CS 6840 Algorithmic Game Theory October 11, 2024

Lecture 20: Potential games and Quality of solutions

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1 Recap

Recall the atomic routing game introduced in previous lectures. We consider n players over a given network, where each player i selects a path $P_i \in \mathcal{P}_i$ from their source node s_i to their destination node t_i, and $\mathcal{P}_i \subseteq 2^E$ denotes the set of all paths between node s_i and node t_i. For a given flow f induced by the paths $(P_1, \ldots, P_n)$ selected by the players, the cost incurred by player i is defined as

$$\text{cost}_i(f) := \sum_{e \in P_i} c_e(f(e)) \tag{1}$$

where $f(e) = |\{i \mid e \in P_i\}|$ denotes the flow on edge e (i.e., the number of players whose selected path includes edge e), and $c_e(f(e))$ denotes the cost on edge e as a function of the flow on that edge. The total cost for a flow f is given by $\text{cost}(f) = \sum_{i=1}^{n} \text{cost}_i(f)$.

We have shown in the previous lecture that this game is a potential game.

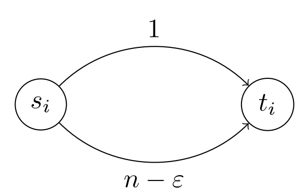
Theorem 1. *The atomic routing game (described above) is a potential game with a potential function Φ given by*

$$\Phi(f) = \sum_e \sum_{k=1}^{f(e)} c_e(k). \tag{2}$$

2 Network cost-sharing games

We now consider an example of an atomic routing game in which the edge costs are shared between players using the same edges. Here, congestion is advantageous since each player's costs decrease with congestion. More formally, for each edge e, we assume that a total cost of $\bar{c}_e$ is split evenly between the players using this edge, such that each player using edge e incurs a cost of $c_e(k) = \frac{\bar{c}_e}{k}$.

We start by examining this game on a 2-node network with n users:



This game possesses two Nash equilibria. The first Nash equilibrium, which also corresponds to the optimal outcome, is when all users pick the upper edge, for a total cost of 1, or a cost of $1/n$ per user. No player is incentivized to change their route because they would incur a cost of $n - \varepsilon$ instead of $1/n$ if they were to deviate. The second Nash equilibrium occurs when all users pick the lower edge, for a total cost of $n - \varepsilon$, or a cost of $1 - \varepsilon/n$ per user. Once again, no player is incentivized to change their route since they would incur an excess cost of ε/n by deviating to the upper edge. It is evident that the second Nash is much worse than the first Nash.

In this case, the price of anarchy does not provide any useful information about the outcomes of this game since the second Nash equilibrium is so much worse than the first. A question we can ask is whether we can quantify the quality of the best Nash compared to the optimal solution, instead of examining the worst Nash. To do so, we define the price of stability.

Definition 1 (Price of Stability)**.** The price of stability is defined as the ratio between the cost of the best Nash equilibrium and the cost of the optimal outcome OPT

$$\text{price of stability} = \min_{f \text{ Nash}} \frac{\text{cost}(f)}{\mathsf{OPT}}. \tag{3}$$

One motivation behind considering the price of stability is in the case of a central planner who seeks to recommend the best Nash equilibrium to selfish users such that the users do not deviate from the recommendation.

Recall from the last lecture that the solution $\arg\min_f \Phi(f)$ is a Nash equilibrium since Φ is a potential function. Using price of stability, we now show that the solution $\arg\min_f \Phi(f)$ is a good Nash equilibrium for the network cost-sharing game.

Theorem 2. *For the network cost sharing game, let $f^\star = \arg\min_f \text{cost}(f)$ denote the total cost minimizing flow and $f = \arg\min_f \Phi(f)$ denote the potential function minimizing flow. Then, it follows that*

$$\text{cost}(f) \leq \mathcal{H}_n \mathsf{OPT} \tag{4}$$

where $\mathcal{H}_n$ denotes the n-th harmonic number $\sum_{i=1}^{n} \frac{1}{n}$.

