CS 6840 Algorithmic Game Theory November 8, 2024

Lecture 31: Second price for multiple items and XOS

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1 Second price for multiple item auctions

For single item auctions, second price is nice because truthful bidding is a dominant strategy. However, it is not the case that all equilibria are good.

Consider an auction where player 1 bids $b_1 \gg v_i$ (for all v_i) and all other players bid $b_i = 0$. It is easy to verify that this is a Nash equilibrium. But the social welfare could be very bad compared to the optimal value. For example, if $v_2 \gg v_1$, the social welfare from this equilibrium could be arbitrarily worse than the optimal social welfare. This shows that the price of anarchy is unbounded.

We can try to resolve this by making a common assumption that there is no over-bidding. That is, no player ever bids more than their valuation. (We should note that although we will make this assumption, it is not so strange to over-bid.)

Generalizing to multi-item separate auctions, we will write this no-over-bidding assumption as follows:

Assumption 1. For all players i and all sets of items S,

$$v_i(S) \geq \sum_{j \in S} b_{ij}.$$

Two lectures ago, we showed that for simultaneous first price auctions, bidding optimally in a given randomized environment is NP-hard. This result also holds for second price auctions. In fact, the same proof works.

Theorem 1. *For simultaneous multi-item second price auctions, finding the optimal bidding strategy is NP-hard.*

We will sketch the proof again below.

Proof sketch. Say you are competing in an auction against one other bidder. Your value for any single item j is $v_j = v \gg 1$, and your value for a set S is $v(S) = \max_{j \in S} v_j$ (or equivalently $v(S) = v$ for $S \neq \emptyset$). Given sets $T_1, \ldots, T_n$, the opponent selects one such set T_i at random and bids 1 on all items $j \in T_i$ and $v' \gg v$ on all other items.

We argued that our strategy should be to bid $1^+ = 1 + \epsilon$ on some set S of items; in fact, it no longer matters how small ϵ should be because if we win the item, we will pay only 1. We also note that we are not over-bidding: since $v \gg 1$, the sum of all our bids does not exceed v.

The rest of the proof is identical to the previous proof for first price, where we show that finding the optimal bidding strategy is as hard as solving a variant of the hitting set problem. ■

2 XOS valuations

So far we have seen one valuation function which was just to take the largest value among all items in our set.

Example 1. $v_i(S) = \max_{j \in S} v_{ij}$.

There is a bigger class of valuations which will see that has this as a special case. The previous example says that we only one item and further items provide no additional value. But this is usually not the case.

We might consider also giving some value to the second largest item in our set:

Example 2. $v_i(S) = \begin{cases} v_{ij} & \text{if } S = \{j\} \\ \max_{\ell \neq j \in S} v_{ij} + \frac{1}{2} v_{i\ell} & \text{otherwise} \end{cases}$

In the general version, we have many options for how to value each item:

Definition 1 (XOS valuation)**.** A valuation function $v_i(S)$ for player i is called XOS if there are values v_{ij}^k such that

$$v_i(S) = \max_k \sum_{j \in S} v_{ij}^k.$$

In the above definition, we interpret v_{ij}^k as option k for the value for player i on item j.

One way to think about XOS valuations is that you are a producer, and you have many things that you could produce. Each thing that you could produce is one of your options (k). If you end up producing a cake, you would place a high value on cake ingredients. But if instead you produce a salad, the high value would be on salad ingredients. And naturally, we will produce the option that gives the highest total value, which is captured in the maximum.

Claim 1. *The previous two examples are special cases of XOS valuations.*

Proof. For Example 1, we will have m options, one for each item, $k = 1, \ldots, m$.

Let

$$v_{ij}^k = \begin{cases} v_{ij} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$v_i(S) = \max_k \sum_{j \in S} v_{ij}^k = \max_{k \in S} v_{ik},$$

so Example 1 is XOS.

