CS 6840 Algorithmic Game Theory October 7, 2024

Lecture 18: Atomic Routing Games

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1 Atomic routing games

In previous lectures, we have assumed that players can move negligible amounts of flow, meaning they can change their strategy with negligible impact on the cost of any edge. We now move on to consider *atomic routing games*, where each player carries a discrete amount of flow that cannot be split up between paths (imagine each player is driving a car, and the entire car must travel on the same path).

Today, we look at a simple case of atomic routing games where each player has exactly one unit of flow. Specifically, our new assumptions are (1) each player has one unit of flow that cannot be split between paths and (2) the cost of edge e when it has x units of flow is $c_e(x) = a_e x + b_e$ with $a_e, b_e \in \mathbb{N}$. The constraints on a_e and b_e ensure c_e is monotonically increasing; we note that while this specific definition of c_e is continuous in x, continuity is no longer a required condition when we have discrete flow.

Like before, each player i has a source-sink pair (s_i, t_i), player i selects a path P_i from s_i to t_i (over which it now must send its entire unit of flow), and the total cost of taking P_i when the overall flow pattern is f is:

$$\text{cost}_{P_i}(f) = \sum_{e \in P_i} c_e(f(e))$$

where $f(e)$ is the number of players i such that $e \in P_i$ (i.e., the total flow on edge e).

2 Price of anarchy in the atomic routing game

Theorem 1. *The price of anarchy for a pure Nash equilibrium on a game with atomic flow with linear delays is upper bounded by* 2.5.

Proof. This proof will be similar to the one we saw in the past few lectures. We first consider the total cost over all players:

$$\sum_i \text{cost}_{P_i}(f) = \sum_e f(e) \cdot c_e(f(e))$$

The **socially optimal flow** $(Q_1, ..., Q_n)$ minimizes this quantity and can be written as follows since $f(e)$ is determined by the profile of paths[1]:

$$\text{OPT} = \min_{(Q_1,...,Q_n)} \sum_e f(e) \cdot c_e(f(e))$$

A **Nash Equilibrium** $(P_1, ..., P_n)$ of the game satisfies the following for every player i:

$$\text{cost}_{P_i}(f) = \sum_{e \in P_i} c_e(f(e))$$

[1]It turns out that this optimization problem is a quadratic integer program, so computing the optimal social welfare directly is intractable.

$$\leq \sum_{e \in Q_i \cap P_i} c_e(f(e)) + \sum_{e \in Q_i \setminus P_i} c_e(f(e)+1)$$
$$\leq \sum_{e \in Q_i} c_e(f(e)+1)$$

Now that we are in an atomic setting, we must add 1 to the flow on edges in $Q_i \setminus P_i$, since i adds a unit of flow to any edge it did not previously take. We use the second inequality to simplify our expression into a single term.

From here, we refer to the flow resulting from the socially optimal flow $(Q_1, ..., Q_n)$ as $f^*(e)$ and the flow resulting from Nash equilibrium $(P_1, ..., P_n)$ as $f(e)$. So the optimal social welfare is $\text{cost}(f^*)$ and the social welfare of the Nash equilibrium is $\text{cost}(f)$. Summing over users to get $\text{cost}(f)$:

$$\begin{aligned}
\text{cost}(f) &= \sum_i \text{cost}_{P_i}(f) \\
&= \sum_i \sum_{e \in P_i} c_e(f(e)) \\
&\leq \sum_i \sum_{e \in Q_i} c_e(f(e)+1) && \text{(Nash eq.)} \\
&\leq \sum_e c_e(f(e)+1) \cdot \sum_{i: e \in Q_i} 1 \\
&= \sum_e c_e(f(e)+1) \cdot f^*(e) \\
&\leq \frac{5}{3} \sum_e c_e(f^*(e)) f^*(e) + \frac{1}{3} \sum_e f(e) c_e(f(e)) && \text{(Lemma 1: see below)} \\
&= \frac{5}{3}\text{cost}(f^*) + \frac{1}{3}\text{cost}(f) \\
\implies \text{cost}(f) &\leq \frac{5}{2}\text{cost}(f^*)
\end{aligned}$$

■

In our proof, we use the following as a "magic fact":

Lemma 1. *For all $a, b \geq 0$ and $x, y \in \mathbb{N}$, if $c(x) = ax + b$ then $y \cdot c(x+1) \leq \frac{5}{3} y \cdot c(y) + \frac{1}{3} x \cdot c(x)$*

Proof. Substituting in the definition of $c(x)$, we see that we need to show the following inequality holds:

$$y(a(x+1)+b) \leq \frac{5}{3} y(ay+b) + \frac{1}{3} x(ax+b)$$

It is clear to see that this inequality always holds for the terms involving b, since $y, b, x \geq 0$:

$$yb \leq \frac{5}{3} yb + \frac{1}{3} xb$$

So it remains to be shown that

$$ya(x+1) \leq \frac{5}{3} ay^2 + \frac{1}{3} ax^2.$$

Since we know that $a \geq 0$, it suffices to show that

$$y(x+1) \leq \frac{5}{3} y^2 + \frac{1}{3} x^2 \tag{1}$$

$$\iff 3xy + 3y \leq 5y^2 + x^2 \tag{2}$$
$$\iff 3y \leq (2y - x)^2 + y(x + y) \tag{3}$$

When $x + y \geq 3$ or $y = 0$, it is clear to see that inequality (3) holds:

$$3y \leq y(x + y) \implies 3y \leq (2y - x)^2 + y(x + y)$$

When $x + 1 \leq y$, it is clear to see that inequality (1) holds:

$$y(x + 1) \leq y^2 \leq \frac{5}{3}y^2 + \frac{1}{3}x^2$$

The only case left to show is $(x, y) = (1, 1)$, which can be verified directly. ■

Note that the inequality is not true when $x = 1.5$ and $y = 1$, since $3 \nleq 0.25 + 2.5$ (inequality (3)). So the magic fact is true for all nonnegative integers but *not* all nonnegative real numbers! It also turns out that the bound we get by using this magic fact is, in fact, a tight bound on the price of anarchy.

3 General framework for PoA proofs

We can extend our PoA proof to different settings if we have an equivalent of our "magic fact" to help us complete the chain of inequalities above. Assume we have some equivalent of our "magic fact":

$$\sum_i \text{cost}_i(s_i^*, s_{-i}) \leq \lambda \ \text{cost}(s^*) + \mu \sum_i \text{cost}_i(s)$$

The existence of such a fact is specific to the type of game and cost function. Then, if we have n players who each play $s_1, ..., s_n$ in a Nash equilibrium and $s_1^*, ..., s_n^*$ in the socially optimal profile,

$$\begin{aligned}
\sum_i \text{cost}_i(s) &\leq \sum_i \text{cost}_i(s_i^*, s_{-i}) && \text{(Definition of Nash eq.)} \\
&\leq \lambda \ \text{OPT} + \mu \sum_i \text{cost}_i(s) && \text{(``Magic fact'')} \\
\implies \text{PoA} &\leq \frac{\lambda}{1 - \mu} && \text{(if } \mu < 1\text{)}
\end{aligned}$$