CS 6840 Algorithmic Game Theory October 9, 2024

Lecture 19: Congestion games and Potential games

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1 Atomic Routing

We first review the problem setting of atomic routing from the last lecture. The atomic routing game is played with $n \in \mathbb{N}$ users, each choosing paths P_i that connect source-destination pairs $s_i \to t_i$, and edgewise costs $c_e(x)$ that depend on the flow x over the edges, which we denote by $f(e) \triangleq \#\{i : e \in P_i\}$. Each player i simply aims to minimize the cost/delay it faces by choosing a path P_i that connects s_i, and t_i. Therefore, we can write the cost of an arbitrary player i or equivalently the cost of the path that this player chooses by

$$\mathrm{cost}_i(f) = \mathrm{cost}_{P_i}(f) = \sum_{e \in P_i} c_e\left(f(e)\right), \tag{1}$$

where the flow f is induced by the paths chosen by players $(P_1, \ldots, P_n)$.

Now, we introduce a fact about the atomic routing game which seems magical at first but is really useful and makes a class of games with very useful properties.

Fact 1. *In an atomic routing game, there exists a function Φ which satisfies*

$$\Phi(P_1, \ldots, P_n) - \Phi(P_1, \ldots, Q_i, \ldots, P_n) = cost_i(P_1, \ldots, P_n) - cost_i(P_1, \ldots, Q_i, \ldots, P_n) \quad \forall i. \tag{2}$$

Although this fact was presented abruptly we will see that it implies very useful results regarding the atomic routing game, and it actually means that the atomic routing game is a *potential* game which we introduce now.

Definition 1. Given a game with a set of strategies S_i, and cost function $c_i : S := \times_j S_j \to \mathbb{R}$ defined for each player, we say that the game is a *potential* game if there is a *potential* function $\Phi : S \to \mathbb{R}$ that satisfies

$$\Phi(s) - \Phi(s', s_{-i}) = c_i(s) - c_i(s', s_{-i}) \quad \forall i, s' \in S_i. \tag{3}$$

Now we formally show that the atomic routing game is a potential game in the following theorem.

Theorem 1. *The atomic routing game admits a potential function regardless of the nature of the cost function on the edges as long as it only depends on the flow on the edge.*

Proof. We prove that the atomic routing game is a potential game by directly constructing a potential function Φ as follows

$$\Phi(f(P)) = \sum_e \sum_{k=1}^{f(e)} c_e(k), \tag{4}$$

where $f(P)$ denotes the flow induced by the path profile $P = (P_1, \ldots, P_n)$ chosen by the players.

Now consider that the flow of player i is removed from the game, i.e., its chosen path P_i is removed and therefore replaced by $\emptyset$, we will now calculate

$$\Phi(f(P_1, \ldots, P_i, \ldots, P_n)) - \Phi(f(P_1, \ldots, \emptyset, \ldots, P_n)). \tag{5}$$

Notice that removing the flow of player i simply reduces the flow on the edges that are a part of the path P_i by one. Therefore, we get

$$\Phi\left(f(P_1,\ldots,P_i,\ldots,P_n)\right)-\Phi\left(f(P_1,\ldots,\emptyset,\ldots,P_n)\right)=\sum_{e\in P_i}\sum_{k=1}^{f(e)}c_e(k)-\sum_{e\in P_i}\sum_{k=1}^{f(e)-1}c_e(k) \tag{6}$$

$$=\sum_{e\in P_i}c_e\left(f(e)\right)\underset{(a)}{=}\text{cost}_i\left(f(P)\right), \tag{7}$$

where (a) simply follows from the definition of the $cost_i$. By the same argument we can argue that

$$\Phi\left(f(P_1,\ldots,\emptyset,\ldots,P_n)\right)-\Phi\left(f(P_1,\ldots,Q_i,\ldots,P_n)\right)=-\text{cost}_i\left(f(P_1,\ldots,Q_i,\ldots,P_n)\right). \tag{8}$$

Since this analysis holds for any player i, and any path profiles we get

$$\Phi\left(f(P)\right)-\Phi\left(f(P_1,\ldots,Q_i,\ldots,P_n)\right)=\text{cost}_i\left(f(P)\right)-\text{cost}_i\left(f(P_1,\ldots,Q_i,\ldots,P_n)\right)\quad\forall i,Q_i,\text{ and }P, \tag{9}$$

which completes the proof. ■

The fact that the atomic routing game is a potential game has very useful implications. First of all, notice that any deviation for an arbitrary player i that reduces their individual cost actually reduces the potential function as well. We provide the results on repeated better response in the following corollary using this simple observation.