Proof. Note that the total cost for a flow f can be rewritten as the sum of all edge costs for edges with nonzero flow

$$\text{cost}(f) = \sum_e f(e) c_e(f(e)) = \sum_{e : f(e) > 0} \bar{c}_e.$$

Similarly, we can also rewrite the potential function evaluated at flow f in terms of a sum over all edges with nonzero flow

$$\Phi(f) = \sum_e \sum_{k=1}^{f(e)} c_e(k) = \sum_{e : f(e) > 0} \bar{c}_e + \frac{\bar{c}_e}{2} + \frac{\bar{c}_e}{3} + \cdots + \frac{\bar{c}_e}{f(e)} = \sum_{e : f(e) > 0} \bar{c}_e \mathcal{H}_{f(e)}.$$

Since $1 \leq \mathcal{H}_{f(e)}$ for $f(e) > 0$, it follows that $\text{cost}(f) \leq \Phi(f)$:

$$\text{cost}(f) = \sum_{e : f(e) > 0} \bar{c}_e \leq \sum_{e : f(e) > 0} \bar{c}_e \mathcal{H}_{f(e)} = \Phi(f).$$

Moreover, using the fact that $\mathcal{H}_{f(e)} \leq \mathcal{H}_n$ for all edges e since $f(e) \leq n$, it also follows that $\Phi(f) \leq \mathcal{H}_n \text{cost}(f)$:

$$\Phi(f) = \sum_{e : f(e) > 0} \bar{c}_e \mathcal{H}_{f(e)} \leq \sum_{e : f(e) > 0} \bar{c}_e \mathcal{H}_n = \mathcal{H}_n \text{cost}(f).$$

Finally, we have that $\Phi(f) \leq \Phi(f^\star)$ by definition of f. Therefore, the desired result follows from these inequalities:

$$\text{cost}(f) \leq \Phi(f) \leq \Phi(f^\star) \leq \mathcal{H}_n \text{cost}(f^\star) = \mathcal{H}_n \mathsf{OPT}.$$

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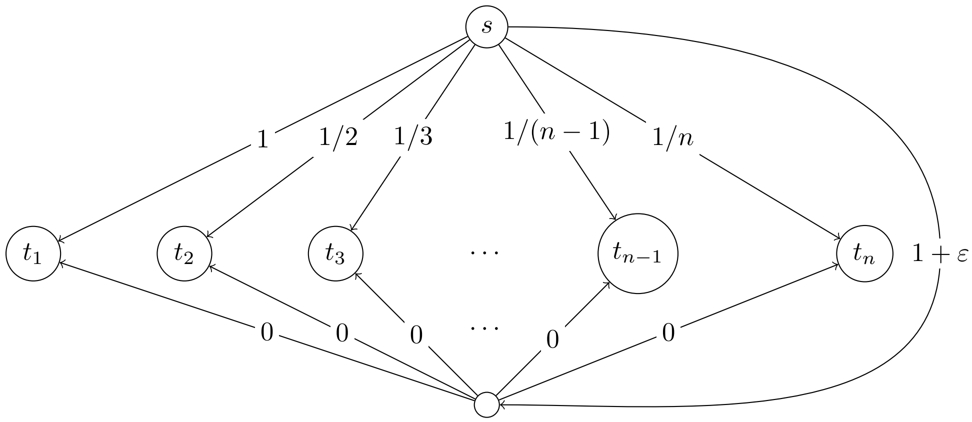
We now show that the bound on the price of stability provided in (4) is a tight bound for network cost-sharing games on directed networks (the bound is not tight for undirected networks). Consider the following example:



In this example, the optimal outcome is attained when all players take the $1+\varepsilon$ cost edge followed by the zero cost edge going to their respective destination node. They incur a total cost of $\mathsf{OPT} = 1+\varepsilon$ or a cost of $(1+\varepsilon)/n$ per player. However, for this outcome, the n-th player is incentivized to deviate their route to take the $1/n$ cost edge connecting source s to their destination t_n directly, since they would incur a cost that is ε/n smaller. In fact, the unique Nash equilibrium occurs when each player i follows the direct edge with cost $1/i$ between the source s and their destination t_i. For this Nash, the players incur a total cost of $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \mathcal{H}_n$.

