for KA, the symbols are $+$, $\cdot$, *, 0, 1 with arities 2, 2, 1, 0, 0, respectively. An *F-algebra* is any structure

$$C = (|C|, +^C, \cdot^C, {}^{*C}, 0^C, 1^C)$$

of signature F. Here $|C|$ is a set, called the *carrier* of C, with distinguished binary operations $+^C : |C|^2 \to |C|$ and $\cdot^C : |C|^2 \to |C|$, a distinguished unary operation ${}^{*C} : |C| \to |C|$, and distinguished constants $0^C \in |C|$ and $1^C \in |C|$. The structure C need not satisfy the axioms of Kleene algebra; indeed, it need not satisfy any equations at all.

The set of regular expressions $\mathsf{Exp}\,\Sigma$ over an alphabet Σ forms an F-algebra. The elements of $\mathsf{Exp}\,\Sigma$ are just the well-formed terms over variables Σ and operators $+$, $\cdot$, *, 0, 1. The distinguished operations are the syntactic ones; for example, $+^{\mathsf{Exp}\,\Sigma}$ is the binary operation that takes regular expressions s and t and produces the regular expression $s + t$. This F-algebra satisfies no equations except identities $s = s$; for example, $(s + t) + u \neq s + (t + u)$ in this algebra, since they are two different terms.

For any two F-algebras C and D, a *homomorphism* from C to D is a map $h : C \to D$ that commutes with all the distinguished operations and constants of F. For the signature of KA, this means that for all $x, y \in C$,

$$\begin{gathered} h(x +^C y) = h(x) +^D h(y) \qquad h(x \cdot^C y) = h(x) \cdot^D h(y) \\ h(x^{*C}) = h(x)^{*D} \qquad h(0^C) = 0^D \qquad h(1^C) = 1^D. \end{gathered} \tag{9}$$

Here the operators and constants on the left-hand sides are interpreted in C and those on the right-hand sides in D. A homomorphism h is

- an *epimorphism* if it is surjective (onto); that is, if for all $y \in D$, there exists an $x \in C$ such that $h(x) = y$;

- a *monomorphism* if it is injective (one-to-one); that is, if for all $x, y \in C$, if $h(x) = h(y)$ then $x = y$;
- an *isomorphism* if it is both an epimorphism and a monomorphism.

An F-algebra D is an *F-subalgebra* of C if $|D| \subseteq |C|$ and the F-operations of D are those of C restricted to domain $|D|$; that is, the inclusion map $|D| \to |C|$ is a monomorphism $D \to C$. An F-algebra D is a *homomorphic image* of C if there is an epimorphism $h : C \to D$.

An *interpretation* in an F-algebra D is just a homomorphism $f : \mathsf{Exp}\,\Sigma \to D$. For example, let $\mathsf{Reg}\,\Sigma$ denote the Kleene algebra of regular sets over alphabet Σ. The *canonical interpretation* over $\mathsf{Reg}\,\Sigma$ is the unique homomorphism $R_\Sigma : \mathsf{Exp}\,\Sigma \to \mathsf{Reg}\,\Sigma$ such that $R_\Sigma(a) = \{a\}$, $a \in \Sigma$. We will show that this interpretation alone characterizes the equational theory of Kleene algebras, as well as all the other classes of algebras mentioned above (Theorem 1).

In general, for any F-algebra D and any set function $f : \Sigma \to |D|$ defined on Σ, f extends uniquely to an interpretation $f : \mathsf{Exp}\,\Sigma \to D$. The values of f on all terms in $\mathsf{Exp}\,\Sigma$ are defined by induction, as there is exactly one way to extend f to domain $\mathsf{Exp}\,\Sigma$ to satisfy (9) when $C = \mathsf{Exp}\,\Sigma$. Because of this property, the structure $\mathsf{Exp}\,\Sigma$ is called the *free F-algebra on generators* Σ. We say *the* free F-algebra because it is unique up to isomorphism. Intuitively, once we know how to interpret the letters in Σ, that uniquely determines the interpretation of any regular expression over Σ.

Let s, t be regular expressions and let $f : \mathsf{Exp}\,\Sigma \to C$ be an interpretation. We write $C, f \vDash s = t$ say that the equation $s = t$ *holds under* f or that f *satisfies* $s = t$ if $f(s) = f(t)$. We write $C \vDash s = t$ say that $s = t$ *holds in* C or that C *satisfies* $s = t$ if $C, f \vDash s = t$ for all interpretations $f : \mathsf{Exp}\,\Sigma \to C$. If $\mathcal{A}$ is a class of algebras or a class of interpretations, we write $\mathcal{A} \vDash s = t$ and say that $s = t$ *holds in* $\mathcal{A}$ if it holds in all members of $\mathcal{A}$. The *equational theory* of $\mathcal{A}$, denoted $\mathsf{EqTh}\,\mathcal{A}$, is the set of equations that hold in $\mathcal{A}$.

