CS 6840 Algorithmic Game Theory September 9, 2024

Lecture 6: First price, second price, and all pay auctions

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1 Auctions as games

We have previously seen Hotelling games as a class of games that is easy to analyze. Auctions consist another area with many applications that also allows for analysis of interesting results.

The simplest auction one can think is the single-item auction. An example is one looking to sell a used car, or a house (in order for these examples to conform to our definitions below, we would have to ignore the rest of the market though).

An important assumption we make is that each participant in the auction has a private value for the good (can be thought of as the willigness to pay for the good / maximum payment they give).

Definition 1. A single-item *Auction* is a game between N players, each of which has a private valuation v_i for an item to be sold. The next steps take place in order.

1. Players bid secretly (e.g. in an envelope). We denote the bid of player i as b_i.
2. A winner from the player pool is selected.
3. Payment(s) are announced and the winner is awarded the item.

The most natural winner selection rule is choosing the highest bidder, however there do exist scenarios where other rules are also considered (e.g. in revenue maximization). We will only consider the natural winner selection rule.

The most interesting ingredient of the above game is the design of the payments. Different payment rules lead to a wide variety of auctions and behaviours of the agents. In this lecture we are going to study three different payments / auctions.

Definition 2. First price, second price and all pay auctions

1. *First price auction*: the winner pays their bid $p = b_i$ (e.g.: real estate)
2. *Second price auction*: the winner pays the second highest bid (e.g. eBay). If $i^* = \arg\max_i b_i$ then $p = \max_{i \neq i^*} b_i$.
3. *All pay auction*: everyone pays their bid. Some examples here are lotteries, contract/prize competitions and also cryptocurrency mining.

Some very interesting differences between these auctions are the below claims.

Claim 1. *Second Price* has the property that every participant always should bid their value.

Claim 2. On the contrary this does not hold for *First Price* and *All Pay.*

Before we present our first interesting result we define the notion of a *dominant strategy.*

Definition 3. A dominant strategy is one that no matter what other players do, the player is as well off as possible.

Theorem 1. *The dominant strategy for player i in the* Second Price Auction *is to bid their valuation, meaning $b_i = v_i$.*

However, we haven't been specific over what a player optimizes for (what's their utility?).

Definition 4. A player's utility in a single-item auction is defined as:

$$u_i(\mathbf{b}) = \mathbb{I}\{i \text{ wins the auction} \mid \mathbf{b}\} \cdot v_i - p_i(\mathbf{b})$$

Where $p_i(\mathbf{b})$ is the payment of player i under the bid profile $\mathbf{b}$ and $\mathbb{I}\{A\} = 1$ if A, else it is 0.

For example in the first and second price auction, only the winner pays an amount p, so the utility is:

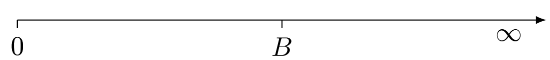
$$u_i(\mathbf{b}) = \begin{cases} v_i - p(\mathbf{b}) & \text{if } i \text{ is the winner} \\ 0 & \text{if not} \end{cases}$$

However in the all-pay auction:

$$u_i(b_i, \mathbf{b_{-i}}) = \begin{cases} v_i - b_i & \text{if } i \text{ is the winner} \\ -b_i & \text{if not} \end{cases}$$

Now we return to Theorem 1 to see why the dominant strategy for any player is to be truthful.

Proof. *(of Theorem 1).* Fix player i and let $B := \max_{j \neq i} b_j$ (the highest bid outside of i). The price to pay if i wins is B, but as we will see a rational player's bid will be independent of this.



- if $B > v_i$
  The player would not want to win, as then $u_i = v_i - B < 0$. If the player "loses" then they get 0 utility, so bidding $b_i = v_i$ is a dominant strategy in this scenario.
- if $B < v_i$
  If the player wins they get a utility of $u_i = v_i - B > 0 = u_i(\text{not winning})$. Bidding $b_i = v_i$ achieves this.
- if $B = v_i$
  The player is indifferent, any strategy leads to 0 utility.

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We now consider the *First Price Auction* and highlight a major difference in these two environments, as we claimed previously.

