CS 6840 Algorithmic Game Theory November 15, 2024

Lecture 33: Equilibrium Analysis for Price of Anarchy

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In this lecture, we will look at a new way to analyze the price of anarchy for games.

1 Introduction

We have seen two ways to analyze the quality of equilibrium in games until now:

- (λ, μ)-smoothness: Obtain inequalities for the costs of players in terms of the socially optimal costs and the equilibrium revenue. Selectively summing these inequalities can give us a bound on the price of anarchy. This extends to learning, and is very prevalent in the community.
- Price of Stability: The equilibrium is optimizing the wrong function, *how wrong is that function?* In the example we covered, we saw that the equilibrium was not optimizing the objective, but *locally* optimizing instead. This gives us the actual price of anarchy result if we can show that the optimum is unique.

In this lecture, we will look at a third style known as **equilibrium analysis**.

2 Problem Setup

We will use the Bandwidth Sharing Game as an example of how to use equilibrium analysis to find the price of anarchy. This is used by companies like Zillow to determine how to divide advertising opportunities on their webpages. The game is defined as follows:

- There is a fixed amount of resources - normalized to 1.
- Each player i has a value $v_i(x_i)$ of advertising on some proportion $x_i \in [0, 1]$ of the resource. We assume that v_i obeys the following properties:
  - **Non-negative**: $v_i(x_i) \geq 0$ for all $x_i \in [0, 1]$. This is a fair assumption, since a player would not participate in the game if they got negative value from it.
  - **Continuous and differentiable**: v_i is continuous and differentiable. This allows us to use calculus in our analysis of the game.
  - **Monotone increasing**: This is a reasonable assumption, since a larger proportion of the resource must provide a higher value.
  - **Concave**: This is a reasonable assumption in the Zillow-like advertising game due to diminishing returns from more ads. However, this might not hold in a general bandwidth sharing problem. For instance, if a player wants to use network bandwith to stream a video, their value will be 0 until they get a high enough bandwidth - which would violate the concavity assumption.

- The game is played in the following way:
  - Each player i submits a bid w_i.
  - The auctioneer (Zillow, for example) collects all bids (All-Pay) and distributes the resource proportionally to the bids.
  - Player i gets $x_i = \frac{w_i}{\sum_j w_j}$ of the resource.
  - Player i's utility is $v_i(x_i) - w_i$.

For example, suppose there are two players with value functions $v_1(x) = 3x$ and $v_2(x) = 6x$. If player 1 bids $w_1 = 1$ and player 2 bids $w_2 = 2$, then player 1 will get $x_1 = \frac{1}{3}$ of the resource and player 2 will get $x_2 = \frac{2}{3}$ of the resource. The utility of player 1 is $v_1(x_1) - w_1 = v_1\left(\frac{1}{3}\right) - 1 = 3 \cdot \frac{1}{3} - 1 = 0$, and the utility of player 2 is $v_2(x_2) - w_2 = v_2\left(\frac{2}{3}\right) - 2 = 6 \cdot \frac{2}{3} - 2 = 2$.

3 Equilibrium Analysis

When trying to determine the price of anarchy, we want to find the ratio of the worst-case Nash equilibrium social cost to the optimal social cost. In the proof, we will:

1. Derive conditions, in terms of x_i, that a Nash equilibrium must satisfy.
2. Demonstrate that restricting the value functions to be of the form $v_i(x_i) = a_i \cdot x_i$ will only make the price of anarchy worse.
3. Use the simplified value function to reason about the worst-case Nash as well as the socially optimum allocation.