Corollary 1. *Repeated better response followed by players where a single player is allowed to deviate at every time-step decreases the potential at each time-step, which implies:*

- *The dynamics do not cycle as the potential function can never increase,*
- *The strategy profile of the players will reach a Nash equilibrium, and end in finite time.*

Note that although the repeated better response will reach a Nash equilibrium and end in finite time, it does not imply that this finite time is computationally efficient. The existence of a potential function also implies that there exists a pure Nash equilibrium which we formalize in the following corollary.

Corollary 2. *The flow f minimizing $\Phi(f)$ is a Nash equilibrium.*

Again note that the flow f minimizing $\Phi(f)$ being a Nash equilibrium does not imply that it is computationally tractable to find the Nash equilibrium since finding $\min\Phi(f)$ is NP-complete.

2 Non-Atomic Setting

Now we turn back to the non-atomic setting where the players are characterized as infinitesimally small parts of a flow. Similar to the atomic setting, we show that the game admits a potential function just as before which we formalize in the following claim

Claim 1. *The non-atomic routing game admits a potential function defined as*

$$\Phi(f)=\sum_e\int_0^{f(e)}c_e(x)dx=:\sum_e\sigma_e\left(f(e)\right),$$

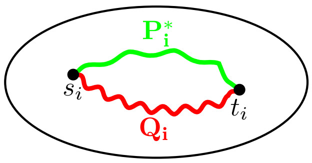
where the last equality follows by defining $\sigma_e\left(f(e)\right):=\int_0^{f(e)}c_e(x)dx$.

Notice that the potential function for the non-atomic setting is analogous to the atomic setting with the summation term replaced by the integral. We omit the proof that this is a valid potential function but note that the proof is analogous to the proof for the atomic setting as well: consider a infinitesimally small flow removed from a path to model an arbitrary player i being removed which corresponds to the derivative of the potential function, characterizing this derivative one can see that it is equal to the individual cost of a player i.

Similar to the atomic setting, the existence of the potential function implies the existence of a pure Nash equilibrium which we formalize in the following theorem.

Theorem 2. *The flow* $f^* = \arg\min_f \Phi(f)$ *is a Nash equilibrium flow.*

Proof. First note that we use P^* to denote the path profile of players that induce the flow f^*.



We will complete the proof by contradiction, therefore assume that there exists a path Q_i such that the cost of an arbitrary player i deviating to follow Q_i is less than the cost of following the path chosen by this player i in the path profile P^* denoted by P_i^*. In other words, assume that there exists Q_i such that

$$\text{cost}_{P_i^*}(f^*) = \sum_{e \in P_i^*} c_e\left(f^*(e)\right) > \sum_{e \in Q_i} c_e\left(f^*(e)\right) = \text{cost}_{Q_i}(f^*). \tag{10}$$

Now consider the effect of player i changing from P_i^* to Q_i on potential function, which will increase the flow on path Q_i by an infinitesimal amount, δ whereas it decreases the flow on path P_i^* by the same amount. The corresponding effect on the cost functions caused by this infinitesimal change can be captured by δ times the derivative of the potential function with respect to the flow over the edges as

$$\Phi(f^*) - \Phi(f(Q_i, P_{-i}^*)) = \sum_{e \in P_i^*} \delta \cdot \sigma_e'\left(f^*(e)\right) - \sum_{e \in Q_i} \delta \cdot \sigma_e'\left(f^*(e)\right) \tag{11}$$

$$= \delta \Big[\sum_{e \in P_i^*} c_e\left(f^*(e)\right) - \sum_{e \in Q_i} c_e\left(f^*(e)\right) \Big] \underset{(a)}{>} 0, \tag{12}$$

where (a) follows from the assumption that $\text{cost}_{P_i^*}(f^*) > \text{cost}_{Q_i}(f^*)$. This is a clear contradiction since the flow $f^* = \arg\min_f \Phi(f)$, therefore we get that there exists no path for any player to deviate to which results in a decrease in experienced cost at the minimizing flow f^* meaning that f^* is a Nash equilibrium. ■

Now that we have shown that the flow $f^* = \arg\min_f \Phi(f)$ is a Nash equilibrium, the remaining question is whether we can compute $\min_f \Phi(f)$. In contrast to the atomic setting we provide a positive answer to this question in the non-atomic setting: Notice that Φ consists of $\sigma_e(y)$, and further notice that $\sigma_e(y)$ is a convex function if the cost functions $c_e(x)$ are monotonically increasing since we have $\sigma_e'(y) = c_e(y)$. Therefore if we have that the cost functions $c_e(x)$ are monotonically increasing functions the following holds

- The minimum value of the potential function $\min_f \Phi(f)$ can be found via convex optimization techniques,
- Notice that the proof we had for Theorem 2 only required that the deviations from the flow f^* by an infinitesimal amount do not decrease the potential function, thus any local optima as well as the global optimum is actually a Nash equilibrium. Since the objective function is now convex when the cost functions $c_e(x)$ are monotonically increasing functions there are no local optima, therefore any Nash equilibrium corresponds to a global optimum.