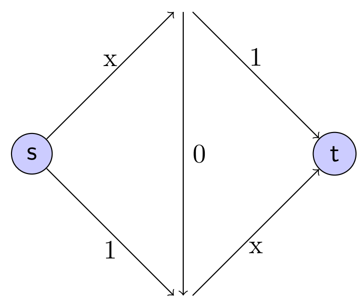
CS 6840 Algorithmic Game Theory September 30th, 2024

Lecture 15: Price of Anarchy in Routing Games

Instructor: Eva Tardos *Scribe: Eliezer Fuentes-Quezada*

1 Recap

We start recalling a very simple example of a routing game. Consider a network with two nodes s and t and two parallel links between them. We will consider rate of traffic or total traffict to be 1.



where the numbers on the edges are the costs/delays on this game. Clearly, Nash Equilibrium is reached when all players decide to use the upper path for the first half and then jump to the lower path, making their cost $2x$. Then with $x = 1$, the total cost for the NE is 2.

On the other hand the optimal solution is to split the traffic evenly between the two paths, making the cost 1.5. So the price of anarchy in this case is $2/1.5 = 4/3$.

2 General Model Network

Let (s_i, t_i) pairs of source and destinations with r_i to be the rate of traffic for each $i \in [n]$. We will consider that each edge was a cost function deppending on x amount of traffic $c_e(x)$. We will assume that $c_e(x)$ is monotone increasing, positive and continuous.

2.1 Possible Solution for Nash Equilibrium

For all paths $p \in P_i$, with P_i the set of paths from s_i to t_i, let f_p be the amount of traffic using path p. Notice that $f_p \geq 0$ and let $r_i = \sum_{p \in P_i} f_p$ for all $i \in [n]$, then the cost for users following path p is

$$c_p(f) = \sum_{e \in p} c_e(f(e)),$$

with $f(e) = \sum_{e \in p} f_p$ for e an edge in path p. Then the total cost of a flow is

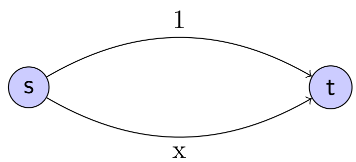
$$cost(f) = \sum_i \sum_{p \in P_i} f_p c_p(f),$$

which is also the Social Welfare. Therefore, the optimal is the minimum of the social welfare over every possible flow f. This is, $\min_f cost(f)$.

On the other hand, the Nash Equilibrium is when player i, given a flow f, chooses the minimum cost $c_p(f)$ over all possible paths $p \in P_i$. This is, for all paths p such that $f_p > 0$, $c_p(f) = \min_{p' \in P_i} c_{p'}(f)$. This definition makes sense as we assume that $c_p(f)$ is continuous.

2.2 Price of Anarchy

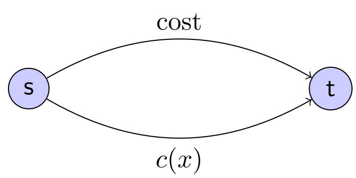
We want to compute $\frac{cost(f)}{OPT}$ with f in Nash Equilibrium. To see how we can find this value we consider the following simple example:



Note that the Nash Equilibrium is that all users use the lower path, making the total cost 1. On the other hand, the optimal solution is to split the traffic evenly, making the total cost $1 \times 1/2 + 1/2 \times 1/2 = 3/4$. So the price of anarchy is $1/0.75 = 4/3$.

Claim: The example just shown is the worst case scenario.

To prove this we define the class of cost functions $\mathcal{C}$ and $\alpha(\mathcal{C})$ as the maximum price of anarchy on two node networks with cost functions in $\mathcal{C}$.



Where the top edge has contant rate of traffic $r > 0$ and the bottom edge has cost function $c(x) \in \mathcal{C}$.

Theorem 1. *For all classes $\mathcal{C}$ and all networks with cost functions in $\mathcal{C}$, Price of Anarchy $\leq \alpha(\mathcal{C})$.*

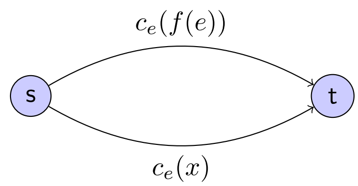
Proof. The proof comes in two steps. First let f be the Equilibrium flow and consider a network with r_i rate of traffic and (s_i, t_i) the source-destination pair. Lets also fix the edge cost $c_e(f(e))$.

Claim: With this cost, f is optimal

$$\begin{aligned} cost(f') &= \sum_p f'_p c_p(f) \\ &\geq \sum_i \sum_{p \in P_i} f'_p \min_{p'' \in P_i} c_{p''}(f) \\ &= \sum_i r_i \min_{p'' \in P_i} c_{p''}(f) = cost(f). \end{aligned}$$

Note that this last line is cost of the equilibrium.

Now, we want to compute the cost for the optimum so we can derive the proce of anarchy. For this, we use $\alpha(\mathcal{C})$ on each edge. Then the $cost = c_e(f)$, cost of teh lower edge as $f_e = c_e(x)$ and the rate of traffic $r = f(e)$



Claim:

$$\alpha(\mathcal{C}) \geq \frac{f(e)c_e(f_e)}{f^*(e)c_e(f^*(e)) + [f(e) - f^*(e)]\, c_e(f(e))}.$$

Where the numerator is the cost of the Nash Equilibrium and $f^*(e)$ is any solution, but the optimal would only make it better ■