3.1 Conditions for Nash

Note that a set of strategies $(w_1, w_2, \ldots, w_n)$ is a Nash equilibrium if no player can unilaterally deviate and improve their utility. That is, for all i:

$$\begin{aligned} & u_i(w_i, w_{-i}) \geq u_i(w_i', w_{-i}) \quad \forall w_i' \\ \implies & w_i = \arg\max_w u_i(w, w_{-i}) \end{aligned}$$

Using the definition of utility, we can expand this to:

$$\begin{aligned} w_i &= \arg\max_w \left(v_i\left(x_i\right) - w_i\right) \\ &= \arg\max_w v_i\left(\frac{w}{w + \sum_{j \neq i} w_j}\right) - w \end{aligned}$$

Since v_i is continuous and differentiable, we can differentiate with respect to w and set the derivative to 0 to find w that maximizes the expression:

$$\frac{d}{dw}\left(v_i\left(\frac{w}{w + \sum_{j \neq i} w_j}\right) - w\right) = 0 \tag{1}$$

Note that the argument of v_i is a function of w and it can be written as:

$$\begin{aligned}
\frac{w}{w+\sum_{j\neq i} w_j} &= \frac{w+\sum_{j\neq i} w_j - \sum_{j\neq i} w_j}{w+\sum_{j\neq i} w_j} \\
&= 1 - \frac{\sum_{j\neq i} w_j}{w+\sum_{j\neq i} w_j} \\
\implies \frac{d}{dw}\left(\frac{w}{w+\sum_{j\neq i} w_j}\right) &= \frac{d}{dw}\left(1 - \frac{\sum_{j\neq i} w_j}{w+\sum_{j\neq i} w_j}\right) \\
&= \frac{\sum_{j\neq i} w_j}{(w+\sum_{j\neq i} w_j)^2}
\end{aligned}$$

Using the result above, and applying chain rule to our original expression (1), we get:

$$v_i'\left(\frac{w}{w+\sum_{j\neq i} w_j}\right)\cdot \frac{\sum_{j\neq i} w_j}{(w+\sum_{j\neq i} w_j)^2} - 1 = 0$$

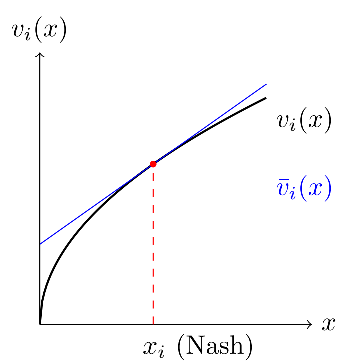
Note that the term $\left(\frac{w}{w+\sum_{j\neq i} w_j}\right)$ is the proportion of the resource that player i gets. That is, $x_i = \frac{w}{w+\sum_{j\neq i} w_j}$. Additionally, setting $p = \sum_j w_j$ (i.e. the price of the auction), we can rewrite the above equation as:

$$\begin{aligned}
v_i'(x_i)\cdot \frac{\sum_{j\neq i} w_j}{(w+\sum_{j\neq i} w_j)^2} &= 1 \\
\implies v_i'(x_i)\cdot \frac{\sum_{j\neq i} w_j}{(w+\sum_{j\neq i} w_j)} &= (w+\sum_{j\neq i} w_j) = p \\
\implies v_i'(x_i)\cdot(1-x_i) &= p
\end{aligned}$$

Where the last step follows from the fact that we are adding the proportions of all players except i, and the sum of all proportions is 1. Therefore, in a Nash equilibrium each player i must satisfy: $v_i'(x_i)\cdot(1-x_i) = p$.

3.2 Simplifying value functions

Let x_i be the proportion of the resource that player i gets in a Nash equilibrium. Set $\bar{v}_i$ to be the linear approximation of v_i around x_i:



Algebraically, this looks like:

$$\bar{v}_i(x) = v_i(x_i) + v_i'(x_i) \cdot (x - x_i)$$

Using the graph, we can make the following observations:

1. We took the approximation around x_i, so:
   (a) The value of $\bar{v}_i(x)$ is the same as $v_i(x)$ at $x = x_i$.
   (b) The slope of $\bar{v}_i(x)$ is the same as the derivative of $v_i(x)$ at $x = x_i$.
2. Since v_i is concave, $\bar{v}_i(x)$ is always at least as high as $v_i(x)$ for all x.

These observations inform the following claims:

Claim 1: The Nash equilibrium under $\bar{v}_i$ is the same as the Nash equilibrium under v_i.

