

CS 6840 Algorithmic Game Theory

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Lecture 39: Bandwidth Sharing and PoA

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1 Announcements

- Course evaluations open today. There will be a ChatGPT question included; please take a moment to fill it out. Responses are anonymous and will not be viewed until after grades are posted.
- Looking for two volunteers to present their final projects (10 minutes each). If interested, please email Prof. Tardos.

2 Introduction

Today we are discussing Bandwidth Sharing and analyzing the Price of Anarchy (PoA) for this setting. This *equilibrium analysis* is different from what we have seen before, and it only applies at equilibria, and does not extend to learning outcomes.

Model Setup

We consider a setting with n users sharing a total bandwidth of B . Each user i submits a bid w_i ; their share of the bandwidth is allocated proportionally:

$$x_i = B \cdot \frac{w_i}{\sum_{j=1}^n w_j}$$

Each user i derives utility $u_i(x)$ from receiving bandwidth x and pays w ; thus, user i 's total value is $u_i(x) - w$.

Assumptions

- The function $u_i(x)$ is assumed to be concave, continuous, differentiable, and strictly monotone increasing. Differentiability and strict monotonicity are assumed only for simplicity in proofs.

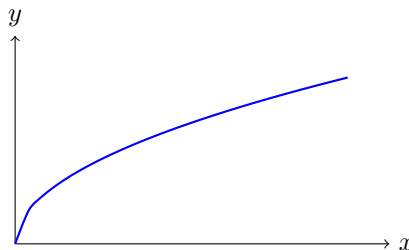


Figure 1: Concave utility function $u_i(x)$.

3 Equilibrium Analysis

Comparison: Fixed-Price Allocation

Consider a simpler alternative where the price p is set externally by a price setter (e.g., by the mechanism or an external market), and each user acts as a price taker. They accept p as given and make their decisions accordingly. Given this fixed price p , the user can purchase as much bandwidth as desired at that unit price, so if they pay w_i , they receive bandwidth $x_i = w_i/p$. The user's objective is then:

$$\max_x (u_i(x) - xp)$$

Because the user is a price taker, they choose x to maximize their utility for the given price p . The optimal choice (for concave u_i) satisfies the first-order condition:

$$u'_i(x) = p$$

Equilibrium Analysis for Proportional Sharing

In the proportional sharing mechanism, each player's bid not only determines their own bandwidth allocation but also influences the effective price faced by everyone.

Without loss of generality, set $B = 1$ (bandwidth can be rescaled). Each user's allocation is then:

$$x_i = \frac{w_i}{\sum_{j=1}^n w_j}$$

Each user chooses w_i to solve:

$$\max_{w_i} \left[u_i \left(\frac{w_i}{\sum_{j=1}^n w_j} \right) - w_i \right]$$

To find optimal w_i , we use the first-order condition:

$$\begin{aligned} \frac{\partial}{\partial w_i} \left[u_i \left(\frac{w_i}{\sum_{j=1}^n w_j} \right) - w_i \right] &= 0 \\ u'_i \left(\frac{w_i}{\sum_{j=1}^n w_j} \right) \left(\frac{1}{\sum_{j=1}^n w_j} - \frac{w_i}{(\sum_{j=1}^n w_j)^2} \right) - 1 &= 0 \quad (\text{by the chain rule}) \end{aligned}$$

Let $p = \sum_{j=1}^n w_j$ and $x_i = \frac{w_i}{\sum_{j=1}^n w_j}$:

$$\begin{aligned} u'_i(x_i) \left(\frac{1}{p} - \frac{x_i}{p} \right) &= 1 \\ u'_i(x_i)(1 - x_i) &= p \end{aligned}$$

In contrast to the fixed-price case where users behave as price-takers and can buy as much as they want at the externally set price p , players here are *price-makers*. When a player increases their bid to obtain more bandwidth, they also raise the total sum of bids, effectively increasing the price per unit for all players, including themselves.

This means that the marginal *cost for additional bandwidth is higher*: not only must they pay more to get a higher share, but the extra bandwidth they receive comes at an increasing per-unit cost, and the bandwidth they already have also becomes more expensive. Thus, in the proportional sharing setting, a player's own decision has an adverse effect on the price they pay even for their *previously allocated* bandwidth, unlike in the fixed-price scenario.

4 Price of Anarchy (PoA)

We now prove the Price of Anarchy to be at most $\frac{4}{3}$.

We do this by applying a series of reductions to the utility functions $u_i(x)$ so that, after replacing them with the resulting utility functions $\bar{u}_i(x)$, the price of anarchy can only become larger, but is nevertheless bounded above by $\frac{4}{3}$.

1. Reduction to affine utility functions.

Without loss of generality, assume each player's utility function is affine: $a_i x + b_i$. More generally, let $u_i(x)$ be any given concave function. At a given Nash equilibrium point x_i , define the tangent (linearization) at x_i as:

$$\bar{u}_i(x) = u_i(x_i) + u_i'(x_i)(x - x_i)$$

This tangent line \bar{u}_i has the same derivative as u_i at x_i . Since the equilibrium conditions depend only on the derivatives at the equilibrium, we can equivalently consider the game with \bar{u}_i in place of u_i . Both the location and value of the Nash equilibrium remain unchanged.

