CS 6840 Algorithmic Game Theory October 30, 2024

Lecture 26: Algorithmic Collusion

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Collusion can be defined as a secret agreement or cooperation especially for an illegal or deceitful purpose. For instance, agreements made between companies to both artificially increase prices when offering competing comparable services is collusion. More concretely, suppose that United and Delta are the only airlines in Ithaca and they decide to both offer higher prices. Then Ithaca residents will just have to pay more, and so United and Delta both make more revenue. This is an example of collusion.

Algorithmic Collusion refers rather to the effect of players using an algorithm. The players don't cooperate or discuss their strategies with each other, and yet by running the algorithm, their strategies produce the same effect as if they had colluded. For instance, in the airline example, algorithmic collusion could refer to the outcome of United and Delta both running some algorithm which tells them to offer higher prices than they would have offered in an equilibrium solution.

1 Pricing Game

Let us formalize this idea with the following game. Consider a game where the same product is sold by two competitors. There is then a universe of buyers of volume 1 who will buy this product no matter the price. All the buyers will purchase the cheapest option. In the case where the competitors offer the same price, then buyers are equally allocated to each competitor. The players must decide on a price for their product in order to maximize their revenue. They have k possible strategies for the choice of price: $\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k} = 1$.

Let us now consider various scenarios of strategy selection.

1.1 Monopoly Pricing

In this scenario, both agree to offer the product at price 1. Since both players offer the product at the same price, each player has a buyer volume of $\frac{1}{2}$. Thus each has revenue $\frac{1}{2}$. Note that this is the maximum combined revenue that can be obtained in this game.

1.2 Nash equilibrium

There are two Nash equilibrium solutions: $\left(\frac{1}{k}, \frac{1}{k}\right)$ and $\left(\frac{2}{k}, \frac{2}{k}\right)$.

Claim 1. All prices in a Nash equilibrium solution are $\leq \frac{2}{k}$.

Proof. Suppose that p is the maximum price offered by the players. WLOG consider the revenue of player 1.

- Case 1: Player 1 plays price p with some probability a, player 2 plays price $< p$ always. Then no buyers purchase from player 1 when they play price p. So player 1 has revenue 0 whenever they play price p.
- Case 2: Both players play p with respective probabilities a and b.

- When player 1 plays price p with probability a, player 1's expected revenue is $b \cdot \frac{1}{2} \cdot p$.
- However, if player 1 plays $p - \frac{1}{k}$ instead of p, their expected revenue is $\geq b \cdot (p - \frac{1}{k}) \cdot 1$

If $p > \frac{2}{k}$, then the revenue associated with playing $p - \frac{1}{k}$ instead of p is better. So it would be preferable to play $p - \frac{1}{k}$ instead of p whenever they were playing p in their past strategy. Using induction, it is easy to see that the prices selected by each player will keep decreasing until they can only play strategies $\leq \frac{2}{k}$. Thus, no price $p > \frac{2}{k}$ can be part of a Nash equilibrium solution. ■

1.3 Stackelberg Equilibrium

Suppose you know that your opponent is a no-swap regret learner. The other player can then be the leader and set their prices first, and the learner will optimize to use the best response strategy and get the Stackelberg value of the game[1].

Proposed strategy of the leader: uniform random across the k prices.

Best Response: Suppose that the learner plays price p. The following table lists their revenue and with what probability they obtain it given the uniformly random price choice of the leader.

Price of Leader	Probability	Revenue of Learner with price p
$> p$	$1 - p$	$1 \cdot p$
$= p$	$\frac{1}{k}$	$\frac{1}{2} \cdot p$
$< p$	$p - \frac{1}{k}$	0

Thus, the total expected revenue of the learner with strategy $p = (1-p)p + \frac{p}{2k}$. Furthermore note that:

$$\max_{p \in \mathbb{R}} \left((1-p)p + \frac{p}{2k} \right) = \frac{2k+1}{4k} = \frac{1}{2} + \frac{1}{4k}$$

Note that $p = \frac{2k+1}{4k}$ is between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{k}$, assuming k is even. It is then easy to check that $\frac{1}{2}$ is a better choice than $\frac{1}{2} + \frac{1}{k}$, which give expected revenues of $\frac{1}{4} + \frac{1}{4k}$ and $\frac{1}{4} + \frac{1}{4k} - \frac{1}{2k^2}$ respectively.

Their expected revenues from playing uniformly at random (leader) and $p = 1/2$ (learner) are:

- Leader Expected Revenue $= \underbrace{0}_{p>1/2} + \underbrace{\frac{1}{k} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{p=1/2} + \underbrace{\frac{1}{k} \sum_{i=1}^{k/2-1} \frac{i}{k}}_{p<1/2} = \frac{1}{4k} + \frac{1}{k} \frac{\frac{k}{2}(\frac{k}{2}-1)}{2k} = \frac{1}{4k} + (\frac{1}{8} - \frac{1}{4k}) = \frac{1}{8}$
- Learner Expected Revenue $= \frac{1}{2}(\frac{1}{2}) + \frac{1}{2}\left(\frac{1}{2k}\right) = \frac{1}{4} + \frac{1}{4k}$

We thus see that the result of using the algorithm ended up being good for both the leader and learner, as they both obtained much higher revenues than in either Nash equilibrium outcomes.

Claim 2. If the leader has revenue $\geq c$, where c is some constant independent of k, then the learner/follower also has revenue $\geq d$, where d is also some constant which is not necessarily equal to c.

[1]See Lecture 24: Danger of No Regret Learning (Oct 23, 2024) for more details.

Proof. Since the leader has expected revenue $\geq c$, we must have that they offer price $> \frac{c}{2}$ at least with probability $\frac{c}{2}$. To see why this is the case, note that for any time that they offer price $\leq c/2$, the maximum revenue that can be obtained is $c/2$. For any time that they offer price $> c/2$, the maximum revenue they can obtain is 1 from offering price 1 and all buyers purchasing their product. Letting q be the probability of offering a price $> c/2$, we get the following inequality:

$$\begin{aligned} & 1 \cdot q + \frac{c}{2}(1-q) > c \\ \Rightarrow \quad & \frac{c}{2} + q(1-\frac{c}{2}) > c \end{aligned}$$

Rearranging, and noting that $c/2 < 1$, we find that $q > c/2$ as desired.

$$q > \frac{\frac{c}{2}}{1-\frac{c}{2}} > \frac{c}{2}$$

Thus, one possible strategy for the learner/follower is the price $\frac{c}{2} - \frac{1}{k}$. With this price, they will obtain an expected revenue of at least $\frac{c^2}{4} - \frac{c}{2k}$ from winning when the leader played price $> \frac{c}{2}$. Thus, the learner also has an expected revenue above some constant. ■

Note that though it was the case that the follower did better than the leader, this is not necessarily always the case. Furthermore, this particular choice of the leader's strategy is not necessarily optimal.

2 Concluding Notes

All buyers purchasing the cheaper option may not be realistic. The paper *Algorithmic Collusion Without Threats* makes some steps towards expanding this toy model game by letting a fraction of buyers purchase the more expensive product based on their preferences.