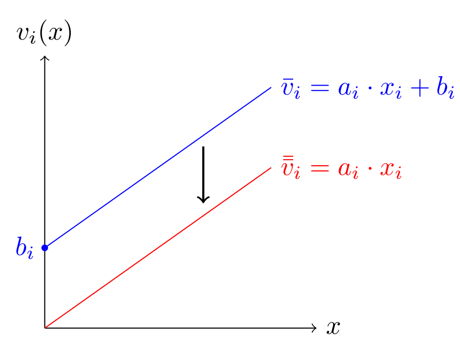
Proof. From observation 1b and the condition we derived for Nash equilibrium, we know that the x_i that satisfies the Nash equilibrium condition for v_i is the same as the x_i that satisfies the Nash equilibrium condition for $\bar{v}_i$. From observation 1a, we know that $v_i(x_i) = \bar{v}_i(x_i)$ for all i - so the social welfare of the Nash equilibrium is the same under both v_i and $\bar{v}_i$. ■

Claim 2: The social welfare under $\bar{v}_i$ is at least as high as the social welfare under v_i at the Nash equilibrium.

Proof. From observation 2, we know that $\bar{v}_i(x)$ is always at least as high as $v_i(x)$ for all x. Irrespective of the proportions x_i* at which the social optimum occurs, the term-wise sum of utilities under $\bar{v}_i$ is at least as high as the term-wise sum of utilities under v_i. So, the social welfare under $\bar{v}_i$ is at least as high as the social welfare under v_i at the Nash equilibrium. ■

So, shifting to a linear value function does not change the Nash equilibrium, but may increase the social optimum. As a result, it can only make the Price of Anarchy worse.

The above analysis shows that we can restrict our attention to value functions of the form $\bar{v}_i(x_i) = a_i \cdot x_i + b_i$. Now, we will show that this can be further simplified to $\bar{\bar{v}}_i(x_i) = a_i \cdot x_i$. Visually, the shift from $\bar{v}_i$ to $\bar{\bar{v}}_i$ looks like:



First, observe that b_i is always non-negative. This is because the original function v_i is non-negative, and the linear approximation is always at least as high as the original function (from observation 2). Then, for all x, we have $\bar{v}_i(x) \geq v_i(x) \geq 0$. So, $b_i = \bar{v}_i(0) \geq v_i(0) \geq 0$.

Now, also note that shifting from $\bar{v}_i$ to $\bar{\bar{v}}_i$ shifts both the social optimum and the Nash equilibrium by an amount that is constant in all x_i. So, the sets of x_i that satisfy the Nash or social optimum conditions do not change, but the value of both the Nash equilibrium and the social optimum decreases. In particular, this decrease is given by: $\sum_i \bar{v}_i(x_i) - \sum_i \bar{\bar{v}}_i(x_i) = \sum_i b_i$. Since each b_i is non-negative, the sum $\sum_i b_i$ is also non-negative.

Let Nash(f) and Opt(f) be the social welfare at the Nash equilibrium and the social optimum under the value function f respectively. Then, we have $\text{Nash}(\bar{\bar{v}}_i) = \text{Nash}(\bar{v}_i) - \sum_i b_i$ and $\text{Opt}(\bar{\bar{v}}_i) = \text{Opt}(\bar{v}_i) - \sum_i b_i$. So, the price of anarchy under $\bar{\bar{v}}_i$ is given by:

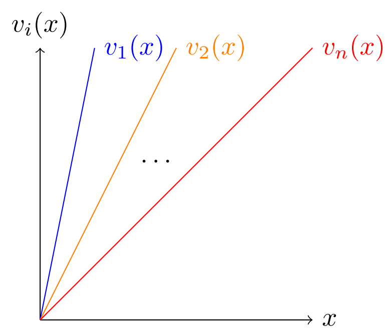
$$\frac{\text{Nash}(\bar{\bar{v}}_i)}{\text{Opt}(\bar{\bar{v}}_i)} = \frac{\text{Nash}(\bar{v}_i) - \sum_i b_i}{\text{Opt}(\bar{v}_i) - \sum_i b_i} \leq \frac{\text{Nash}(\bar{v}_i)}{\text{Opt}(\bar{v}_i)}$$