For Example 2, we keep the same m options and add $m(m-1)$ more options, one for each ordered pair of distinct items (k_1, k_2):

$$v_{ij}^{(k_1,k_2)} = \begin{cases} v_{ik_1} & \text{if } j = k_1 \\ v_{ik_2}/2 & \text{if } j = k_2 \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to see that with these options, the valuation under the XOS definition is exactly the one in Example 2. ■

Next lecture, we will show that everything we did before with the valuation in Example 1 also works more generally for XOS valuations. Intuitively, we know how to handle a single option because it is just the sum of items which we can treat separately. And if our list of options is not exponentially large, we can do this for every option and choose the best.

3 Submodular valuations

A more natural class of valuations is submodular valuations, which capture the idea that more items have diminishing additional value.

Definition 2 (Submodular)**.** A valuation v is submodular if for any sets $A \subseteq B$ and any element $j \notin B$,

$$v(A \cup \{j\}) - v(A) \geq v(B \cup \{j\}) - v(B).$$

Theorem 2. *Submodular valuations are XOS.*

Proof. Suppose player i has a submodular valuation function v'. We will show this is XOS by creating a set of values v_{ij}^{π} for every permutation π of element that satisfy the XOS definition.

We will define the set $A_j^{\pi} = \{k : \pi(k) < \pi(j)\}$ to be the elements that came before j in the permutation π.

Then we define $v_{ij}^{\pi} = v'(A_j^{\pi} \cup \{j\}) - v'(A_j^{\pi})$.

We will prove the theorem by proving two claims.

Claim 2.

$$v'(S) \leq \max_{\pi} \sum_{j \in S} v_{ij}^{\pi}$$

Claim 3.

$$v'(S) \geq \max_{\pi} \sum_{j \in S} v_{ij}^{\pi}$$

These two claims together show that the two terms are equal and so the submodular valuation is indeed XOS, proving the theorem.

Proof of Claim 2. To show this, we just need to find one permutation such that the inequality holds. Say the elements in S are $\{j_1, j_2, \ldots, j_k\}$. Then consider the permutation π where all elements in S appear before all elements not in S. In particular, $\pi(j_1) = 1$, $\pi(j_2) = 2, \ldots$ and so on.

Then using the definition of v_{ij}^{π},

$$\begin{aligned}\sum_{j \in S} v_{ij}^{\pi} &= v'(\{j_1\}) + \left[v'(\{j_1, j_2\}) - v'(\{j_1\})\right] + \cdots + \left[v'(S) - v'(S \setminus \{j_k\})\right] \\ &= v'(S).\end{aligned}$$

Since $v'(S) = \sum_{j \in S} v_{ij}^{\pi}$ for this π, we have $v'(S) \leq \max_{\pi} \sum_{j \in S} v_{ij}^{\pi}$ as desired. □

Proof of Claim 3. The claim is equivalent to the statement that $v'(S) \geq \sum_{j \in S} v_{ij}^{\pi}$ for all permutations π.

So let's take any arbitrary permutation π, and define π' as the permutation with all elements of S moved to the beginning. In particular, the first $|S|$ elements in π' should be the elements of S in the order that they appear in π, and the last $m - |S|$ elements should be the remaining elements in the order they appeared in π.

Now, for any $j \in S$, we notice that $A_j^{\pi'}$ contains only elements of S, while A_j^{π} contains those same elements along with some elements not in S. In other words, $A_j^{\pi'} \subseteq A_j^{\pi}$. Then using the definition of submodularity, we get:

$$\begin{aligned} v_{ij}^{\pi} &= v'(A_j^{\pi} \cup \{j\}) - v'(A_j^{\pi}) \\ &\leq v'(A_j^{\pi'} \cup \{j\}) - v'(A_j^{\pi'}) \\ &= v_{ij}^{\pi'} \end{aligned}$$

Finally summing over all elements $j \in S$,

$$\sum_{j \in S} v_{ij}^{\pi} \leq \sum_{j \in S} v_{ij}^{\pi'} = v'(S)$$

where the last equality is because the elements of S are at the beginning of π' and the same argument as in the previous proof. $\square$

$\blacksquare$