Theorem 1. *The following classes of algebras all have the same equational theory: Kleene algebras, star-continuous Kleene algebras, closed semirings,* S*-algebras,* N*-algebras,* R*-algebras, language models, and relational models. Moreover, an equation $s = t$ over alphabet Σ is a member of this theory iff it holds under the canonical interpretation* $R_\Sigma : \mathsf{Exp}\,\Sigma \to \mathsf{Reg}\,\Sigma$.

One can see from this theorem that the equational theory of Kleene algebras is quite robust indeed. If the equational theory were all that we were interested in, there would not be much more to say.

Some Constructions

We will not be able to complete the proof of Theorem 1 today. Some parts of the theorem follow immediately from inclusion relationships among the classes of interpretations, but others are more difficult.

First we note that if $\mathcal{A}$ and $\mathcal{B}$ are two classes of algebras or classes of interpretations and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathsf{EqTh}\,\mathcal{B} \subseteq \mathsf{EqTh}\,\mathcal{A}$, since any equation that holds in all members of $\mathcal{B}$ must perforce hold in all members of $\mathcal{A}$.

Also, the following lemma asserts that equations are preserved in subalgebras and homomorphic images.

Lemma 2. *Let C and D be F-algebras.*

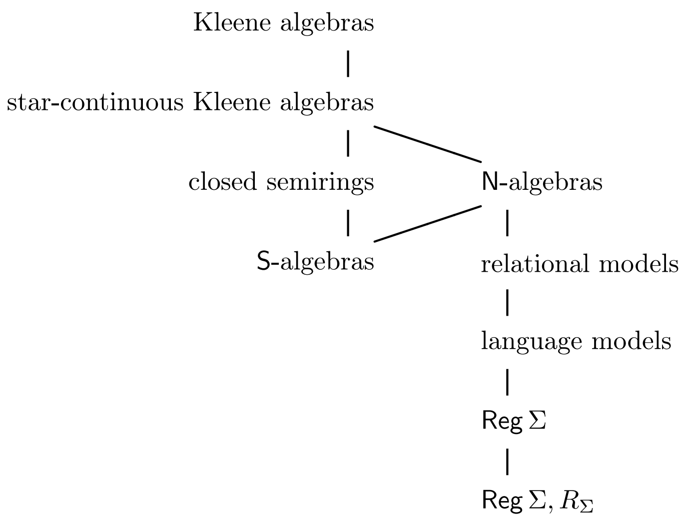
(i) *If $h : D \to C$ is a monomorphism and $C \vDash s = t$, then $D \vDash s = t$.*

(ii) *If $h : C \to D$ is an epimorphism and $C \vDash s = t$, then $D \vDash s = t$.*

Proof. (i) Let $f : \mathsf{Exp}\,\Sigma \to D$ be an arbitrary interpretation. Then $h \circ f : \mathsf{Exp}\,\Sigma \to C$ is an interpretation, as the composition of two homomorphisms is a homomorphism. Since $C \vDash s = t$, we have $h(f(s)) = h(f(t))$. Since h is injective, $f(s) = f(t)$. As f was arbitrary, $D \vDash s = t$.

(ii) Let $f : \mathsf{Exp}\,\Sigma \to D$ be an arbitrary interpretation. For each $a \in \Sigma$, $f(a) \in |D|$. Since h is surjective, there exists an element of $|C|$, call it $g(a)$, such that $h(g(a)) = f(a)$. This defines a map $g : \Sigma \to |C|$ such that $h(g(a)) = f(a)$ for all $a \in \Sigma$. Then g extends uniquely to an interpretation $g : \mathsf{Exp}\,\Sigma \to C$, and since $C \vDash s = t$, we have $g(s) = g(t)$. Moreover, $h \circ g : \mathsf{Exp}\,\Sigma \to D$ is an interpretation, since the composition of homomorphisms is a homomorphism. But $h \circ g$ and f agree on Σ, and since the extensions are unique, they agree on all of $\mathsf{Exp}\,\Sigma$, thus $f(s) = h(g(s)) = h(g(t)) = f(t)$. As f was arbitrary, $D \vDash s = t$. □

We have already established the following inclusions among the classes mentioned in Theorem 1:



If class $\mathcal{A}$ occurs above $\mathcal{B}$ in this diagram and there is a path from $\mathcal{A}$ down to $\mathcal{B}$, then $\mathsf{EqTh}\,\mathcal{A} \subseteq \mathsf{EqTh}\,\mathcal{B}$. Note that for the two lowest entries in this diagram, the upper one $\mathsf{Reg}\,\Sigma$ refers to the equations that hold under any interpretation in $\mathsf{Reg}\,\Sigma$, whereas the lower one $\mathsf{Reg}\,\Sigma, R_\Sigma$ refers to the equations that hold under the canonical interpretation only.

