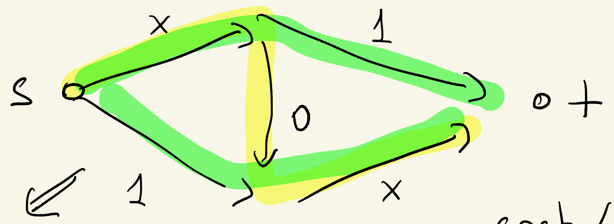
Price of Anarchy in routing games

Example:



cost / delays

r = 1 unit of traffic

delay = 2 Nash

opt ½ – ½ top & bottom

delay 1.5

General model

Network

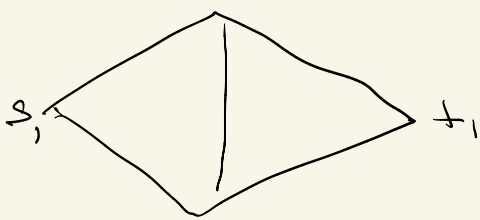
$s_i - t_i$ pairs source & dest. $i = 1, \dots, n$

r_i = rate of traffic

edge: cost fn $c_e(x)$ = cost on e with x amount of traffic

c is non increasing, ≥ 0 & continuous

Solutions:



all path P s_i to t_i $f_P \geq 0$ amount of traffic using path P

$f_P \geq 0$ all P

$$\sum_{P\ s_i - t_i} f_P = r_i \quad \text{all } i$$

cost for user following path P

$$c_P(f) = \sum_{e \in P} c_e(f(e))$$

$$f(e) = \sum_{e \in P} f_P \quad \text{edge } e$$

total cost of a flow

$$\boxed{\mathrm{cost}(f) = \sum_P f_P \, c_P(f)}$$

OPT $\min \mathrm{cost}(f)$

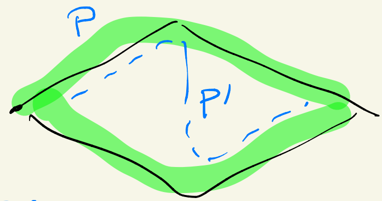
f feasible solution

Eqilibrium flow f

– feasible

– all $s_i - t_i$ path P with $f_P > 0$

$$c_P(f) = \min_{P' s_i \to t_i} c_{P'}(f)$$



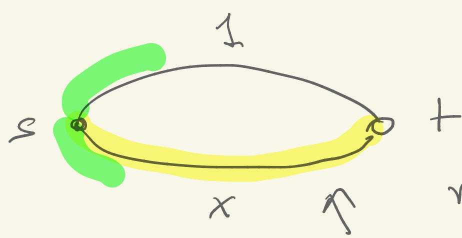
definition makes sense as we assumed $c_e(x)$ continuous

Question: price of Anarchy

$$\max_{f \text{ Nash}} \frac{\text{cost}(f)}{\text{OPT}}$$

f Nash

Example



$r = 1$

Nash cost: 1

OPT $\frac{1}{2}$ - $\frac{1}{2}$

cost $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

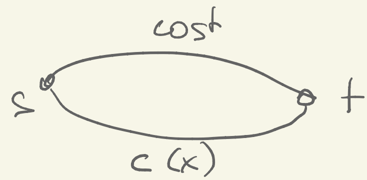
Define a class cost function $\mathcal{C}$

$\alpha(\mathcal{C})$ = max PoA on 2-node networks

top link const > 0

rate $r > 0$

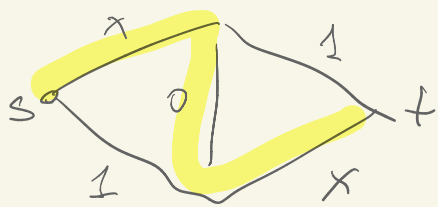
bottom link $c \in \mathcal{C}$



<u>Thm:</u> all classes $\mathcal{C}$ all networks

$$\text{PoA} \le \alpha(\mathcal{C})$$

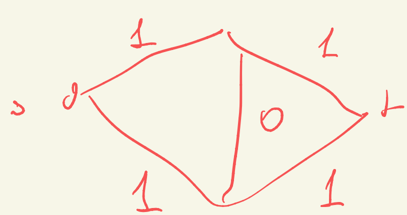
<u>Proof:</u> f equilibrium flow



Consider network & r_i, $s_i - t_i$ as above

fixed edge cost $c_e(f(e))$

e.g



Claim: with these cost f is optimal.

$$\mathrm{cost}(f') = \sum_P f'_P \, c_P(f') \underset{\text{cost fixed}}{=} \sum f'_{P} \, c_P(f)$$

$$\geq \sum_i \sum_{P_{s_i - t_i}} f'_P \min_{P''_{s_i - t_i}} c_{P''}(f)$$

$$= \sum_i r_i \min_{P''_{s_i - t_i}} c_{P''}(f)$$

$$\xrightarrow{\text{Eqilib.}} = \mathrm{cost}(f)$$

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use $\alpha(e)$ on each edge

cost $= c_e(f)$, fn is: $c_e(x)$

$r = f(e)$

Claim:

$$\alpha(e) \geq \frac{f(e)\, c_e(f_e)}{f^*(e)\, c_e(f^*(e)) + \left[f(e) - f^*(e)\right] c_e(f(e))}$$

Nash of this nework

a solution, OPT only make it better