Theorem 2. *In a* First Price Auction *bidding the true value is not the dominant strategy in general.*

Proof. Consider a player i with a high enough valuation such that:

$$v_i > B = \arg\max_{j \neq i} b_j$$

If the player bids their valuation they would have utility 0, as they pay their bid; but suppose they bid just above B: $b_i = B + \epsilon(v_i - B)$ with $\epsilon \in (0, 1)$. Their utility now is $(1 - \epsilon)(v_i - B) > 0$. The player with the highest value always has incentive to bid less in a *First Price Auction.* ■

We move on to the *All Pay Auction* and consider a very simple scenario that distinguishes this auction as inherently different to the previously discussed ones.

Theorem 3. *Consider an* All Pay Auction *with two players of identical values (who have common knowledge of this). In this game no pure Nash equilibria exist.*

Proof. Suppose the common value is 1 wlog. Now suppose $b_1 < b_2$. Both would prefer to have chosen a different bid. The winner, player 2 would want to bid less in order to pay less. The loser would want to bid more if $b_2 < 1$ to obtain the item and if $b_2 = 1$ they would want to bid 0.

Now when $b_1 = b_2 < 1$ both would prefer to up their bid. When $b_1 = b_2 = 1$ we assume either that they split the item (both getting a value of 1/2) or that the winner is selected at random. In both cases the utility is $-1/2$, so they both prefer to bid 0. ■

However there exists a Mixed Nash Equilibrium that is simple in description. It can also be shown that it is unique (we won't prove that).

Theorem 4. *In the above example, if both players bid $b_i \in [0, 1]$ uniformly at random they are at an equilibrium.*

Proof. Consider any of the two players, ie player 1 and suppose they chose any pure strategy $b_1 = x$:

$$\begin{aligned}
\mathbb{E}[u_1(x, b_2)] &= \mathbb{E}[\text{payment}] + \mathbb{E}[\text{value gotten}] \\
&= -x + \mathbb{P}(\text{win}) \cdot v_1 \\
&= -x + \mathbb{P}(b_2 < x) \cdot 1 \\
&= -x + x = 0
\end{aligned}$$

Any randomized strategy by 1 under the fixed behaviour of 2 also leads in expected utility of 0 (can think of it as a weighted some of fixed bids). This means that player 1's utility under the MNE is also 0 in expectation and so has no incentive to deviate. ■

Another very interesting example with a surprising MNE is described below.

Theorem 5. *Consider a single-item* All Pay Auction *with two players, having valuations $v_1 = 1$ and $v_2 = 2$ respectively. There exists a unique Mixed Nash Equilibrium, where $b_2 \in [0, 1]$ uniformly at random and with equal chance $b_1 = 0$ or $b_1 \in [0, 1]$ uniformly at random.*

Proof. We will only prove that this mixed strategy set is an MNE. It is easy to see by a similar analysis to the previous, that any pure strategy deviation from player 1 leads to the same expected utility as in the mixed strategy described (that is, both achieve expected utility 0).

Now examine player 2 and consider a pure deviation $b_2 = x$, then:

$$\begin{aligned}\mathbb{E}[u_2(b_1, x)] &= -x + 2\mathbb{P}(b_1 < x) \\ &= -x + 2\left(\frac{1}{2} + \frac{1}{2} \cdot x\right) \\ &= 1\end{aligned}$$

Consider player 2's expected utility under the (proposed) MNE.

$$\begin{aligned}\mathbb{E}[u_2(b_1, b_2)] &= \mathbb{E}[\text{payment}] + \mathbb{E}[\text{value gotten}] \\ &= -\frac{1}{2} + \frac{1}{2}\mathbb{E}[\text{value gotten} \mid b_1 = 0] + \frac{1}{2}\mathbb{E}[\text{value gotten} \mid b_1 \text{ uniform}] \\ &= -\frac{1}{2} + \mathbb{P}(\text{get item} \mid b_1 = 0) + \mathbb{P}(\text{get item} \mid b_1 \text{ uniform}) \\ &= -\frac{1}{2} + 1 + \frac{1}{2} = 1\end{aligned}$$

Above we used that:

$$\mathbb{P}(\text{get item} \mid b_1 \text{ uniform}) = \frac{1}{2} \text{ (by symmetry)}$$

We conclude that both players have no incentive to deviate from this mixed strategy set, hence it is an MNE. ■