Due to concavity, the tangent always lies above the function, i.e., $\bar{u}_i(x) \geq u_i(x)$ for all x . Therefore, concavity $\implies \bar{u}_i(x) \geq u_i(x) \implies$ optimum may only increase \implies PoA can only become larger

That is, moving to the tangent upper-bounds the worst-case PoA.

2. Normalization by shifting.

We can, without loss of generality, shift each affine utility by subtracting b_i , so that $\bar{u}_i(x) = a_i x$ (i.e., $b_i = 0$ for all players). This constant shift does not affect the *location* of the equilibrium or the optimum, because the slopes remain the same.

The PoA can only become larger, because the Nash and optimal welfare decrease by the same constant $\sum_i b_i$.

To show this, let x_i be the Nash equilibrium and x_i^* be the social optimum points for player i under the $\bar{u}_i(x) = a_i x + b_i$ utility function. Then, for all $b_i \geq 0$, we have:

$$\text{PoA} \leq \frac{\sum_{i=1}^n a_i x_i^* + b_i}{\sum_{i=1}^n a_i x_i + b_i} = \frac{\sum_{i=1}^n a_i x_i^* + \sum_{i=1}^n b_i}{\sum_{i=1}^n a_i x_i + \sum_{i=1}^n b_i} \leq \frac{\sum_{i=1}^n a_i x_i^*}{\sum_{i=1}^n a_i x_i}$$

To see this, recall the general fact: if $\alpha/\beta \geq \gamma/\delta$, then $\frac{\alpha+\gamma}{\beta+\delta}$ is between α/β and γ/δ . Here, the constant terms subtracted from both numerator and denominator have ratio 1, and since PoA is at least 1, removing these constants can only increase the ratio.

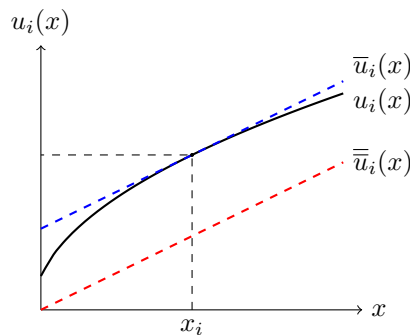


Figure 2: Concave value function u_i , its linear approximation \bar{u}_i at the Nash point x_i (green), and the further downward shift $a_i x$ (red).

Optimal Allocation

Under the assumption that $\bar{u}_i(x) = a_i x$, then the optimum is to allocate all the bandwidth to the player with the steepest slope a_i . Without loss of generality, assume $a_1 \geq a_2 \geq \dots \geq a_n$, so the optimum is to allocate all the bandwidth to player 1. So, $x_1 = 1$ and $x_i = 0$ for all $i \neq 1$.

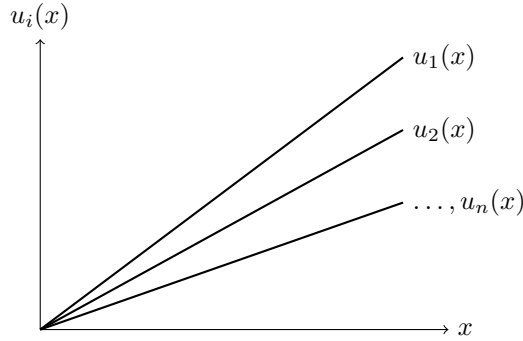


Figure 3: Ordered linear value functions $u_i(x) = a_i x$ with $a_1 \geq a_2 \geq \dots \geq a_n$.

Nash Allocation

For a player to receive any bandwidth, i.e, $x_i > 0$, they must have $a_i \geq p$. Otherwise, the player will not be interested in bidding.

If player 1 receives bandwidth x_1 , then rest of the bandwidth is allocated to players with $a_i \geq p$. So,

$$\sum_{i=2}^n \bar{u}_i(x_i) = \sum_{i=2}^n x_i a_i \geq \sum_{i=2}^n x_i p = (1 - x_1)p$$

At Nash, the Nash condition for player 1 is:

$$\bar{u}'_1(x_1)(1 - x_1) \implies p = a_1(1 - x_1)$$

Therefore, the total utility is at least:

$$\sum_{i=1}^n \bar{u}_i(x_i) \geq \underbrace{a_1 x_1}_{\text{utility for player 1}} + \underbrace{(1 - x_1)p}_{\text{players with } a_i \geq p} = a_1 x_1 + a_1(1 - x_1)^2$$

To bound the worst-case PoA, we minimize the utility at Nash:

$$\min_x \left(a_1 x + a_1(1 - x)^2 \right) \implies x = 1/2$$

So the utility at Nash is at least:

$$\sum_{i=1}^n \bar{u}_i(x_i) \geq a_1 \frac{1}{2} + a_1 \frac{1}{4} = \frac{3}{4} a_1$$

So the PoA $\leq \frac{1}{3/4} = \frac{4}{3}$.