Therefore, if we denote the Nash flow by f, we have that

$$\text{cost}(f) = \mathcal{H}_n \leq \mathcal{H}_n(1+\varepsilon) = H_n \mathsf{OPT}.$$

As $\varepsilon \to 0$, the inequality turns into equality, demonstrating that the bound in (4) is tight.

3 Linear congestion routing

Last Monday, we considered a routing game with affine cost functions $c_e(x) = a_e x + b_e$. For a given flow f in this game, the total cost is

$$\text{cost}(f) = \sum_e f(e)[a_e f(e) + b_e]$$

and the potential function evaluated at f is

$$\Phi(f) = \sum_e \sum_{k=1}^{f(e)} [a_e k + b_e].$$

We have previously shown that the price of anarchy was bounded by 2.5. It can also be shown that the price of stability in this game is upper bounded by 2.

Theorem 3. *For the linear congestion routing game, let $f^\star = \arg\min_f \text{cost}(f)$ denote the total cost minimizing flow and $f = \arg\min_f \Phi(f)$ denote the potential function minimizing flow. Then, it follows that*

$$\text{cost}(f) \leq 2\mathsf{OPT}. \tag{5}$$

Proof. Note that we can rewrite the total cost as

$$\text{cost}(f) = \sum_e f(e)[a_e f(e) + b_e] = \sum_e a_e f(e)^2 + b_e f(e)$$

and the potential function as

$$\Phi(f) = \sum_e \sum_{k=1}^{f(e)} [a_e k + b_e] = \sum_e a_e \left(\sum_{k=1}^{f(e)} k \right) + b_e f(e).$$

Hence, using the fact that $\sum_{k=1}^{f(e)} k = \frac{f(e)(f(e)+1)}{2}$ and the fact that

$$\frac{f(e)^2}{2} \leq \frac{f(e)(f(e)+1)}{2} \leq f(e)^2,$$

it follows that $\Phi(f) \leq \text{cost}(f)$ and $\Phi(f) \geq \frac{1}{2}\text{cost}(f)$. Hence, using the fact that $\Phi(f) \leq \Phi(f^\star)$ by definition of f, the desired result follows from the following string of inequalities

$$\text{cost}(f) \leq 2\Phi(f) \leq 2\Phi(f^\star) \leq 2\text{cost}(f^\star).$$

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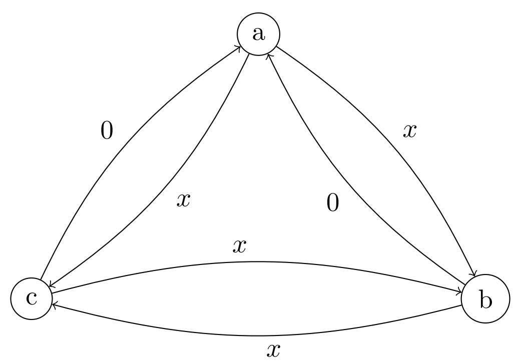
While we do not have an example of a linear congestion routing game where the price of stability upper bound of 2 is attained, we do have an example where a price of anarchy of 2.5 is attained. Consider the following bidirected triangle network example:



Here, x denotes the congestion on the associated edge (i.e., the number of agents using that edge). In this example, we consider four agents with source-destination pairs given by (a, b), (a, c), (b, c), and (c, b). Each agent has two routing options: a direct path from their source to destination or a two-edge path.

The optimal outcome (which is a Nash equilibrium) occurs when each agent takes the direct path connecting their source to their destination, causing them to incur a cost of $x = 1$ each, for a total cost of 4. The outcome where each agent follows the two-edge indirect path between their source and destination is also a Nash equilibrium. In this latter outcome, the edges $a \to b$ and $a \to c$ have congestion of two users, resulting in two players incurring a cost of 2 each and two other players incurring a cost of 3 each, for a total cost of 10. Hence, the second Nash equilibrium has a total cost that is 2.5 times higher than the optimal outcome, attaining the price of anarchy bound of 2.5.