First we observe that the equational theories of the S-algebras and the N-algebras coincide. Recall that the N-algebras are the subsets of S-algebras closed under the Kleene algebra operations considered as F-algebras. We have $\mathsf{EqTh}\,\mathsf{N} \subseteq \mathsf{EqTh}\,\mathsf{S}$, since every S-algebra is a subalgebra of itself, therefore is an N-algebra. Conversely, by Lemma 2(i), any equation holding in an S-algebra A holds in any subalgebra of A; therefore $\mathsf{EqTh}\,\mathsf{S} \subseteq \mathsf{EqTh}\,\mathsf{N}$.

This observation says that the equational theories of the following classes of interpretations are linearly ordered by inclusion as follows: Kleene algebras, star-continuous Kleene algebras, closed semirings, S-algebras, N-algebras, relational models, language models, $\mathsf{Reg}\,\Sigma$, and $\mathsf{Reg}\,\Sigma, R_\Sigma$.

We might also add R-algebras to this list. Recall that R-algebras are those algebras that satisfy all the same equations as N-algebras, thus $\mathsf{EqTh}\,\mathsf{R} = \mathsf{EqTh}\,\mathsf{N}$. It will turn out that all the algebras in the diagram above are R-algebras, since they all share the same equational theory, so the class of R-algebras sits at the very top of the diagram above and at the head of the list in the previous paragraph.

However, the concept of R-algebra is not very interesting or useful. Conway [5, p. 102] gives a four-element R-algebra R_4 that is not a star-continuous Kleene algebra. The elements of R_4 are $\{0, 1, 2, 3\}$, and the operations are given by the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

$*$	
0	1
1	1
2	3
3	3

To show that R_4 is an R-algebra, by Lemma 2(ii) it suffices to construct an epimorphism $h : \mathsf{Reg}\,\Sigma \to R_4$, since any equation that holds in an F-algebra also holds in all its homomorphic images. Take $h(\varnothing) \stackrel{\text{def}}{=} 0$, $h(\{\varepsilon\}) \stackrel{\text{def}}{=} 1$, and for any other set A,

$$h(A) \stackrel{\text{def}}{=} \begin{cases} 2, & \text{if } A \text{ is finite,} \\ 3, & \text{if } A \text{ is infinite.} \end{cases}$$

One can verify easily that this is an epimorphism, therefore R_4 is an R-algebra. It is not a star-continuous Kleene algebra, since $2^n = 2$ for all n, but $2^* = 3$. It is also easily shown that all finite Kleene algebras are star-continuous, therefore R_4 is not a Kleene algebra either.

The family $\mathsf{Reg}\,\Sigma$ of regular events over an alphabet Σ gives an example of a star-continuous Kleene algebra that is not a closed semiring. If A is nonregular, the countable set $\{\{x\} \mid x \in A\}$ has no supremum. However, the power set of Σ^* does form a closed semiring.

To construct a closed semiring that is not an S-algebra, we might take the countable and co-countable subsets of ω_1 (the first uncountable ordinal) with operations of set union for $\sum$, set intersection for $\cdot$, $\varnothing$ for 0, ω_1 for 1, and $A^* = \omega_1$.

To complete the picture, we should construct a Kleene algebra that is not star-continuous. Let ω^2 denote the set of ordered pairs of natural numbers and let $\bot$ and $\top$ be new elements. Order these elements such that $\bot$ is the minimum element, $\top$ is the maximum element, and ω^2 is ordered lexicographically in between. Define $+$ to give the supremum in this order. Define $\cdot$ as follows:

$$x \cdot \bot = \bot \cdot x = \bot \qquad x \cdot \top = \top \cdot x = \top,\ x \neq \bot \qquad (a,b) \cdot (c,d) = (a+c, b+d).$$

Then $\bot$ is the additive identity and $(0,0)$ is the multiplicative identity. Finally, define

$$a^* = \begin{cases} (0,0), & \text{if } a = \bot \text{ or } a = (0,0), \\ \top, & \text{otherwise.} \end{cases}$$

It is easily checked that this is a Kleene algebra. We verify the axiom

$$ax \le x \Rightarrow a^*x \le x$$

explicitly. Assuming $ax \le x$, we wish to show $a^*x \le x$. If $a = \bot$ or $a = (0,0)$, then $a^* = (0,0)$ and we are done, since $(0,0)$ is the multiplicative identity. If $x = \bot$ or $x = \top$, we are done. Otherwise, $a > (0,0)$ and $x = (u,v)$, in which case $ax > x$, contradicting the assumption.

This Kleene algebra is not star-continuous, since $(0,1)^* = \top$, but

$$\sum_n (0,1)^n = \sum_n (0,n) = (1,0).$$

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