This follows from the fact that for any two non-negative numbers a and b, $\frac{a-c}{b-c} \leq \frac{a}{b}$ for $c \geq 0$. For verification, set $a = 7$, $b = 8$, and $c = 6$. Then, we obtain $\frac{1}{2} \leq \frac{7}{8}$. So, the price of anarchy under $\bar{\bar{v}}_i$ is worse than the price of anarchy under $\bar{v}_i$.

We can conclude, from the above analysis, that the price of anarchy under linear value functions of the form $\bar{\bar{v}}_i(x_i) = a_i \cdot x_i$ is at least as bad as the price of anarchy under the original value functions.

3.3 Worst-case Nash and Optimum

We can now work with a simplified value function of the form $v_i(x_i) = a_i \cdot x_i$. For notational convenience, order the players such that $a_1 \geq a_2 \geq \ldots \geq a_n$. Then, plotted on the same graph, the value functions look like:



The social welfare is given by:

$$\begin{aligned} SW(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} (v_i(x_i) - w_i) + \sum_{i=1}^{n} w_i \\ &= \sum_{i=1}^{n} a_i \cdot x_i - \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i \\ &= \sum_{i=1}^{n} a_i \cdot x_i \end{aligned}$$

Since player 1 has the highest a_i value, the optimal allocation in this case would be to give the entire resource to player 1. The social welfare in this case is given by $a_1 \cdot 1 = a_1$.

Now, let's consider a Nash equilibrium. The social welfare is given by:

$$SW(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} a_i \cdot x_i$$

Let $N^+ = \{i \mid x_i > 0\} \setminus \{1\}$. That is, N^+ is the set of players not including player 1 that get a non-zero proportion of the resource in the Nash equilibrium. Then, we can rewrite the above expression as:

$$SW(x_1, x_2, \dots, x_n) = a_1 \cdot x_1 + \sum_{i \in N^+} a_i \cdot x_i$$

Then, every player i in N^+ satisfies 2 conditions - $a_i \cdot (1 - x_i) = p$ and $x_i > 0$. Both of these conditions are true if and only if $a_i > p$ for any $i \in N^+$. Then, the social welfare at the Nash equilibrium is given by:

$$\begin{aligned} SW(x_1, x_2, \dots, x_n) &= a_1 \cdot x_1 + \sum_{i \in N^+} a_i \cdot x_i \\ &\geq a_1 \cdot x_1 + \min_{i \in N^+} a_i \cdot \left(\sum_{i \in N^+} x_i \right) \\ &= a_1 \cdot x_1 + \min_{i \in N^+} a_i \cdot (1 - x_1) \\ &\geq a_1 \cdot x_1 + p \cdot (1 - x_1) \\ &= a_1 \cdot x_1 + (a_1 \cdot (1 - x_1)) \cdot (1 - x_1) \\ &= a_1 \cdot \left(x_1 + (1 - x_1)^2 \right) \end{aligned}$$

Minimizing the above expression with respect to x_1, we get $1 + 2 \cdot (1 - x_1) \cdot -1 = 0 \implies x_1 = \frac{1}{2}$. Substituting this value back in, we get:

$$\begin{aligned} SW(x_1, x_2, \dots, x_n) &\geq a_1 \cdot \left(\frac{1}{2} + \left(1 - \frac{1}{2} \right)^2 \right) \\ &= a_1 \cdot \left(\frac{1}{2} + \frac{1}{4} \right) \\ &= \frac{3}{4} \cdot a_1 \end{aligned}$$

So, the minimum social welfare at a Nash equilibrium is at least $\frac{3}{4} \cdot a_1$. Using this, and that the social optimum is a_1, we can conclude that the price of anarchy is at most $\frac{4}{3}$.